



# **SUPERCONDUCTING QUANTUM MATERIALS & SYSTEMS CENTER**

## SQMS Summer Internship Report Modeling quantum state transport with SQMS devices

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### **Abstract**

To scale up the Hilbert space of architectures for quantum computation involving microwave cavities, we need to control more and more photons states.

A big step forward in this direction would be to connect two different cavities from a quantum mechanics point of view.

In this report, we will discuss which experimental approach would be feasible to transfer a quantum state from one cavity to another.

To perform this task, we will take advantage of the secondary modes of the cavities, discussing how a variable coupling between the cavity modes could help us in this process.

In the end, we will see how we can create entanglement between the cavities from this procedure.

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# 1 Introduction

Quantum computing could be a breakthrough technology for our world.

Although promising, we are still in the early stages of this revolution; to perform increasingly complex algorithms, we must scale the number of qubits under our control.

Thanks to their excellent facilities, which can produce High-Q cavities, SQMS devotes much research to performing quantum computation using these cavities.

With the photons inside these cavities, we can build and control a qudit state. A qudit is a  $d$ -level system with a Hilbert space which can represent the Hilbert state of a joint system of qubits.

The more photons we can control, the higher the dimension of the space, which allows us to run algorithms with increased complexity.

One way to address scalability could be to connect different cavities quantum mechanically. Being able to transfer and entangle states between these could allow us to decide when and where to store information and to communicate between the nodes of this quantum network.

With that in mind, in this report, we discuss various approaches that could be implemented experimentally to pursue this aim.

We tested our ideas with QuTip, a python library that lets us simulate a quantum system's evolution. The simulations are based on experimental measurements carried out on SQMS devices.

The report will be divided into different sections.

To begin our discussion, we need to implement the conversion between stationary information and a propagating form of that information in a single node. This is made possible by exploiting secondary modes of the cavity, which are strongly coupled with the transmission line.

The mechanism, called Q-Switch, and the second section deals with that.

In the third section, we will begin to talk about two different cavities, discussing how we can model this system.

Finally, we explore methods of connecting the two systems.

In the last section, we will briefly discuss how we can entangle two cavities with our implemented procedure.

During this report we will use natural units setting  $\hbar = 1$ .

## 2 Q-Switch

Our purpose is to discuss which could be the best way to build a quantum network with SQMS devices.

Before studying how we can make the network's nodes able to communicate with one another, we focus on the single node: the conversion between stationary information and a propagating form of that information is the first essential mechanism we must implement.

This is made possible by taking advantage of the cavity's secondary modes, which are strongly coupled to the transmission line.

We will transfer a quantum state from a storage mode with high Q to an output mode with low Q. This is why we give this process the name Q-Switch.

To complete this task, we need to drive the system with a four-wave mixing-based pumping scheme that uses the transmon as a conversion element.

### 2.1 Four wave mixing as memory-output conversion

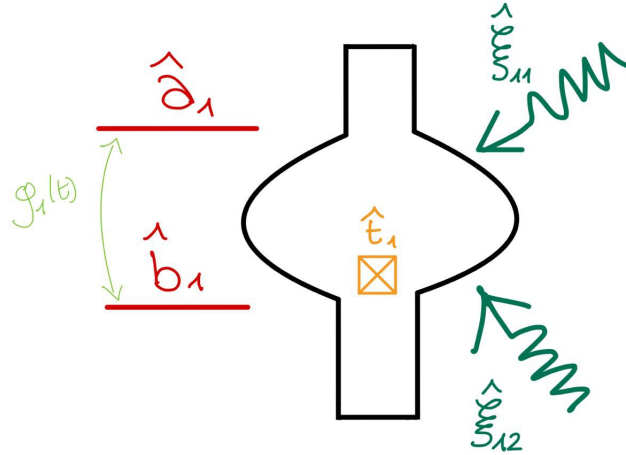
We wish to implement a mechanism in which the Hamiltonian takes the form:

$$\hat{H} = g(\hat{a}\hat{b}^\dagger + \hat{a}^\dagger\hat{b}) \quad (2.1)$$

This would provide oscillation in the system as a Jaynes-Cummings model. However, the output is strongly coupled to a transmission line, so the state will not return to the memory.

In this section, we will describe the coupling between the output mode and the external world as a coupling to a bath, adding a right dissipator to the Lindblad equation.

This scenario is equivalent to the following:



**Figure 1:** In this section, the coupling to the external world is treated as a coupling to a bath.

We need a four-wave mixing process to complete our task, where two driving modes will produce this interaction.

The full Hamiltonian is:

$$\hat{H} = \hat{H}_a + \hat{H}_b + \hat{H}_t + \hat{H}_p \quad (2.2)$$

$$\hat{H}_a = \omega_a \hat{a}^\dagger \hat{a} \quad \hat{H}_b = \omega_b \hat{b}^\dagger \hat{b} \quad (2.3)$$

$$\hat{H}_t = \omega_t \hat{t}^\dagger \hat{t} - E_j (\cos(\hat{\phi}) + \hat{\phi}^2/2) \quad (2.4)$$

$$\hat{H}_p = \epsilon_1(t)(e^{i\omega_1 t} \hat{a}^\dagger + e^{-i\omega_1 t} \hat{a}) + \epsilon_2(t)(e^{i\omega_2 t} \hat{b}^\dagger + e^{-i\omega_2 t} \hat{b}) \quad (2.5)$$

If we go to a displaced and resonant frame (see B for details) and we apply rotating wave approximation, we find that, besides the terms that are already present in the undriven case, this process reveals the additional terms:

$$\tilde{H} = |\xi_1|^2 (2\chi_{aa}\hat{a}^\dagger\hat{a} + \chi_{ab}\hat{b}^\dagger\hat{b} + \chi_{at}\hat{t}^\dagger\hat{t}) + |\xi_2|^2 (2\chi_{bb}\hat{b}^\dagger\hat{b} + \chi_{ab}\hat{a}^\dagger\hat{a} + \chi_{bt}\hat{t}^\dagger\hat{t}) + g(e^{-i\delta t}\hat{a}^\dagger\hat{b} + e^{i\delta t}\hat{a}\hat{b}^\dagger) \quad (2.6)$$

Where  $\chi_{kl} = -E_j\phi_k^2\phi_l^2$ ,  $\Delta_1 = \omega_a - \omega_1$ ,  $\Delta_2 = \omega_b - \omega_2$ ,  $\delta = \Delta_1 - \Delta_2$  and:

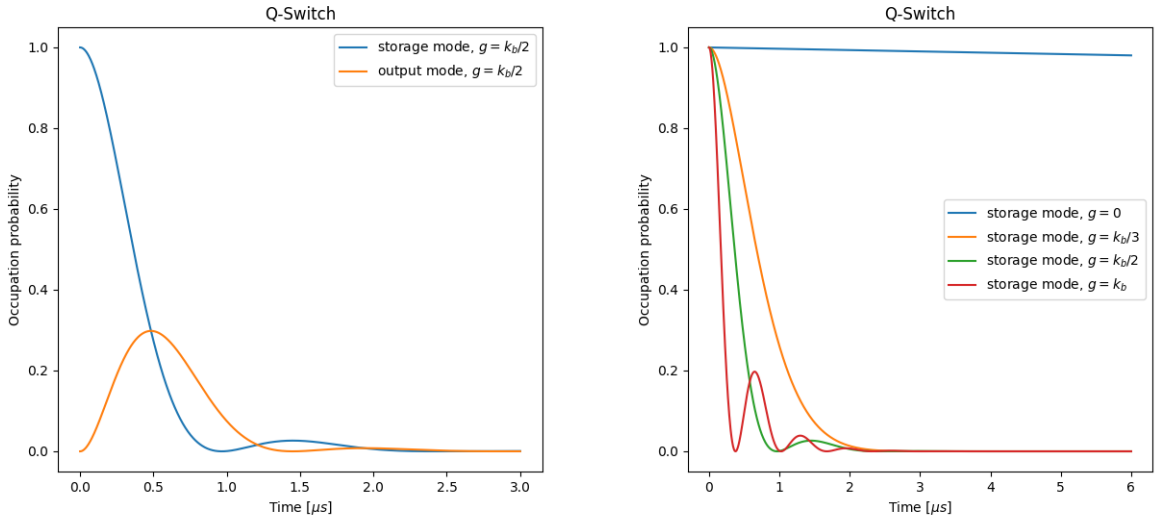
$$g(t) = \chi_{ab}\xi_1(t)\xi_2(t) \quad \xi_i = \frac{\epsilon_i}{\Delta_i} \quad (2.7)$$

The first two terms in eq(2.6) correspond to Stark shifts that will influence the resonant condition of the conversion process. The most important term for our purpose is the last one.

### 2.1.1 Simulations with constant coupling between the modes

We start from a fock state  $|1\rangle$  in storage mode and let the system evolve via a Lindblad equation. In this way, we treat the transmission line as an external bath.

For simplicity, we can drop the stark shift terms and we can consider a constant coupling  $g$  (so both  $\xi_1$  and  $\xi_2$  are constant).



**Figure 2:** Left: occupation probability for Storage mode (5.012 GHz with  $T_1=301 \mu s$ ) and Output mode (9.246 GHz with  $T_1=200 ns$ ),  $g=k_b/2=5$  MHz and  $\chi_{ab}/2\pi = -60 kHz$   
Right: occupation probability for Storage mode (5.012 GHz with  $T_1=301 \mu s$ ) for different coupling values, the best compromise between a fast process and the absence of oscillations is  $g = k_b/2$ .

### 3 Quantum communication between two cavities

We want to focus on two different cavities connected via a transmission line.

Firstly, we must understand how a transmission line is coupled to a single cavity.

Previously, we used a master-equation treatment to evaluate the evolution of the system. In this way, we described the external system as a heat bath.

Now we will explicitly consider this in the Hamiltonian, so as part of our system.

The total Hamiltonian will be:

$$\hat{H} = \hat{H}_{sys} + \hat{H}_{tl} + \hat{H}_{int} \quad (3.1)$$

Where  $\hat{H}_{sys}$  is the cavity's Hamiltonian,  $\hat{H}_{tl}$  the Hamiltonian of the transmission line, and  $\hat{H}_{int}$  the interaction's hamiltonian:

$$\hat{H}_{sys} = \omega_b \hat{b}^\dagger \hat{b} \quad (3.2)$$

$$\hat{H}_{tl} = \int_{-\infty}^{\infty} d\omega k(\omega) \hat{c}^\dagger(\omega) \hat{c}(\omega) \quad (3.3)$$

$$\hat{H}_{int} = \gamma(t) \int_{-\infty}^{\infty} d\omega k(\omega) [\hat{c}(\omega) \hat{b}^\dagger + h.c.] \quad (3.4)$$

Where  $c(\omega)$  are the boson annihilation operators:

$$[\hat{c}(\omega), \hat{c}^\dagger(\omega')] = \delta(\omega - \omega') \quad (3.5)$$

The choice of the letters b and c is due to the subject of the next section.

If we want to consider two cavities, we can generalize to:

$$\hat{H} = \hat{H}_{cav1} + \hat{H}_{cav2} + \hat{H}_{tl} + \hat{H}_{int1} + \hat{H}_{int2} \quad (3.6)$$

With:

$$\hat{H}_{cavi} = \omega_{b_i} \hat{b}_i^\dagger \hat{b}_i \quad (3.7)$$

$$\hat{H}_{tl} = \int_{-\infty}^{\infty} d\omega k(\omega) \hat{c}^\dagger(\omega) \hat{c}(\omega) \quad (3.8)$$

$$\hat{H}_{inti} = \gamma_i(t) \int_{-\infty}^{\infty} d\omega k(\omega) [\hat{c}(\omega) \hat{b}_i^\dagger + h.c.] \quad (3.9)$$

In these equations, we are supposed to control over time the coupling between the transmission line and the cavities.

#### 3.1 Switch on and switch off the coupling

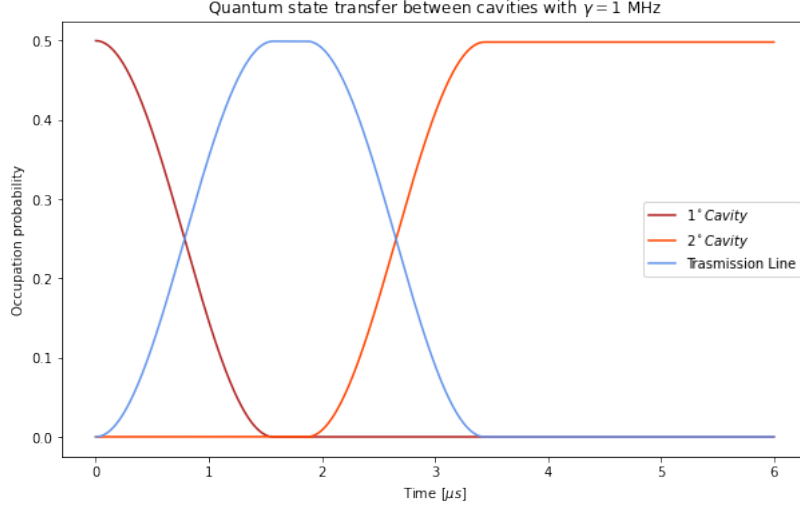
We can make a further assumption: we can put this coupling to 0 whenever we want.

We want to simulate a simple model to understand the system better: we consider the two cavities to be in resonance and we treat the transmission line as a single mode with the cavity's frequency.

Then the Hamiltonian is:

$$\hat{H} = \gamma_1(t) [\hat{c} \hat{b}_1^\dagger + h.c.] + \gamma_2(t) [\hat{c} \hat{b}_2^\dagger + h.c.] \quad (3.10)$$

In this case we can choose to implement:  $\gamma_i(t) = \gamma_i [\Theta(t - t_i^{in}) - \Theta(t - t_i^{fin})]$ .

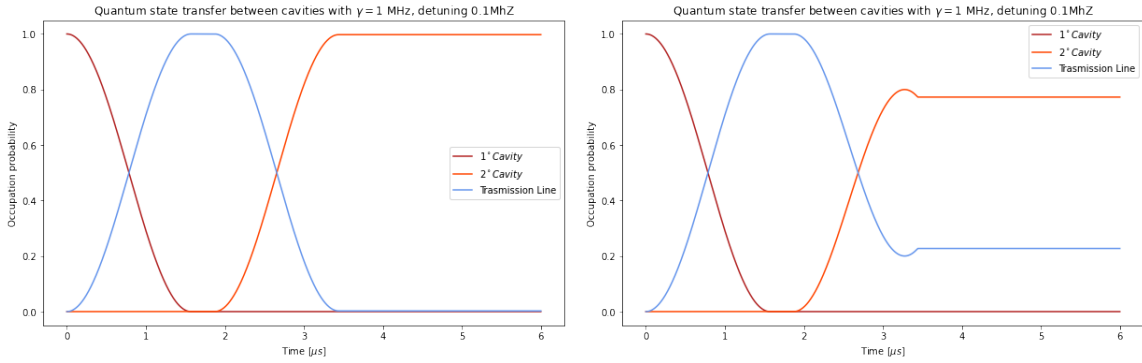


**Figure 3:** Transfer of a fock state  $|1\rangle$  between two cavities with  $\gamma_i(t) = \gamma[\Theta(t - t_i^{in}) - \Theta(t - t_i^{fin})]$   
Parameters:  $t_1^{in} = 0$ ,  $t_1^{fin} = \frac{\pi}{2\gamma}$ ,  $t_2^{in} = t_1^{fin} + 0.4 \mu s$ ,  $t_2^{fin} = t_2^{in} + \frac{\pi}{2\gamma}$ ,  $\gamma = 1 MHz$

If we want to add some detuning between the two cavities in this simple model, we consider the Hamiltonian:

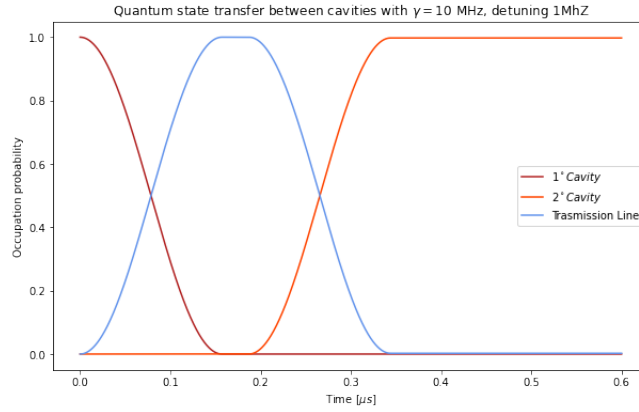
$$\hat{H} = \gamma_1(t) [\hat{c}\hat{b}_1^\dagger + h.c.] + \gamma_2(t) [\hat{c}\hat{b}_2^\dagger e^{i\Delta t} + h.c.] \quad (3.11)$$

Where  $\Delta = \omega_{b_2} - \omega_{b_1}$



**Figure 4:** On the left 0.1 MHz of detuning between the two cavities, while 1 MHz on the right

This will depend on the strength of the coupling with the transmission line. Making it bigger, this will show a better performance for 1 MHz:



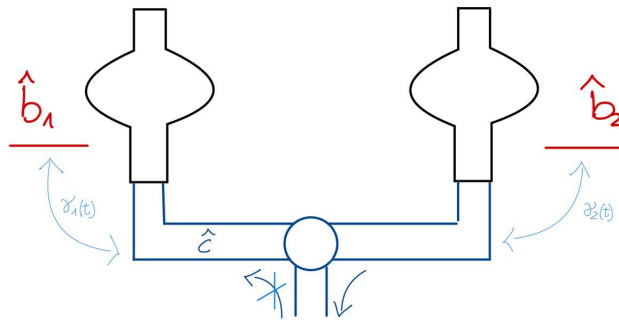
**Figure 5:** 1 MHz of detuning between the two cavities

### 3.2 One way transfer: some ideas

This procedure seems very good, therefore are still some concerns about that: is it feasible to switch off the coupling between the cavity and the transmission line during the dynamic evolution of the system? Can we control the coupling during the entire process?

If we cannot do that, we need to implement a one-way transfer somehow.

One idea that came to us at the beginning of our discussion, was to insert a circulator into the system:



**Figure 6:** Scheme of the system with a circulator

This is a passive, three-port device that transmits an input signal in one direction: a signal that enters from the left side in figure (6) is transmitted to the right side, while a signal from the right goes into the center port where it is measured.

Unfortunately, it is not simple to model the circulator at the Hamiltonian level as Qutip requires. Also, the loss this apparatus will bring to the system will affect the fidelity.

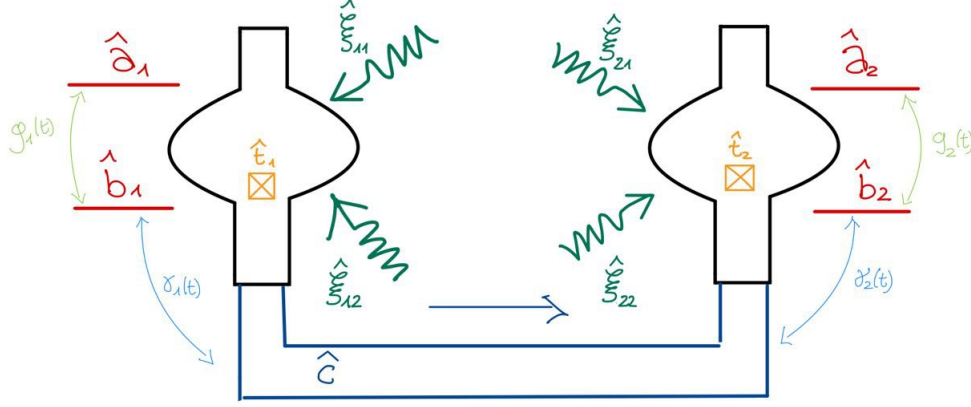
Another idea is to shape the coupling over time to maximize the absorption in the second cavity. This could be done by studying the Quantum Langevin equation.

Therefore, this could not be possible if we also cannot control the coupling with the transmission line over time.



## 4 Quantum state transfer between two cavities with Q-Switch

We consider the entire system:



**Figure 7:** Scheme of the system

The total Hamiltonian is the sum of the different terms for each part.

We can repeat the transformations that we have used in the previous sections to obtain the Hamiltonian:

$$\hat{H} = H_{Q-S1} + H_{Q-S2} + H_{CT} \quad (4.1)$$

Where dropping stark shifts:

$$H_{Q-Si} = g_i(t)(\hat{a}_i^\dagger \hat{b}_i + \hat{a}_i \hat{b}_i^\dagger) \quad g_i(t) = \chi_{ab} \xi_{i1} \xi_{i2} \quad (4.2)$$

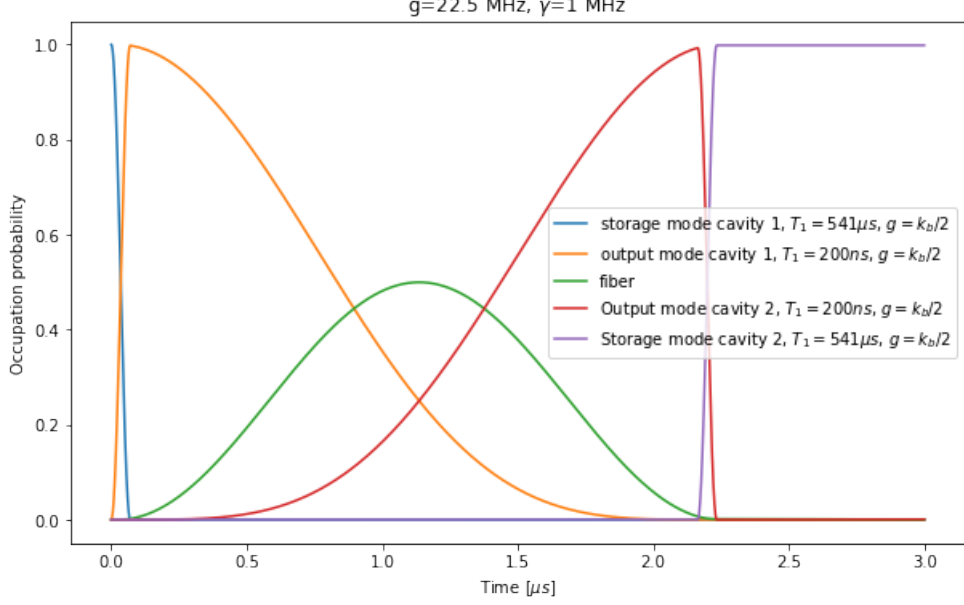
We consider the transmission line as a single mode and we suppose that it has the same frequency as the output modes in the cavities:

$$\hat{H}_{CT} = \gamma_1 [\hat{c}(\hat{b}_1^\dagger - \xi_{12} e^{-i\Delta_{12}}) + h.c.] + \gamma_2 [\hat{c}(\hat{b}_2^\dagger - \xi_{22} e^{-i\Delta_{22}}) + h.c.] \quad (4.3)$$

With  $\Delta_{i2} = \omega_{b_i} - \omega_{i2}$ .

### 4.1 Constant pulses

If we suppose the coupling  $\gamma_1 = \gamma_2 = \gamma$  with the transmission line to be constant, we can engineer the pulses of the Q-Switch to happen in a narrow interval of time compared to  $\frac{\pi}{\gamma}$ . In this way, we can use the Q-Switch process as a switch to transfer the state from one cavity to another:



**Figure 8:** At  $t=0$  we apply Q-Switch on the first cavity, then at  $t = \frac{2\pi}{3\gamma}$  we apply the second Q-Switch. Couplings:  $\gamma_1 = \gamma_2 = 1 \text{ MHz}$ ,  $g_i(t) = g[\Theta(t - t_i^{in}) - \Theta(t - t_i^{fn})]$ ,  $g = 22 \text{ MHz}$ ,  $\Delta t_i = \frac{\pi}{2g}$

## 4.2 Shaping the pulses

In the Q-switch, we can also shape the pulse to optimize the process.

Since the conversion process depends on both pumps, only one of them needs to vary in time to produce any particular  $g(t)$ .

Another significant effect of the application of pumps is ac-Stark shifts that shift the frequency of the modes. The Stark shifts during the transfer process are a function of both pump amplitudes:

$$H_{stark} = \delta_a(t)\hat{a}^\dagger\hat{a} + \delta_b(t)\hat{b}^\dagger\hat{b} \quad (4.4)$$

Each Stark shift depends on each pump amplitude:

$$\delta_a(t) = 2\chi_{aa}|\xi_1(t)|^2 + \chi_{ab}|\xi_2|^2 \quad \delta_b(t) = 2\chi_{bb}|\xi_2(t)|^2 + \chi_{ab}|\xi_1|^2 \quad (4.5)$$

We can now write down the equations of motion for the modes  $\hat{a}$  and  $\hat{b}$ , and, in particular, we will consider the expectation values of the field operators of the sender:

$$\dot{a} = -ig(t)b(t) - i\delta_a(t) \quad (4.6)$$

$$\dot{b} = -ig(t)b(t) - i\delta_b(t) \quad (4.7)$$

$$b(t) = \frac{b_{out}}{k_{out}} \quad (4.8)$$

Where  $k_{out}$  is the coupling of the output mode to the transmission line.

In these equations, we are assuming that there is no incoming field.

In this case, for a chosen  $b_{out}(t)$  and  $\xi_2$ , the only undetermined quantities are  $a(t)$  and  $\xi_1(t)$ .

With some algebra we can eliminate  $a(t)$  and leaving with an equation of  $g(\xi_1)$  in terms of  $b$ .

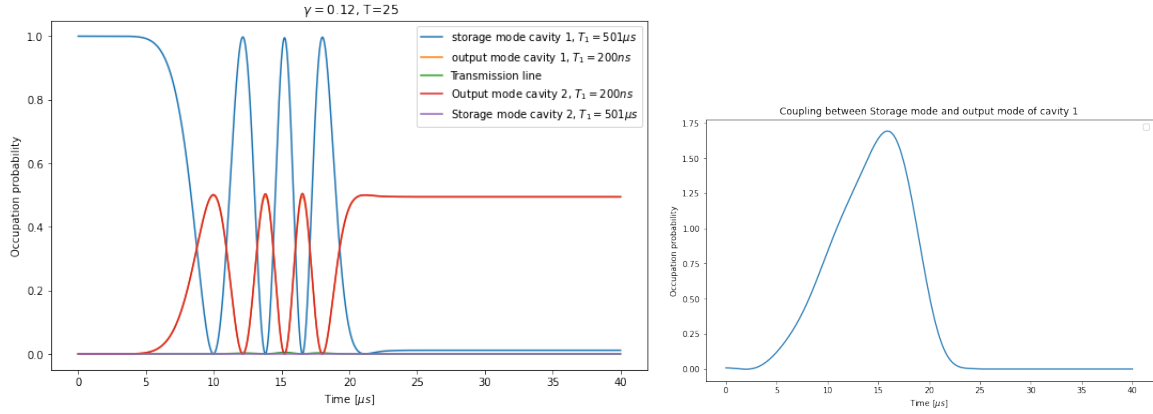
Following the steps of [1], we can solve this equation by imposing:

$$|b_{out}(t)| = 1 - \cos^2\left(\frac{\pi t}{T}\right) \quad (4.9)$$

Where  $T$  is the pulse length.

We wanted to see how this would affect our procedure: we solved the equation numerically, and then, after fitting the results with a function, we tried to evolve the system under this coupling for the modes within the first cavity while the Storage mode-Output mode coupling of the second one was off.

We tried this for different solutions due to different initial parameters. One example is that:

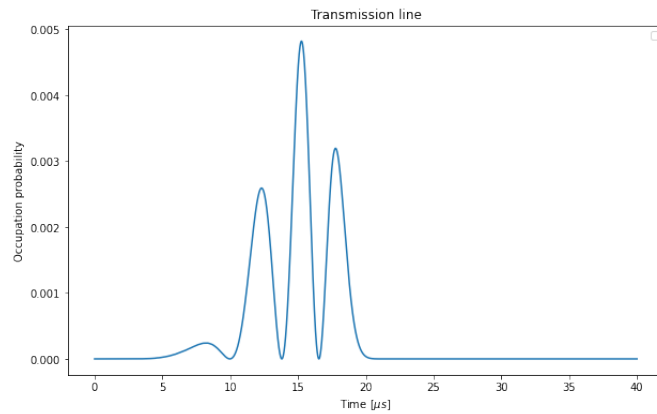


**Figure 9:** Left: occupation probability for the evolution of the system. The output modes of the cavities coincide.

Right: time dependence of the coupling between the modes of the first cavity

We note that after the pulse is over, the state does not change anymore, even if the output modes are still coupled to the transmission line.

Therefore the signal into the transmission line is different from what we expected:



**Figure 10:** Transmission line population

Since we began to discuss this in the last week of the internship, we have not had the chance to deepen this as we would have wished.

Therefore this approach looks very promising, and we could also do the same for the receiver, trying to optimize all processes.

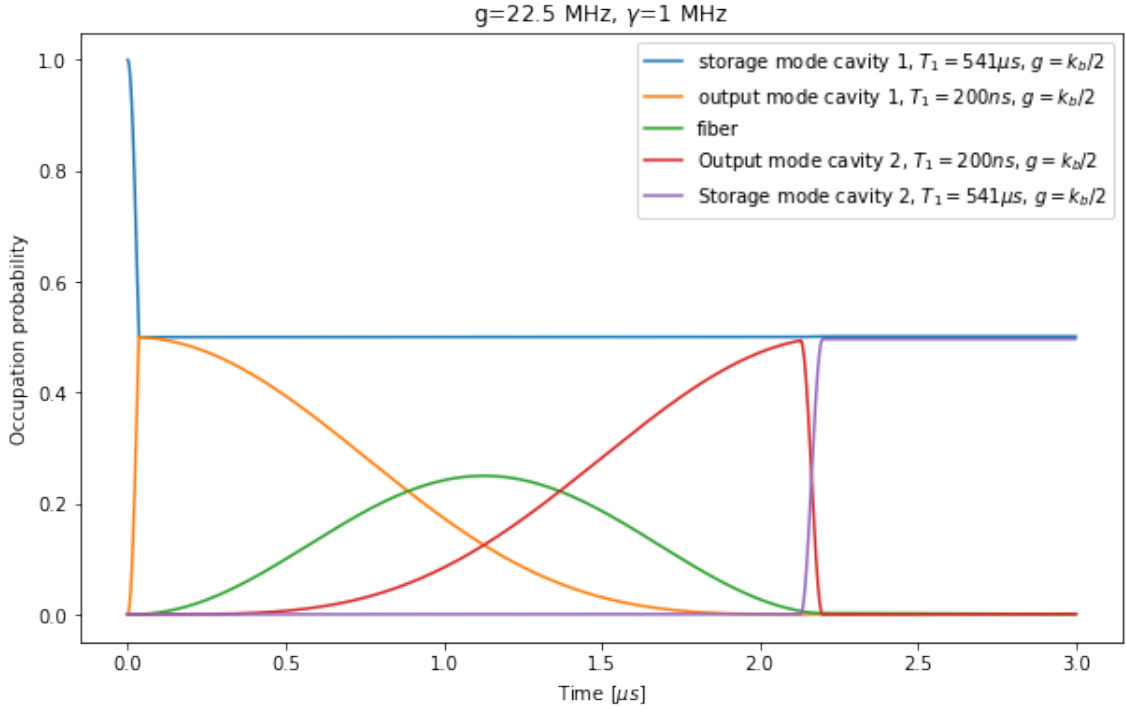
## 5 Entanglement

The next step of this discussion will be how to entangle the two cavities.

To complete this task, we can use different approaches.

We can start from a superposition of state inside the storage mode and repeat the process we have described.

Another way to accomplish that could be by stopping the process at the right time: suppose we are starting from the fock state  $|1\rangle$  inside the storage mode. We perform the Q-Switch until the fraction of the population we desire moves into the output mode and then we can follow the procedure.



**Figure 11:** The Q-switch process stopped

If we perform the partial trace on output modes and transmission line, we obtain the reduced density matrix for the storage modes:

$$\rho_{HQ2}^{HQ1} = \begin{pmatrix} 0.002 & 0 & 0 & 0 \\ 0 & 0.497 & 0.499 & 0 \\ 0 & 0.499 & 0.501 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.1)$$

Which is very similar to the density matrix of a Bell state:

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \implies \rho_{\Psi^+} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5.2)$$

So it shows an entangled state between the two cavities.

## A Changing frame

We apply an unitary transformation to a quantum system:

$$|\psi\rangle \rightarrow |\tilde{\psi}\rangle = U |\psi\rangle \quad (\text{A.1})$$

From the Schrödinger equation we can compute the new Hamiltonian as:

$$\hat{H} \rightarrow \tilde{\hat{H}} = \hat{U} \hat{H} \hat{U}^\dagger - i \dot{\hat{U}} \hat{U}^\dagger \quad (\text{A.2})$$

We present two basic examples that we have used in our discussion.

### A.1 Rotating frame

We go into a rotating frame, performing the unitary operation  $U = e^{-i\omega \hat{a}^\dagger \hat{a} t}$ :

$$\hat{H} \rightarrow \tilde{\hat{H}} = \hat{U} \hat{H} \hat{U}^\dagger - i \dot{\hat{U}} \hat{U}^\dagger \quad (\text{A.3})$$

$$= \hat{U} \hat{H} \hat{U}^\dagger - \omega \hat{a}^\dagger \hat{a} \quad (\text{A.4})$$

To understand the  $\hat{U} \hat{H} \hat{U}^\dagger$  component, let us first discuss how the ladder operators transform using the BCH formula:

$$\hat{U} \hat{a} \hat{U}^\dagger = a + i\omega t [\hat{a}^\dagger \hat{a}, \hat{a}] - \frac{(\omega t)^2}{2} [\hat{a}^\dagger \hat{a}, [\hat{a}^\dagger \hat{a}, \hat{a}]] + \dots = \sum_{k=0} \frac{(i\omega t)^k}{k!} [\hat{a}^\dagger \hat{a}, \cdot]^k(\hat{a}) \quad (\text{A.5})$$

But since  $[\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a} \implies [\hat{a}^\dagger \hat{a}, \cdot]^k(\hat{a}) = \hat{a}$ :

$$\hat{U} \hat{a} \hat{U}^\dagger = \sum_{k=0} \frac{(i\omega t)^k}{k!} \hat{a} = e^{i\omega t} \hat{a} \quad (\text{A.6})$$

So we obtain the useful relation:

$$\hat{U} f(\hat{a}, \hat{a}^\dagger) \hat{U}^\dagger = f(\hat{a} e^{i\omega t}, \hat{a}^\dagger e^{-i\omega t}) \quad (\text{A.7})$$

And finally:

$$\tilde{\hat{H}} = \hat{H}(\hat{a} e^{i\omega t}, \hat{a}^\dagger e^{-i\omega t}) - \omega \hat{a}^\dagger \hat{a} \quad (\text{A.8})$$

### A.2 Displaced frame

We apply the transformation:

$$\hat{U} = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} \quad (\text{A.9})$$

Using the BCH formula:

$$\hat{U} \hat{a} \hat{U}^\dagger = \sum_{k=0} \frac{1}{k!} [\alpha \hat{a}^\dagger - \alpha^* \hat{a}, \cdot]^k \hat{a} \quad (\text{A.10})$$

The commutator is trivial:

$$[\alpha \hat{a}^\dagger - \alpha^* \hat{a}, \hat{a}] = \alpha \implies [\alpha \hat{a}^\dagger - \alpha^* \hat{a}, \cdot]^k(\hat{a}) = 0 \quad (\text{A.11})$$

This means the BCH expansion stops after the first term, producing:

$$\hat{U} \hat{a} \hat{U}^\dagger = \hat{a} + \alpha \quad (\text{A.12})$$

So, in general:

$$\hat{U} f(\hat{a}, \hat{a}^\dagger) \hat{U}^\dagger = f(\hat{a} + \alpha, \hat{a}^\dagger + \alpha^*) \quad (\text{A.13})$$

Supposing that  $\alpha$  is time dependent:

$$\tilde{\hat{H}} = \hat{H}(\hat{a} + \alpha, \hat{a}^\dagger + \alpha^*) - i(\dot{\alpha} \hat{a}^\dagger - \dot{\alpha}^* \hat{a}) \quad (\text{A.14})$$

## B Q-Switch Hamiltonian

The full Hamiltonian is:

$$\hat{H} = \hat{H}_a + \hat{H}_b + \hat{H}_t + \hat{H}_p \quad (\text{B.1})$$

$$\hat{H}_a = \omega_a \hat{a}^\dagger \hat{a} \quad \hat{H}_b = \omega_b \hat{b}^\dagger \hat{b} \quad (\text{B.2})$$

$$\hat{H}_t = \omega_t \hat{t}^\dagger \hat{t} - E_j (\cos(\hat{\phi}) + \hat{\phi}^2/2) \quad (\text{B.3})$$

$$\hat{H}_p = \epsilon_1(t)(e^{i\omega_1 t} \hat{a}^\dagger + e^{-i\omega_1 t} \hat{a}) + \epsilon_2(t)(e^{i\omega_2 t} \hat{b}^\dagger + e^{-i\omega_2 t} \hat{b}) \quad (\text{B.4})$$

We firstly move to a rotating frame applying the transformations:

$$U_{11} = e^{-\omega_1 \hat{a}^\dagger \hat{a} t} \quad U_{12} = e^{-\omega_2 \hat{b}^\dagger \hat{b} t} \quad U_{13} = e^{-\omega_t \hat{t}^\dagger \hat{t} t} \quad (\text{B.5})$$

We use the relation in (A.1), and we expand the cosine term up to the fourth order of  $\phi$  to obtain:

$$\hat{H}' = \hat{H}'_a + \hat{H}'_b + \hat{H}'_t + \hat{H}'_p \quad (\text{B.6})$$

$$\hat{H}'_a = \Delta_1 \hat{a}^\dagger \hat{a} \quad \hat{H}'_b = \Delta_2 \hat{b}^\dagger \hat{b} \quad (\text{B.7})$$

$$\hat{H}'_t = -\frac{E_j}{4!} \left[ \phi_a (\hat{a}^\dagger e^{-i\omega_1 t} + \hat{a} e^{i\omega_1 t}) + \phi_b (\hat{b}^\dagger e^{-i\omega_2 t} + \hat{b} e^{i\omega_2 t}) + \phi_t (\hat{t}^\dagger e^{-i\omega_1 t} + \hat{t} e^{i\omega_1 t}) \right]^4 \quad (\text{B.8})$$

$$\hat{H}'_p = \epsilon_1(t)(\hat{a}^\dagger + \hat{a}) + \epsilon_2(t)(\hat{b}^\dagger + \hat{b}) \quad (\text{B.9})$$

Where  $\Delta_1 = \omega_a - \omega_1$  e  $\Delta_2 = \omega_b - \omega_2$ .

Now we move into a displaced frame applying the transformations:

$$U_{21} = D\left(-\frac{\epsilon_1(t)}{\Delta_1}\right) = D(-\xi_1(t)) \quad U_{22} = D\left(-\frac{\epsilon_2(t)}{\Delta_2}\right) = D(-\xi_2(t)) \quad (\text{B.10})$$

Using (A.2):

$$\hat{H}'' = \hat{H}''_a + \hat{H}''_b + \hat{H}''_t + \hat{H}''_p + i(\dot{\xi}_1 \hat{a}^\dagger - \dot{\xi}_1 \hat{a}) + i(\dot{\xi}_2 \hat{b}^\dagger - \dot{\xi}_2 \hat{b}) \quad (\text{B.11})$$

$$\hat{H}''_a = \Delta_1 (\hat{a}^\dagger - \xi_1)(\hat{a} - \xi_1) \quad \hat{H}''_b = \Delta_2 (\hat{b}^\dagger - \xi_2)(\hat{b} - \xi_2) \quad (\text{B.12})$$

$$\hat{H}''_t = -\frac{E_j}{4!} \left\{ \phi_a \left[ (\hat{a}^\dagger - \xi_1) e^{-i\omega_1 t} + (\hat{a} - \xi_1) e^{i\omega_1 t} \right] \right. \quad (\text{B.13})$$

$$\left. + \phi_b \left[ (\hat{b}^\dagger - \xi_2) e^{-i\omega_2 t} + (\hat{b} - \xi_2) e^{i\omega_2 t} \right] + \phi_t (\hat{t}^\dagger e^{-i\omega_1 t} + \hat{t} e^{i\omega_1 t}) \right\}^4 \quad (\text{B.14})$$

$$\hat{H}''_p = \epsilon_1(t)(\hat{a}^\dagger + \hat{a} - 2\xi_1) + \epsilon_2(t)(\hat{b}^\dagger + \hat{b} + 2\xi_2) \quad (\text{B.15})$$

Summing the terms:

$$\hat{H}'' = \Delta_1 \hat{a}^\dagger \hat{a} + \Delta_2 \hat{b}^\dagger \hat{b} - E_j \left\{ \phi_a \left[ (\hat{a}^\dagger - \xi_1) e^{-i\omega_1 t} + (\hat{a} - \xi_1) e^{i\omega_1 t} \right] + \right. \quad (\text{B.16})$$

$$\left. + \phi_b \left[ (\hat{b}^\dagger - \xi_2) e^{-i\omega_2 t} + (\hat{b} - \xi_2) e^{i\omega_2 t} \right] + \phi_t (\hat{t}^\dagger e^{-i\omega_1 t} + \hat{t} e^{i\omega_1 t}) \right\}^4 - \quad (\text{B.17})$$

$$- i(\dot{\xi}_1 \hat{a}^\dagger - \dot{\xi}_1 \hat{a}) - i(\dot{\xi}_2 \hat{b}^\dagger - \dot{\xi}_2 \hat{b}) - \Delta_1 \xi_1^2 - \Delta_2 \xi_2^2 \quad (\text{B.18})$$

Now we apply the last two transformations:

$$U_{31} = e^{-i\Delta_1 \hat{a}^\dagger \hat{a} t} \quad U_{32} = e^{-i\Delta_2 \hat{b}^\dagger \hat{b} t} \quad (\text{B.19})$$

We obtain:

$$\hat{H}''' = -E_j \left\{ \phi_a \left[ \hat{a}^\dagger e^{-i\omega_a t} + \hat{a} e^{i\omega_a t} - \xi_1 e^{i\omega_1 t} - \xi_1 e^{-i\omega_1 t} \right] + \right. \quad (\text{B.20})$$

$$\left. + \phi_b \left[ \hat{b}^\dagger e^{-i\omega_b t} + \hat{b} e^{i\omega_b t} - \xi_2 e^{i\omega_2 t} - \xi_2 e^{-i\omega_2 t} \right] + \phi_t (\hat{t}^\dagger e^{-i\omega_1 t} + \hat{t} e^{i\omega_1 t}) \right\}^4 \quad (\text{B.21})$$

$$- i(\dot{\xi}_1 \hat{a}^\dagger e^{-i\Delta_1 t} + \dot{\xi}_1 \hat{a} e^{i\Delta_1 t}) - i(\dot{\xi}_2 \hat{b}^\dagger e^{-i\Delta_2 t} + \dot{\xi}_2 \hat{b} e^{-i\Delta_2 t}) - \Delta_1 \xi_1^2 - \Delta_2 \xi_2^2 \quad (\text{B.22})$$

Now we calculate the product, and we apply rotating wave approximation. We also drop the last two terms in the last equation since they would be constant.

Besides the terms that are already present in the undriven case, this reveals the additional terms:

$$\hat{H} = \xi_1^2 (2\chi_{aa} \hat{a}^\dagger \hat{a} + \chi_{ab} \hat{b}^\dagger \hat{b} + \chi_{at} \hat{t}^\dagger \hat{t}) + \xi_2^2 (2\chi_{bb} \hat{b}^\dagger \hat{b} + \chi_{ab} \hat{a}^\dagger \hat{a} + \chi_{bt} \hat{t}^\dagger \hat{t}) + g(e^{-i\delta t} \hat{a}^\dagger \hat{b} + e^{i\delta t} \hat{a} \hat{b}^\dagger) \quad (\text{B.23})$$

Where  $\chi_{kl} = -E_j \phi_k^2 \phi_l^2$ ,  $\delta = \Delta_1 - \Delta_2$  and:

$$g(t) = \chi_{ab} \xi_1(t) \xi_2(t) \quad (\text{B.24})$$

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