What is a Qubit?

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1. Introduction:

- Classical bit

- Bit : Binary digit : 0 or 1

  Yes or No ; True or False

- Binary string : A collection of bits

2. Quantum Bit or Qubit

Objectives: Introduce Dirac notation and discuss the general qubit state

Dirac Notation:

Dirac Notation

- $|\psi\rangle$, $|f\rangle$, $|g\rangle$ — **kets**
- $\langle\psi|$, $\langle f|$; $\langle g|$ — Bra

state 0 can be denoted $|0\rangle$

1 $|1\rangle$

Definition

A qubit is a state that is a superposition of two orthonormal state : $|0\rangle$ $|1\rangle$

- A qubit : $|\psi\rangle$
- Superposition : Linear combination : Complex Vector space
- $|0\rangle$ and $|1\rangle$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex numbers.

- Normalization : $|\alpha|^2 + |\beta|^2 = 1$

$|\alpha|^2 = \alpha^* \alpha$

$|\alpha|$: Magnitude of α

Measurement

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Probabilistic

$\text{Prob}(|\psi\rangle \to |0\rangle) = |\langle 0|\psi\rangle|^2 = |\alpha|^2$

$\text{Prob}(|\psi\rangle \to |1\rangle) = |\langle 1|\psi\rangle|^2 = |\beta|^2$

- Collapse

- Start with a state $|\psi\rangle$
- Make a measurement and you find an outcome $|\theta\rangle$
- The state stays at the outcome $|\theta\rangle$

Activity – Classify Candidate States as Valid or Invalid

Candidate	Valid (V) or Invalid (I)
$\|\Psi> = \frac{1}{2}\|0> + \frac{1}{2}\|1>$	I
$\|\Psi> = \frac{1}{\sqrt{2}}\|0> + \frac{1}{\sqrt{2}}\|1>$	V
$\|\Psi> = \frac{1}{\sqrt{2}}\|0> - \frac{1}{\sqrt{2}}\|1>$	V
$\|\Psi> = \frac{1}{\sqrt{10}}\|0> + \frac{3}{\sqrt{10}}\|1>$	V
$\|\Psi> = \frac{1}{\sqrt{10}}\|Tail> + \frac{3}{\sqrt{10}}\|Head>$	V

2. Quantum Bit or Qubit

Objectives: Discuss normalization of the state and Measurement

Activity: Guided example – The Quantum Coin

Problem: The quantum state of a spinning coin can be written as a superposition of Head and Tail, using head as |1> and tail has |0>. Suppose the state is

$$|coin> = \frac{1}{\sqrt{2}}(|0> \; +|1>)$$

- Upon measurement, what is the probability of getting head? What is the probability of getting head?

Question for you:

- Two measurements on the same coin are done consecutively. The first measurement yields Head. What is the probability the second measurement would yield tail.

$$|\psi\rangle = \alpha\,|\text{Head}\rangle + \beta\,|\text{tail}\rangle \qquad P(H) = |\alpha|^2$$
$$P(T) = |\beta|^2$$

- Measurement : Outcome Head

$$\boxed{|\psi\rangle = |\text{Head}\rangle}$$

Probability : $|\langle \text{tail}|\psi\rangle|^2 = |\langle \text{tail}|\text{Head}\rangle|^2$

$$= 0$$

Activity: Solve the problem below about a weighted Quantum Coin

A weighted coin has twice the probability of landing on head then on tail. Write down quantum state of the coin.

Activity: Measuring a qubit – Answer questions about measurements

A qubit is prepared in an unknown state. It is measured with an outcome $|0>$

- ❑ Which of the following could be its initial state: $|0>$, $\frac{1}{\sqrt{10}}$|0> + $\frac{3}{\sqrt{10}}$|1>, $\frac{1}{\sqrt{2}}$|0> + $\frac{3}{\sqrt{2}}$|1>,
- ❑ If you tried to measure the same qubit the second time, can you narrow down what the initial state was?
- ❑ Another qubit is prepared in the same unknown state (the original one). It is measured in the state $|1>$. What can you say about the unknown state?

3. Matrix/Vector Representation

Objective: Discuss the vector representation of a state

Vector Representation

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

2D space $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|0\rangle \xrightarrow{\text{map}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

- Change the state of a qubit by multiplying it with Matrices.
- Any Matrix? No
- Projections and Unitary Matrices

1) Projections : Measurement.

Unitary Matrices :

Definition

A matrix U is unitary if its conjugate transpose (Hermitian conjugate) is its own inverse.

- Matrix
- Conjugate Transpose : $U^\dagger$ (Hermitian conjugate)

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- * Transpose of U : U^T

$$U^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \leftarrow$$

- * Take the conjugate of every entry in U^T

$$U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$$\boxed{U^{\dagger} = U^{-1}}$$ Then U is unitary.

Properties of Unitary Matrices

* They keep to Normalization

start $|\psi_1\rangle$ you know $\langle\psi_1|\psi_1\rangle = 1$

$$|\psi_2\rangle = U|\psi_1\rangle \qquad \langle\psi_2|\psi_2\rangle = 1$$

* $\mathrm{Det}(U) = e^{i\varphi} \Rightarrow |\mathrm{Det}(U)| = 1$

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det(U) = ad - cb$$

$$|\det(U)| = 1$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\det(X) = 0\times 0 - 1\times 1$

$\hookrightarrow = -1$

$= e^{i\pi}$

$\det(X) = |-1|$

$= 1$

* They always have an inverse.

4. Action on a Qubit State

Discuss evolving the state of a qubit through unitary operations.

Practice – Take Conjugate the Transpose of the following Matrices

- $M = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$
- $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
- $V = \begin{pmatrix} i & 2i+5 \\ 3i & -i \end{pmatrix}$

Activity: Classify each candidate action on a qubit as Valid or Invalid

$H^2 = H H = H H^\dagger = \mathbb{1}$ $H^\dagger = H^{-1}$ Unitary $H^\dagger = (H^T)^* = (H^*)^T$

Matrix	Useful Information	Valid (V) or Invalid (I)
H	$H^\dagger = H$ and $H^2 = I$ (Identity matrix)	V because $H^\dagger = H^{-1}$
Y	$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	V
L	Det(L) = 0 … (It does not have an inverse)	I
K	K has 4 rows and 5 columns	I
M	Det(M) = 5	I

$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ Hadamard matrix

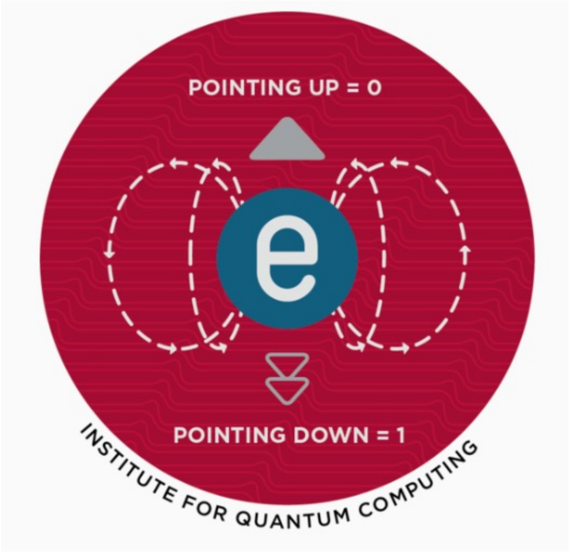
Physical Realization: What is a Qubit Made out of?

- 1. **Spin Qubit:**

We can use the spin of an atom/electron/nucleus to as a qubit.

Example:

Spin up = $|0>$ and Spin down = $|1>$



https://uwaterloo.ca/institute-for-quantum-computing/quantum-101/quantum-information-science-and-technology/what-qubit

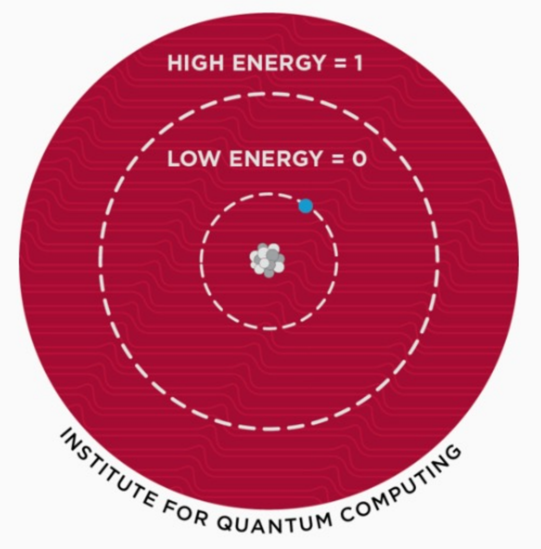
Physical Realization: What is a Qubit Made out of?

2. **Trapped Atoms and Ions:**

We can use the energy levels in atoms/ions to make a qubit

Example:

Lowest energy = $|0>$ and First excited energy = $|1>$



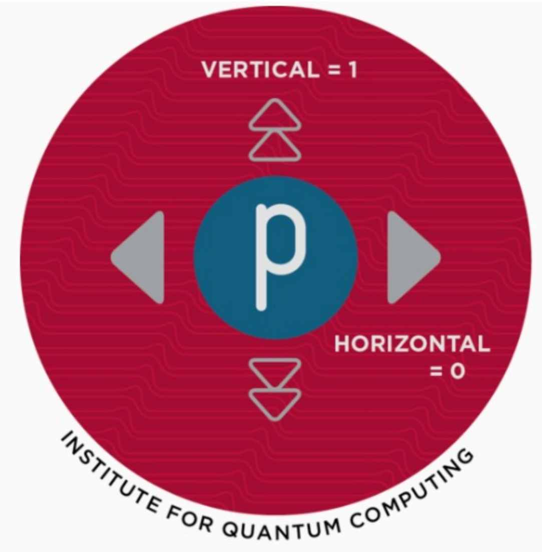
Physical Realization: What is a Qubit Made out of?

3. **Photon:** Polarization Qubit

- The polarization is the direction of the Electric Field.
- It turns out that if you know the direction of the photon, the polarization has two possibilities: **Horizontal and vertical.**
- **So, we can make a qubit with superposition of polarization.**

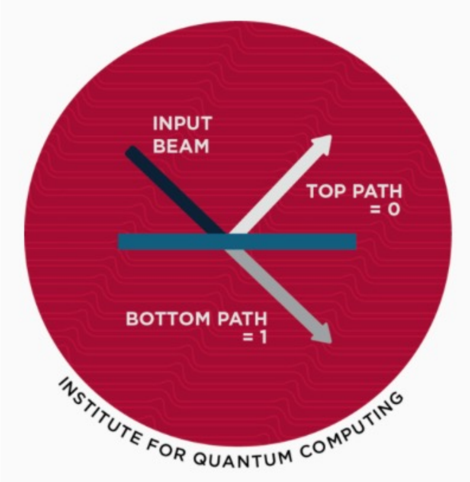
Example:

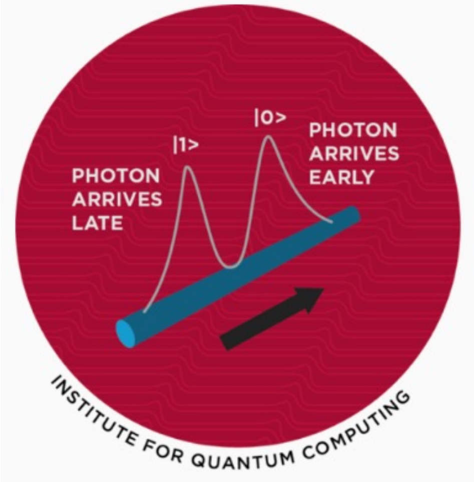
Vertical = $|0 >$ and Horizontal = $|1 >$



Physical Realization: What is a Qubit Made out of?

Other types of Photon qubits





Physical Realization: What is a Qubit Made out of?

3. **Superconducting Qubit**

- We can design an electrical circuit so that the current flow in a superconductor has two possibilities – Clockwise and anti-clockwise.
- So, we can make a qubit of superposition of these states

Example:

Clockwise = $|0>$ and Anti-clockwise = $|1>$

Physical Realization: What is a Qubit Made out of?

Company	Qubit Type	Number of Qubits
Google	Superconducting	Sycamore: 54 qubits
IBM	Superconducting	Eagle: 127 qubits Osprey: 423 qubits (release schedule 2022)
Xanadu	Photonics	216 qubits
Microsoft	Topological	
Ionq	Trapped ions	100+ qubits

Summary

- We use Dirac – Notation (Bra – Ket) to specify quantum states
- A qubit state is given by a superposition of states |0> and |1>
- Then, a general qubit state is $|\Psi> = \alpha|0> \; + \beta|1>$
- α and β are complex numbers with $|\alpha|^2 + |\beta|^2 = 1$
- It is convenient to use vector notation to represent a qubit state

THE END