Analytic Differential Phase Calculations Using Feynman Diagrams — Higher Order Quantum Corrections and Finite Wave Function Width Effects for 1D Interferometry

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Overview

- Computing scattering amplitudes in QFT is fun — can we use the same math to compute quantities relevant to what we're measuring in the lab?

- Been working on this question on and off for ~2 years — interesting results in the last few months

- We've used this new formalism to perturbativly compute $\Delta\phi$ terms higher order in $\hbar$, and to compute the dependence of $\Delta\phi$ on the waist of an atom wave function at the time of the first $\pi/2$ pulse

The Semiclassical Formalism

- Powerful method for computing differential phases through perturbative calculations of classical trajectories
- Very difficult to compute terms higher order than $\hbar^{-1}$, or finite wave function waist effects following this approach

$$\Delta\phi_{\text{tot}} = \Delta\phi_{\text{propagation}} + \Delta\phi_{\text{separation}} + \Delta\phi_{\text{laser}}$$

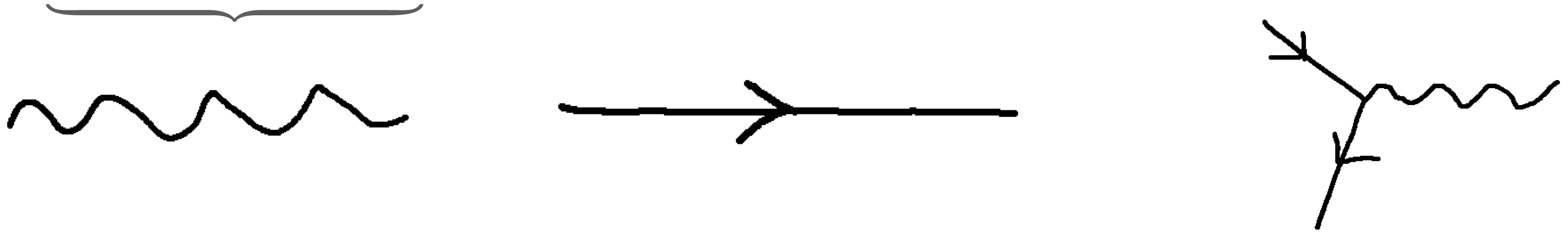
	Phase shift	Size (rad)	Fractional size
1	$-k_{\text{eff}} g T^2$	-2.85×10^{8}	1.00
2	$k_{\text{eff}} R_e \Omega_y^2 T^2$	6.18×10^{5}	2.17×10^{-3}
3	$-k_{\text{eff}} T_{zz} v_z T^3$	1.58×10^{3}	5.54×10^{-6}
4	$\frac{7}{12} k_{\text{eff}} g T_{zz} T^4$	-9.21×10^{2}	3.23×10^{-6}
5	$-3 k_{\text{eff}} v_z \Omega_y^2 T^3$	-5.14	1.80×10^{-8}
6	$2 k_{\text{eff}} v_x \Omega_y T^2$	3.35	1.18×10^{-8}
7	$\frac{7}{4} k_{\text{eff}} g \Omega_y^2 T^4$	3.00	1.05×10^{-8}
8	$-\frac{7}{12} k_{\text{eff}} R_e T_{zz} \Omega_y^2 T^4$	2.00	7.01×10^{-9}
9	$-\frac{\hbar k_{\text{eff}}^2}{2m} T_{zz} T^3$	7.05×10^{-1}	2.48×10^{-9}
10	$\frac{3}{4} k_{\text{eff}} g Q_{zzz} v_z T^5$	9.84×10^{-3}	3.46×10^{-11}
11	$-\frac{7}{12} k_{\text{eff}} Q_{zzz} v_z^2 T^4$	-7.66×10^{-3}	2.69×10^{-11}
12	$-\frac{7}{4} k_{\text{eff}} R_e \Omega_y^4 T^4$	-6.50×10^{-3}	2.28×10^{-11}
13	$-\frac{7}{4} k_{\text{eff}} R_e \Omega_y^2 \Omega_z^2 T^4$	-3.81×10^{-3}	1.34×10^{-11}
14	$-\frac{31}{120} k_{\text{eff}} g^2 Q_{zzz} T^6$	-3.39×10^{-3}	1.19×10^{-11}
15	$-\frac{3\hbar k_{\text{eff}}^2}{2m} \Omega_y^2 T^3$	-2.30×10^{-3}	8.06×10^{-12}
16	$\frac{1}{4} k_{\text{eff}} T_{zz}^2 v_z T^5$	2.19×10^{-3}	7.68×10^{-12}
17	$-\frac{31}{360} k_{\text{eff}} g T_{zz}^2 T^6$	-7.53×10^{-4}	2.65×10^{-12}
18	$3 k_{\text{eff}} v_y \Omega_y \Omega_z T^3$	2.98×10^{-4}	1.05×10^{-12}
19	$-k_{\text{eff}} \Omega_y \Omega_z y_0 T^2$	-7.41×10^{-5}	2.60×10^{-13}
20	$-\frac{3}{4} k_{\text{eff}} R_e Q_{zzz} v_z \Omega_y^2 T^5$	-2.14×10^{-5}	7.50×10^{-14}
21	$\frac{31}{60} k_{\text{eff}} g R_e Q_{zzz} \Omega_y^2 T^6$	1.47×10^{-5}	5.17×10^{-14}
22	$\frac{3}{2} k_{\text{eff}} T_{zz} v_z \Omega_y^2 T^5$	-1.42×10^{-5}	5.00×10^{-14}
23	$-\frac{7}{6} k_{\text{eff}} T_{zz} v_x \Omega_y T^4$	1.08×10^{-5}	3.81×10^{-14}
24	$-2 k_{\text{eff}} T_{xx} \Omega_y x_0 T^3$	-6.92×10^{-6}	2.43×10^{-14}
25	$-\frac{7\hbar k_{\text{eff}}^2}{12m} Q_{zzz} v_z T^4$	-6.84×10^{-6}	2.40×10^{-14}
26	$-\frac{7}{6} k_{\text{eff}} T_{xx} v_x \Omega_y T^4$	-5.42×10^{-6}	1.90×10^{-14}
27	$-\frac{31}{60} k_{\text{eff}} g T_{zz} \Omega_y^2 T^6$	4.90×10^{-6}	1.72×10^{-14}
28	$k_{\text{eff}} T_{xx} v_z \Omega_y^2 T^5$	4.75×10^{-6}	1.67×10^{-14}
29	$\frac{3\hbar k_{\text{eff}}^2}{8m} g Q_{zzz} T^5$	4.40×10^{-6}	1.55×10^{-14}
30	$\frac{31}{360} k_{\text{eff}} R_e T_{zz}^2 \Omega_y^2 T^6$	1.63×10^{-6}	5.74×10^{-15}
31	$-\frac{31}{90} k_{\text{eff}} g T_{xx} \Omega_y^2 T^6$	-1.63×10^{-6}	5.74×10^{-15}
32	$\frac{\hbar k_{\text{eff}}^2}{8m} T_{zz}^2 T^5$	9.78×10^{-7}	3.43×10^{-15}
33	$-\frac{\hbar k_{\text{eff}} \alpha B_0 (\partial_z B) T^2}{m}$	-7.67×10^{-8}	2.69×10^{-16}
34	$\frac{31}{60} k_{\text{eff}} g S_{zzzz} v_z^2 T^6$	-7.52×10^{-8}	2.64×10^{-16}
35	$-\frac{1}{4} k_{\text{eff}} S_{zzzz} v_z^3 T^5$	3.64×10^{-8}	1.28×10^{-16}
36	$\frac{31}{72} k_{\text{eff}} T_{zz} Q_{zzz} v_z^2 T^6$	-3.13×10^{-8}	1.10×10^{-16}

When including into the system Lagrangian terms corresponding to Coriolis and centrifugal forces, spherical earth gravity gradients, Zeeman shifts from earth's B field, computing trajectories perturbativly in T, and plugging those into the semiclassical formalism, one produces the terms on the left. When expressed in terms of the recoil velocity $v_r = \hbar k_{\text{eff}}/m$, and all terms in the table are $\propto \hbar^{-1}$

[1] **Light-pulse Atom Interferometry**. Jason M. Hogan, David M. S. Johnson and Mark A. Kasevich [arXiv:0806.3261]

Diagrammatic Perturbation Theory — QFT

$$\mathscr{L} = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}\; + \underbrace{i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - m\bar{\Psi}\Psi} + \underbrace{e\bar{\Psi}\gamma^{\mu}\Psi A_{\mu}}$$



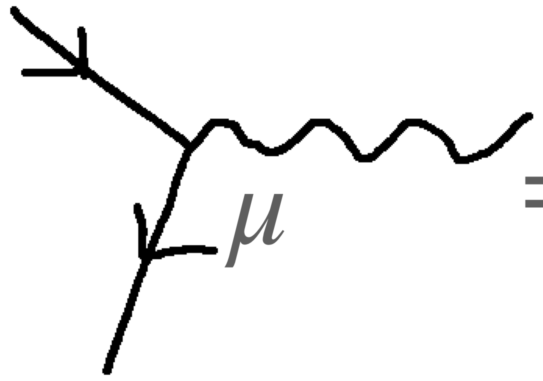
Rules relevant for leading order $e^- e^- \to e^- e^-$ calculation:

"Incoming"

$= u_{s_i}[p_i]$

"Outgoing"

$= \bar{u}_{s_i}[p_i]$

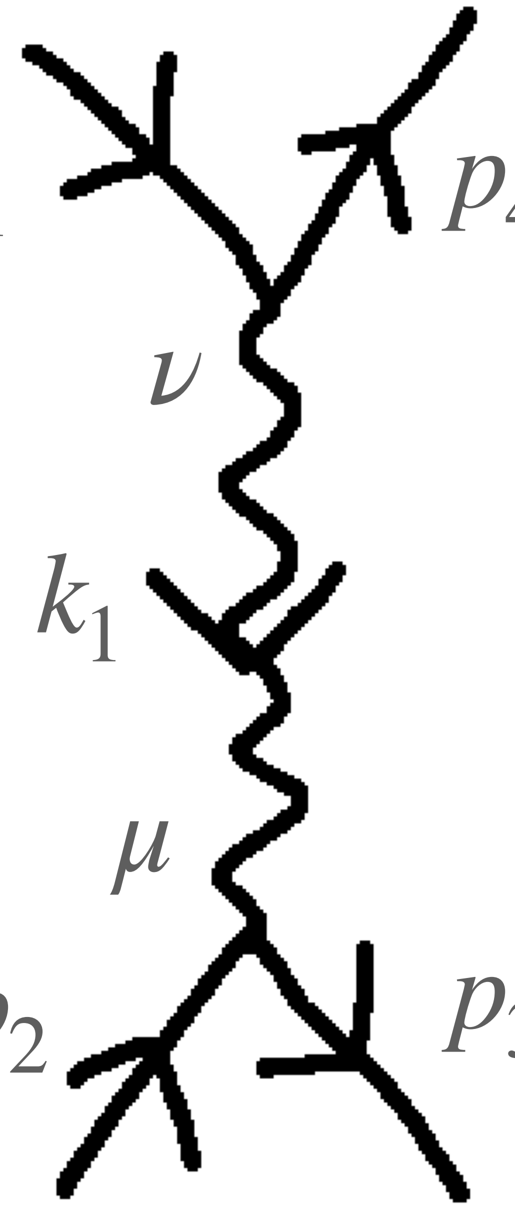


$= ie\gamma^{\mu}\delta[\Sigma_i p_i]$

$= \dfrac{-ig^{\mu\nu}}{k_i^2 - i\epsilon}$

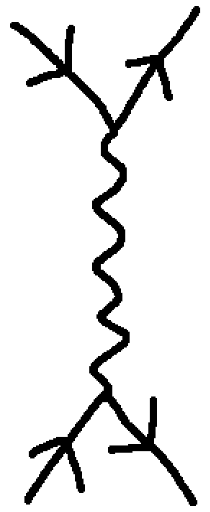
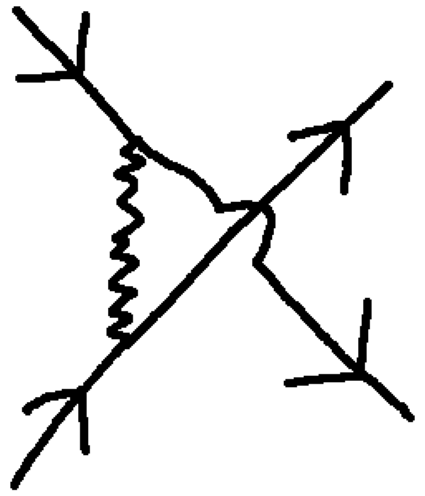
- Integrate over all internal momenta
- Start at the external edge of an outgoing e^- and work backwards

One of the two leading order diagrams:



$$= \int_{-\infty}^{\infty} dk_1 \delta[p_1 - k_1 - p_4]\delta[p_2 + k_1 - p_3]$$

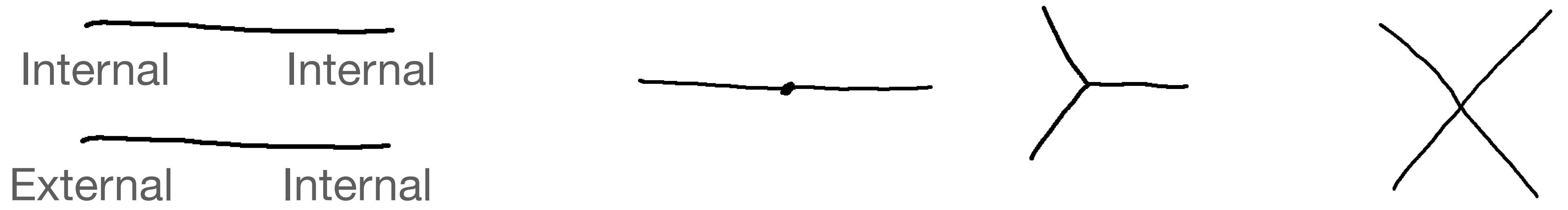
$$\left(\bar{u}_{s_3}[p_3]ie\gamma_{\mu}u_{s_2}[p_2]\right)\frac{-ig^{\mu\nu}}{k_1^2 - i\epsilon}\left(\bar{u}_{s_4}[p_4]ie\gamma_{\nu}u_{s_1}[p_1]\right)$$

$i\mathcal{T} =$  $+$  $+\mathcal{O}[e^4]$

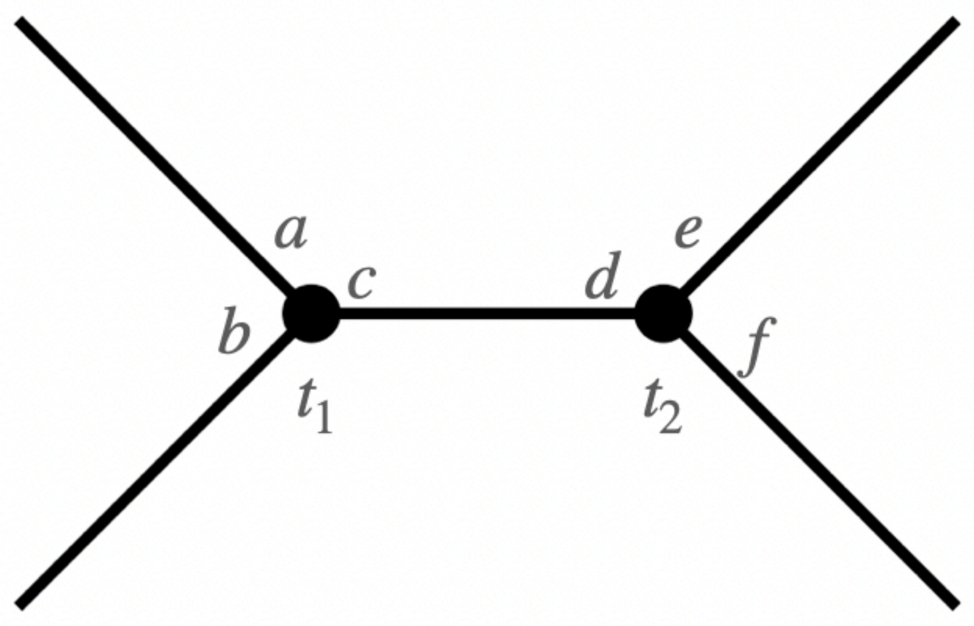
[2] **Quantum Field Theory**. Mark Srednicki [https://web.physics.ucsb.edu/~mark/ms-qft-DRAFT.pdf]

Diagrammatic Perturbation Theory — AI in 1D

$$L = \underbrace{\frac{1}{2}m(\partial_t z)^2 + (-mg + J[t])z}\, - \underbrace{\frac{1}{2}mT_{zz}z^2}\, - \underbrace{\frac{1}{3!}mQ_{zzz}z^3}\, - \underbrace{\frac{1}{4!}mS_{zzzz}z^4} + \ldots$$

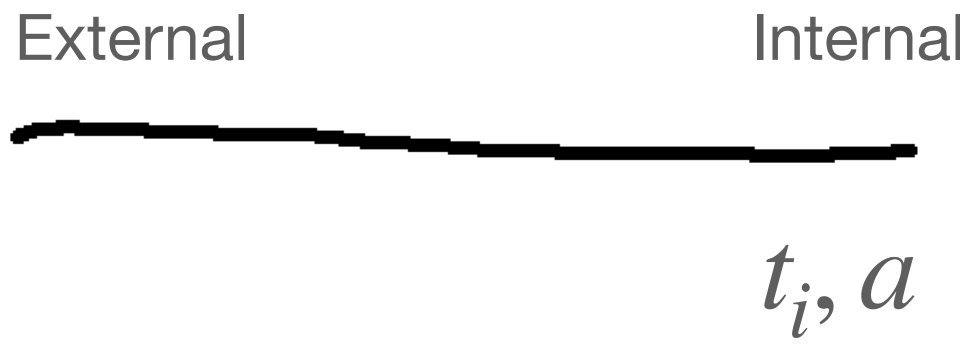


One of the Q_{zzz}^2 diagrams:

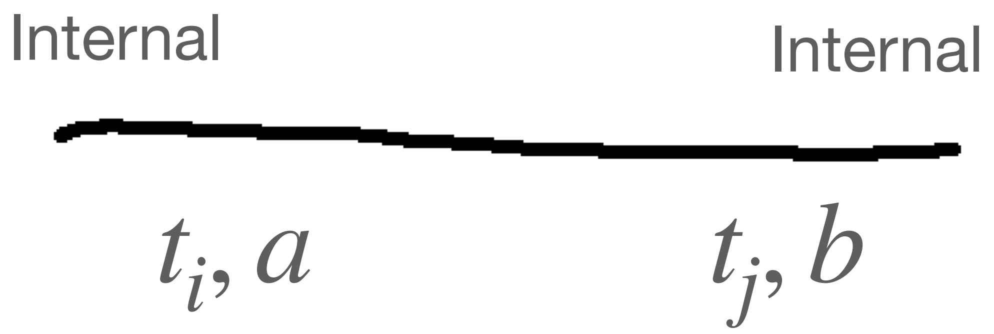


$$= \frac{1}{8}\varepsilon_{abc}\varepsilon_{def}\int_0^{2T} dt_1 \int_0^{2T} dt_2\ \langle 1\rangle_a \langle 1\rangle_b \langle 12\rangle_{cd} \langle 2\rangle_e \langle 2\rangle_f$$

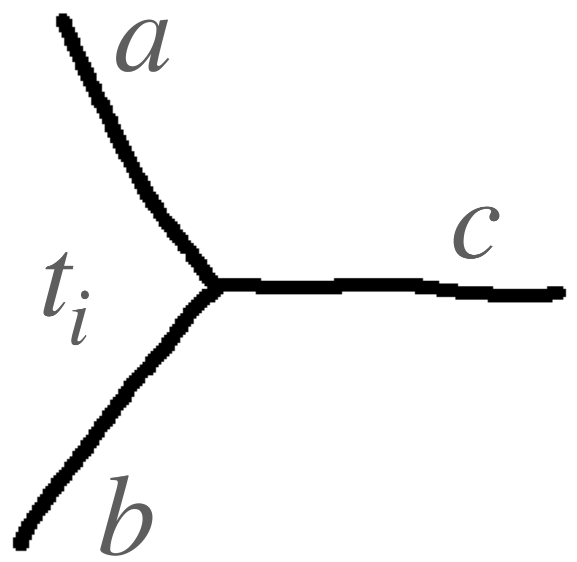
Rules relevant for computing Q_{zzz} terms:



$= \langle i\rangle_a$



$= \langle ij\rangle_{ab}$



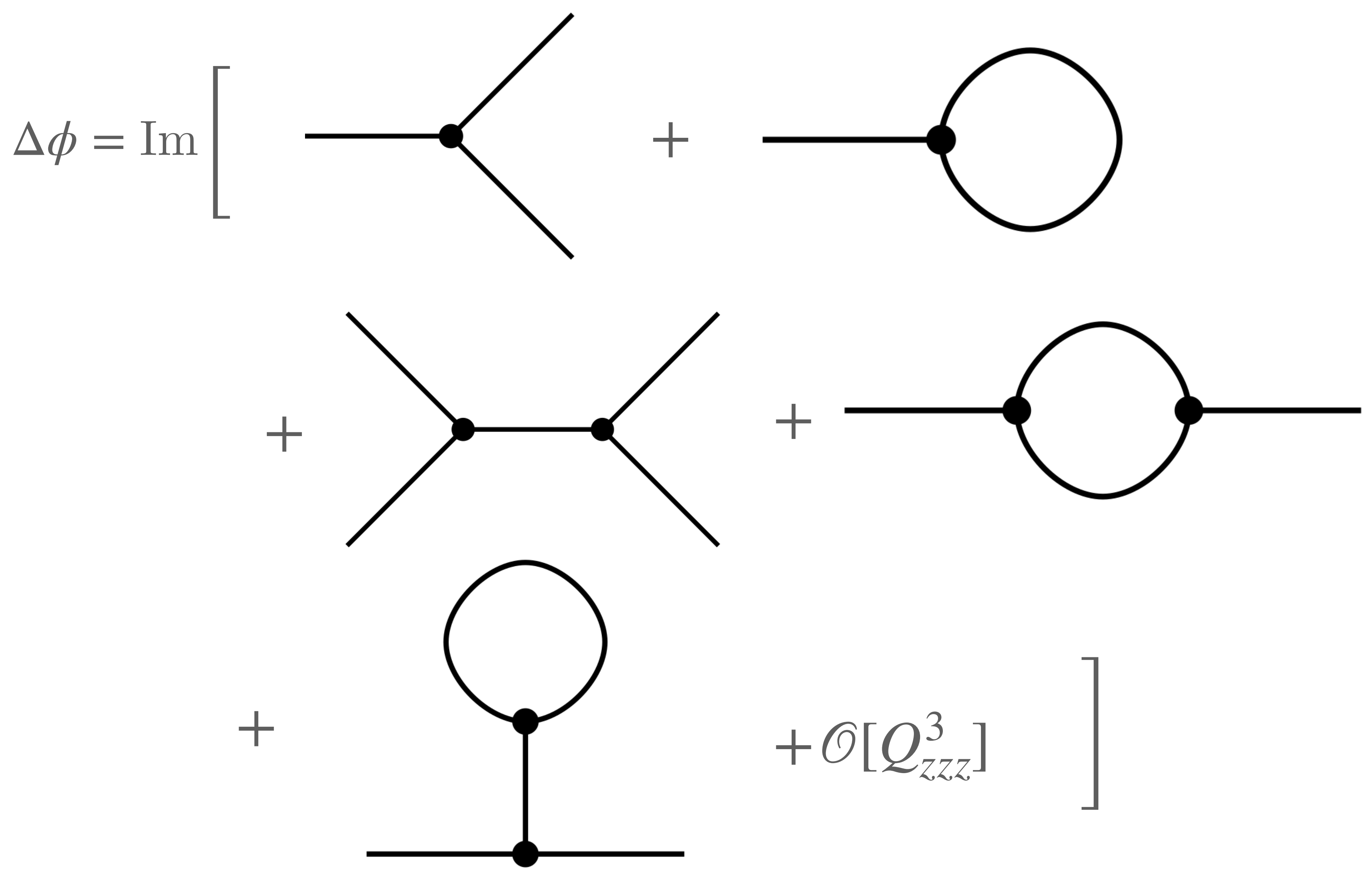
$$= \varepsilon_{abc}\int_0^{2T} dt_i$$

$$\varepsilon_{abc} \propto Q_{zzz}$$

$$\langle i\rangle_1 = \frac{i}{\hbar} z_{\mathrm{cl}}^{\uparrow}[t_i]$$

$$\langle i\rangle_2 = -\frac{i}{\hbar} z_{\mathrm{cl}}^{\downarrow}[t_i]$$

Latin indices are summed over and run from 1 to 2



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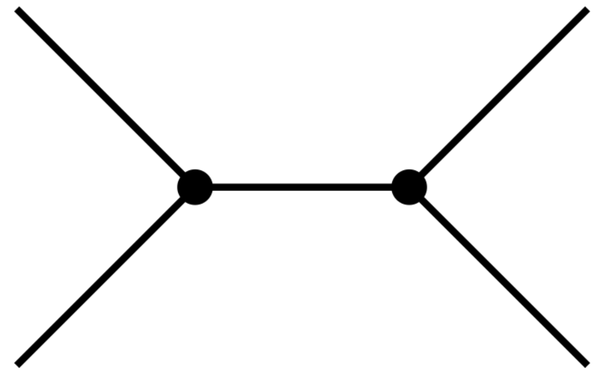
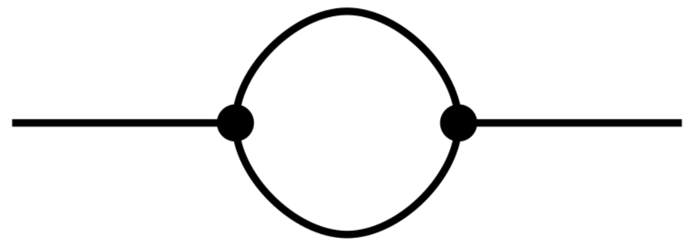
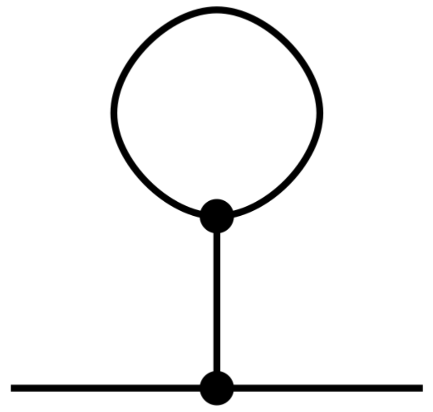
Diagrammatic Perturbation Theory — AI in 1D

Im[]

$$= \frac{m}{\hbar}Q_{zzz}^2\Big(-\frac{511}{21600}g^3v_rT^{10}+\frac{17}{288}g^2v_r^2T^9-\frac{703}{10080}gv_r^3T^8+\frac{1}{36}v_r^4T^7+\frac{17}{144}g^2v_rv_zT^9$$
$$-\frac{1397}{6720}gv_r^2v_zT^8+\frac{17}{144}v_r^3v_zT^7-\frac{1397}{6720}gv_rv_z^2T^8+\frac{3}{16}v_r^2v_z^2T^7+\frac{1}{8}v_rv_z^3T^7+\frac{127}{1120}g^2v_rz_0T^8$$
$$-\frac{9}{40}gv_r^2z_0T^7+\frac{1}{8}v_r^3z_0T^6-\frac{9}{20}gv_rv_zz_0T^7+\frac{31}{72}v_r^2v_zz_0T^6+\frac{31}{72}v_rv_z^2z_0T^6-\frac{31}{90}gv_rz_0^2T^6$$
$$+\frac{5}{16}v_r^2z_0^2T^5+\frac{5}{8}v_rv_zz_0^2T^5+\frac{7}{24}v_rz_0^3T^4\Big)$$

Im[]

$$= Q_{zzz}^2\Big(-\frac{31}{1440}\frac{1}{\beta}gv_rT^6+\frac{1}{32}\frac{1}{\beta}v_r^2T^5+\frac{1}{16}\frac{1}{\beta}v_rv_zT^5+\frac{7}{48}\frac{1}{\beta}v_rz_0T^4$$
$$-\frac{127}{1120}\beta gv_rT^8+\frac{1}{8}\beta v_r^2T^7+\frac{1}{4}\beta v_rv_zT^7+\frac{31}{90}\beta v_rz_0T^6\Big)$$

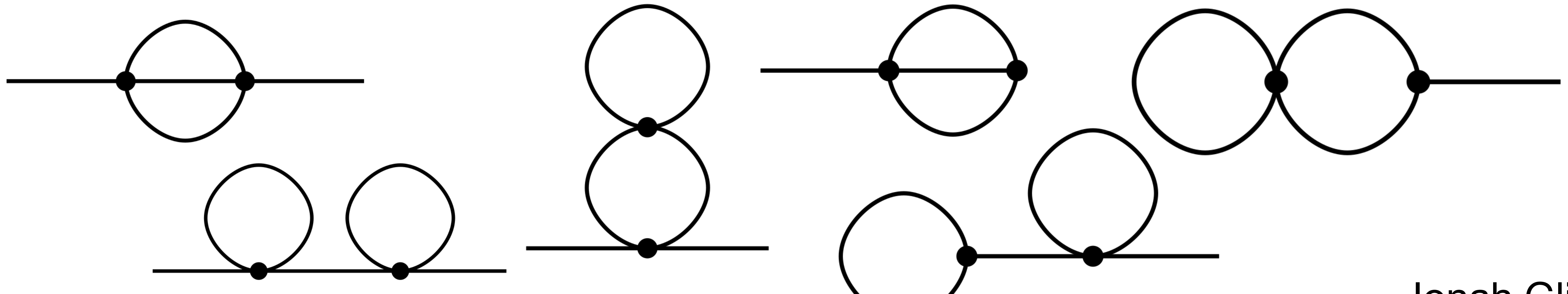
Im[]

$$= Q_{zzz}^2\Big(-\frac{31}{480}\frac{1}{\beta}gv_rT^6+\frac{3}{64}\frac{1}{\beta}v_r^2T^5+\frac{3}{32}\frac{1}{\beta}v_rv_zT^5+\frac{7}{96}\frac{1}{\beta}v_rz_0T^4$$
$$-\frac{127}{1344}\beta gv_rT^8+\frac{1}{16}\beta v_r^2T^7+\frac{1}{8}\beta v_rv_zT^7+\frac{31}{360}\beta v_rz_0T^6\Big)$$

$\beta = \dfrac{\hbar}{mw_0^2}$ is a parameter which sets the rate at which the free wave function expands

- This method reproduces the terms that emerge from the semi-classical formalism, and in addition, computes terms $\propto \hbar^0, \hbar^1, \hbar^2, \ldots$ and dependence on w_0

- Writing these terms in terms of β, an n loop diagram will be $\propto (\hbar/m)^{n-1}$
- One could make the case that terms $\propto \hbar^0$ are still ‘semiclassical’, but two loops diagrams are definite higher order quantum corrections
- These two loop diagrams emerge at third order Q_{zzz}, second order in S_{zzzz}, first order in $Q_{zzz}S_{zzzz}$



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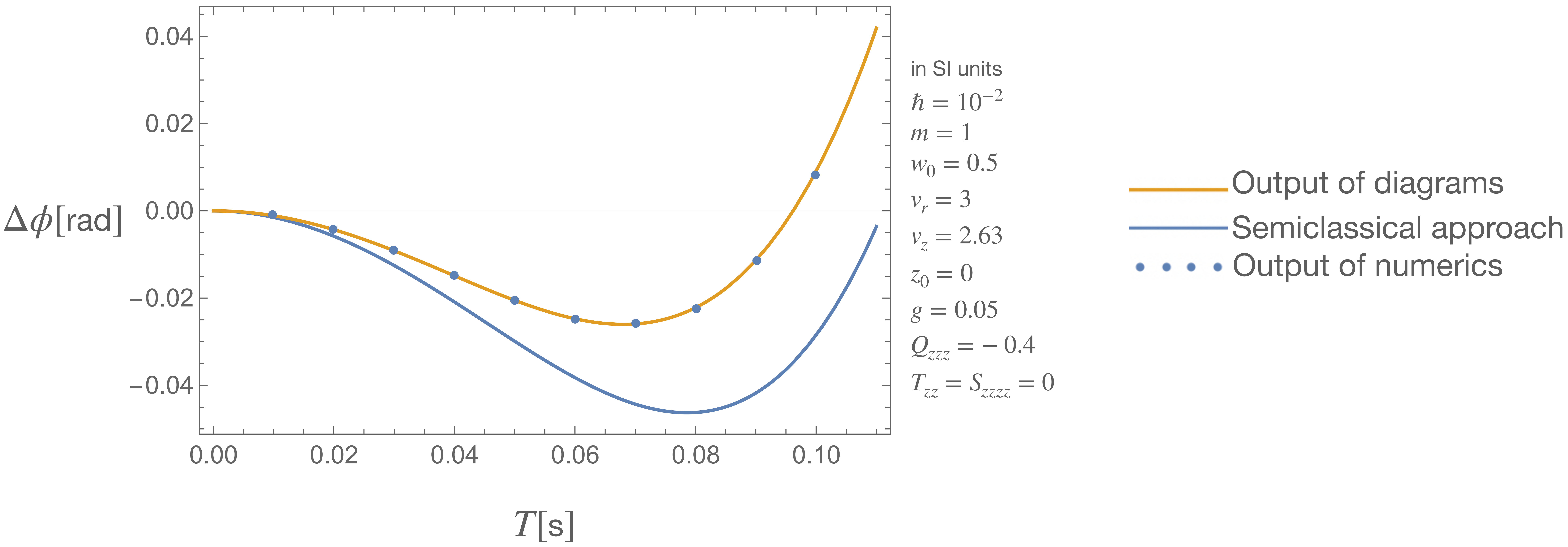
Agreement with Split Step Numerics



We start with a wave function at time t = 0 of the form $\psi[z_a] = \left(\frac{2}{\pi w_0^2}\right)^{1/4} \exp\left[-\frac{1}{w_0^2} z_a^2\right] \exp\left[i \frac{m}{\hbar} v_z z_a\right]$ and impose an interferometer sequence on it, evaluated numerically in Matheamtica via the 'split step' method

Literature on split step numerics:

[3] **Solution of the Schrodinger Equation by a Spectral Method.** M. J. Feit, J. A. Fleck, A. Steiger [*J. Compute. Phys., 47:412, 1982*]

[4] **Modeling of Precision Light-Pulse Atom Interferometers with Distorted Wavefronts**. S. J. Seckmeyer [Master's Thesis]

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Conclusion

- I only went through Q_{zzz} diagrams, but we've computed additional diagrams involving T_{zz}, S_{zzzz}, and their cross couplings.
- We've used these rules to compute an expression for the differential phase to all orders in $\hbar$ and w_0 and first order in a general potential $V[z]$ by summing over all the 1 vertex diagrams, which simplifies to the known semi-classical expression in the $\hbar \to 0$ limit and agrees with the few split step simulation we've run
- Working on diagram rules for accounting for rotation of the earth (external edges pick up x,y,z labels, latin indices run from 1 to 4, …)
- The structural similarity of the free particle Schrodinger equation and the paraxial wave equation means this same formalism can be applied to characterizing the profile of beams that have been aberrated by non-parabolic lenses — I'm currently working on getting measurements of $\langle x^2 \rangle - \langle x \rangle^2$ in the lab to agree with expressions coming out of diagrammatic formalism, but already some very fun mathematical relations have come to light, which I haven't seen in literature
- I'm pretty sure there's a way to talk diagrammatically about transition amplitudes to do with lattice accelerations where the large lattice depth limit is treated as the unperturbed case and shallow lattice depth corrections are treated perturbativly