UCDAVIS
UNIVERSITY OF CALIFORNIA

QMAP

Hamiltonian Truncation on the Lattice

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Work in progress with Ben Guthrie, Pavel Press, Joseph Takach

Motivation

Resurgence of interest in Hamiltonian methods in numerical QFT:

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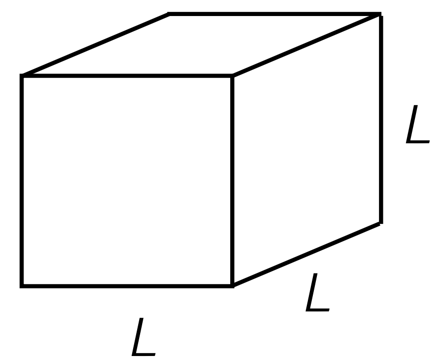
- Tensor network states
- Quantum computing
- Hamiltonian truncation

Hamiltonian Truncation...

(Rayleigh-Ritz variational method)

Goal: approximately diagonalize H in finite volume $H|\mathcal{E}\rangle = \mathcal{E}|\mathcal{E}\rangle$



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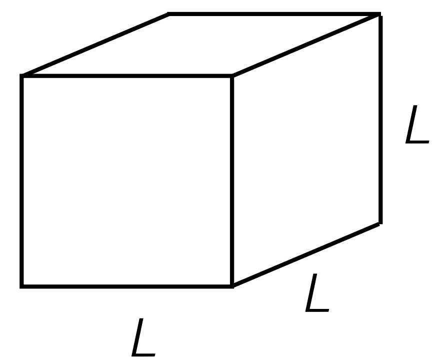

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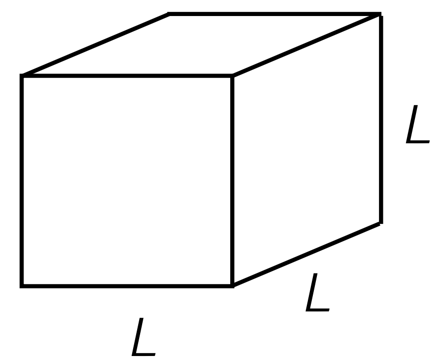

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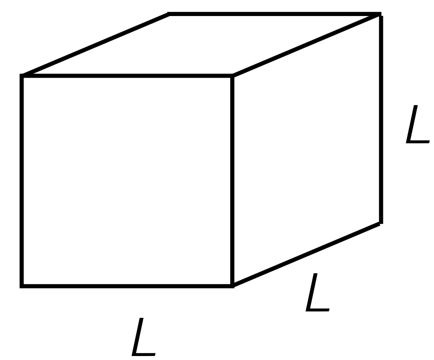

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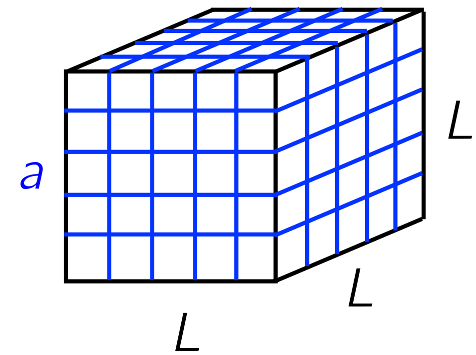
- Compute low-lying spectrum of truncated Hamiltonian:

$$(H_{\text{eff}})_{n',n} = \langle n'|H|n\rangle$$

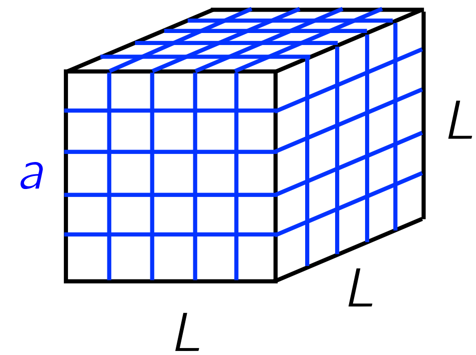
...on the Lattice



...on the Lattice

$$H_0 = \sum_x \left[\tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\overline{m}^2\phi_x^2\right]$$

$a = 1$



$$V = \sum_x \left[\tfrac{1}{2}(m^2 - \overline{m}^2)\phi_x^2 + \tfrac{1}{2}\sum_{i=1}^{d}(\phi_{x+i} - \phi_x)^2 + \frac{\lambda}{4!}\phi_x^4\right]$$

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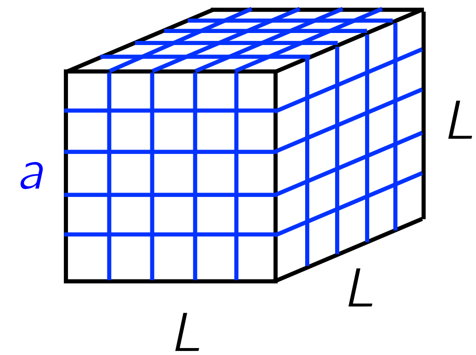

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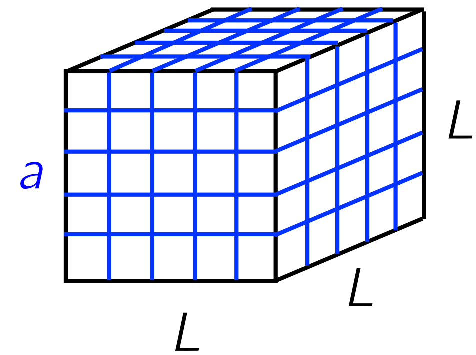

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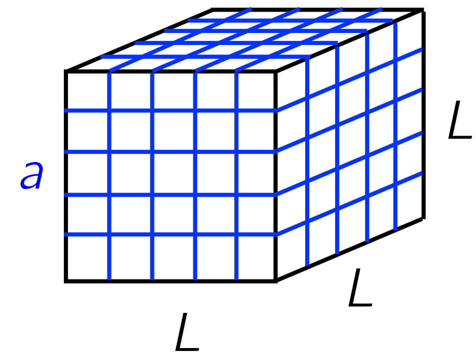

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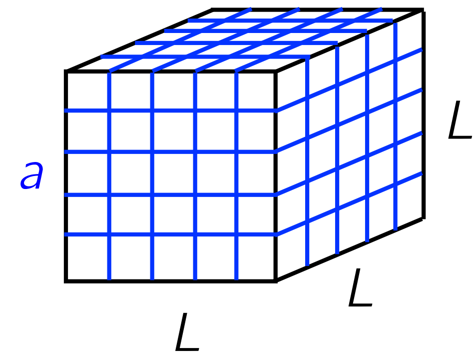

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$\Rightarrow\ \ \overline{m} \sim 1$ for optimal convergence

Lattice Truncation

- H_0 eigenbasis for $\mathcal{S}_\infty$:

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- Simple error estimate:

$$\delta\mathcal{E}^{(k)} \simeq \tfrac{1}{2}\big|\mathcal{E}^{(k+1)} - \mathcal{E}^{(k)}\big|$$

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- $\underbrace{E_{\max} \gg \frac{L^d}{a^{d+1}}}_{\substack{\text{requires } N \gtrsim e^{\# L^d} \\ \text{“IR catastrophe”}}} \quad \Rightarrow \quad \underbrace{\text{error} \sim \frac{1}{N^{\#}}}_{\substack{\text{theory} \\ + \text{ experiment}}}$

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$n =$ degrees of freedom/site
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$L = 40, \;\; m = 0.1, \;\; \overline{m} = 1.1:$

$$|\langle 0|\Omega\rangle|^2 \sim \begin{cases} 0.1 & n = 1, d = 1, \\ 10^{-27} & n = 8, d = 3. \end{cases}$$

Truncate QCD?

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What, me worry? There's more to life than QCD...

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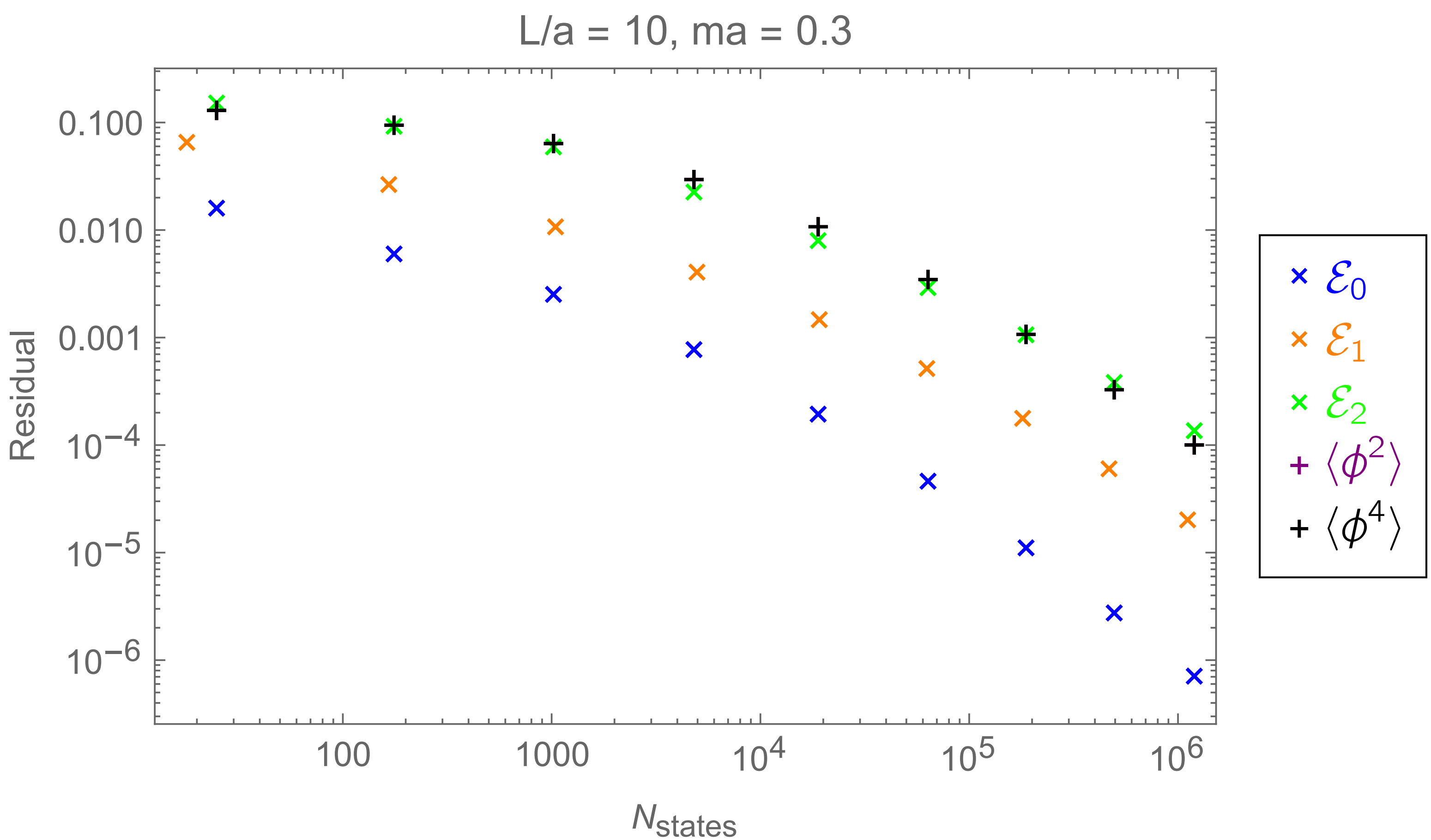
- Qualitative problem
- Finite size scaling: $L =$ physical scale $\sim \dfrac{1}{m_{\text{phys}}}$
- Monte Carlo is exponentially expensive for some theories

  $\theta \neq 0, \ \rho \neq 0, \ \ldots$

2D Free Field Theory



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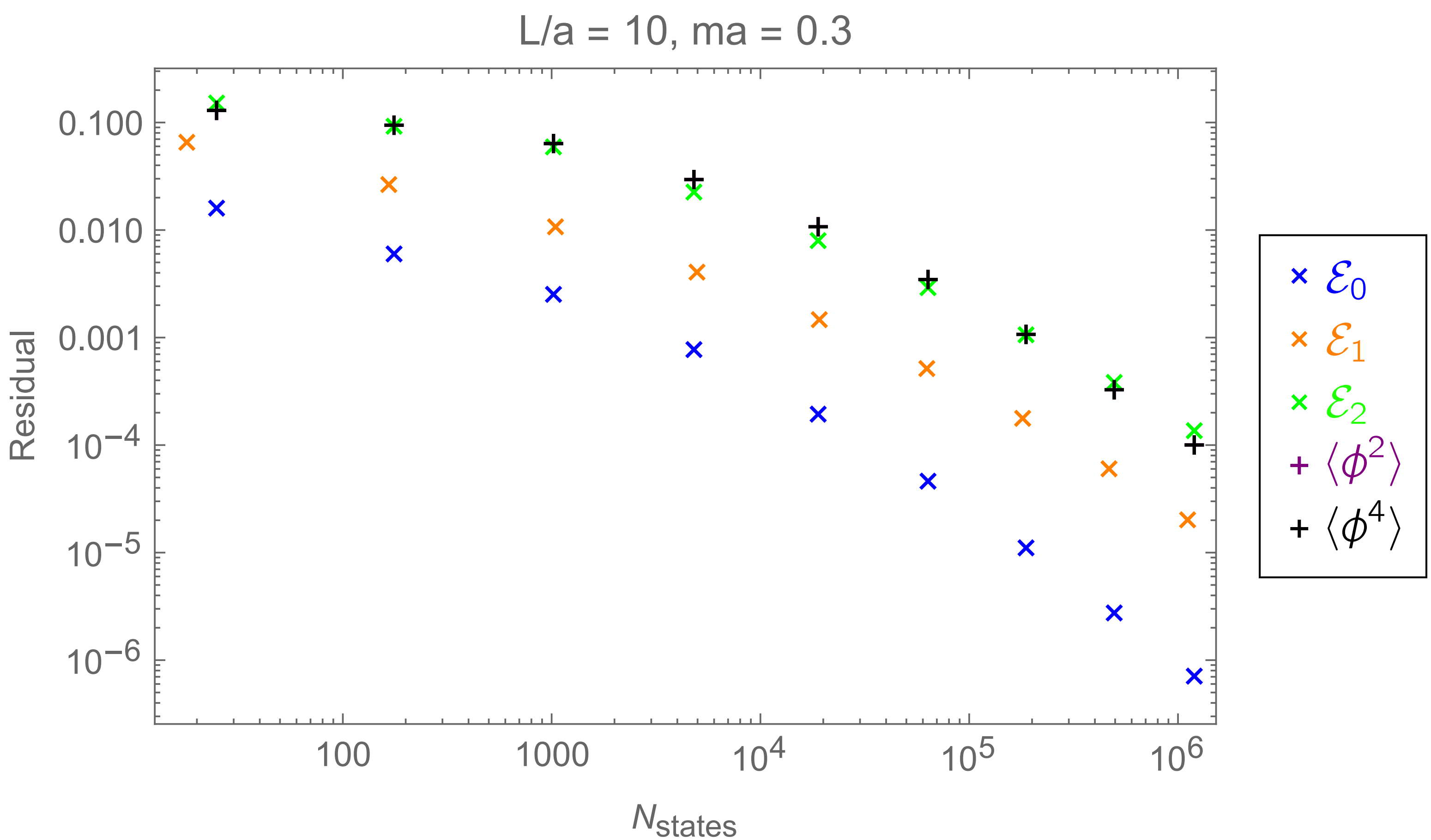

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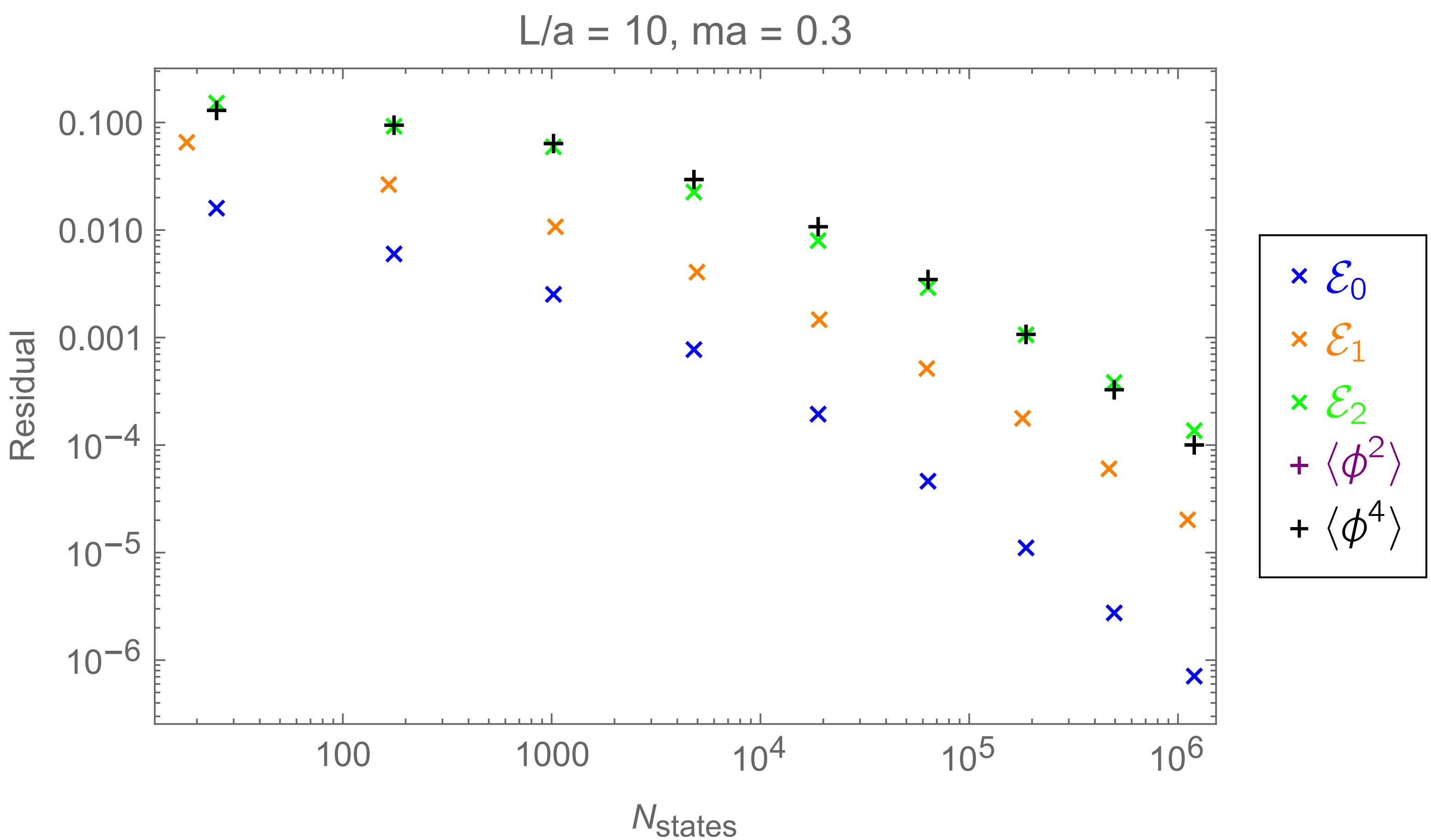

- Power law convergence $\quad \delta\mathcal{E}/\mathcal{E} \propto N^{-\alpha} \quad \alpha \simeq 1.2$
- cost $\sim NL \log N$

  $\sim$ 1 min on laptop running Julia/C++

2D ϕ^4 Critical Coupling

Method	Year	α_c
MPS[20]	2013	11.064(20)
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$$\Rightarrow \text{UV matching error} \sim \left(\frac{\lambda a^2}{4\pi}\right)^3 \sim 10^{-4} \quad \text{(in preparation)}$$

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- $\lambda a^2 = 1$

  $\Rightarrow$ UV matching error $\sim \left(\frac{\lambda a^2}{4\pi}\right)^3 \sim 10^{-4}$ (in preparation)

- $L/a = 10$

  $\Rightarrow$ Reasonable accuracy in 10 minutes on laptop

Binder Cumulant

$$B = \frac{\langle \phi^4 \rangle}{\langle \phi^2 \rangle^2} \to \begin{cases} 3 & m^2/\lambda \to +\infty, \\ 1 & m^2/\lambda \to -\infty \end{cases}$$

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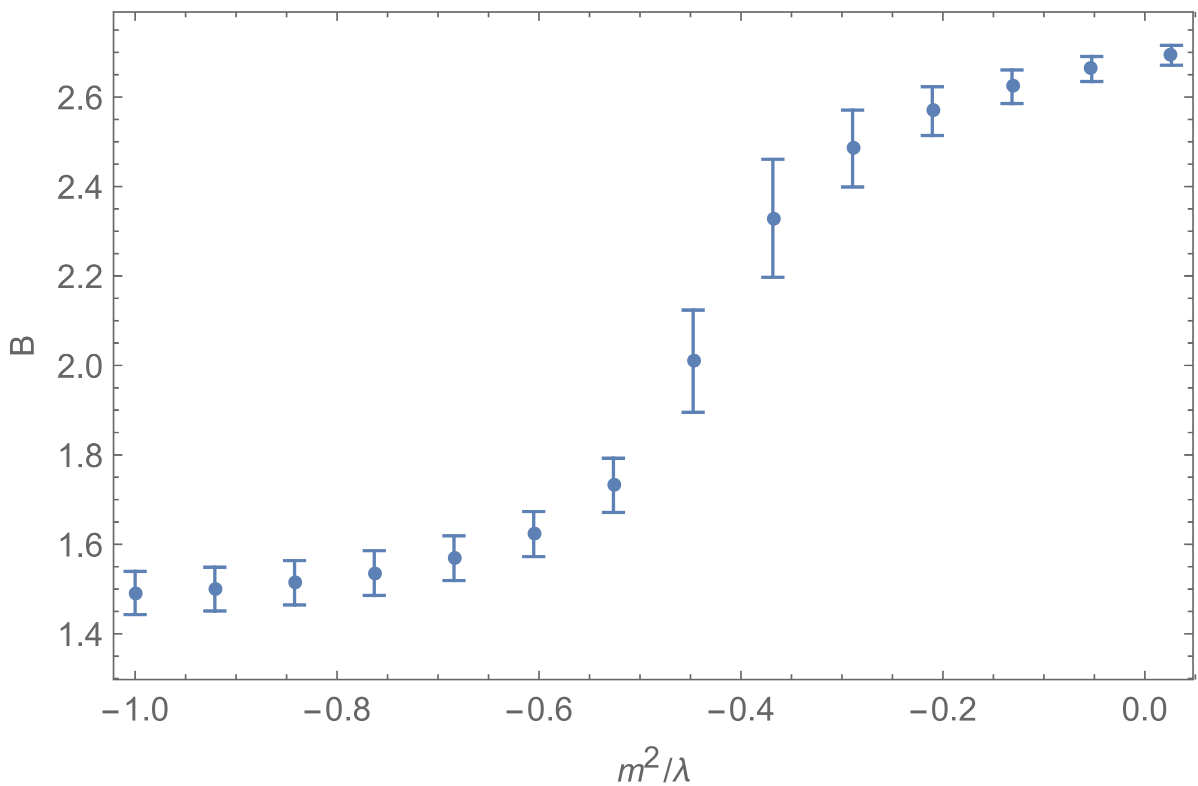

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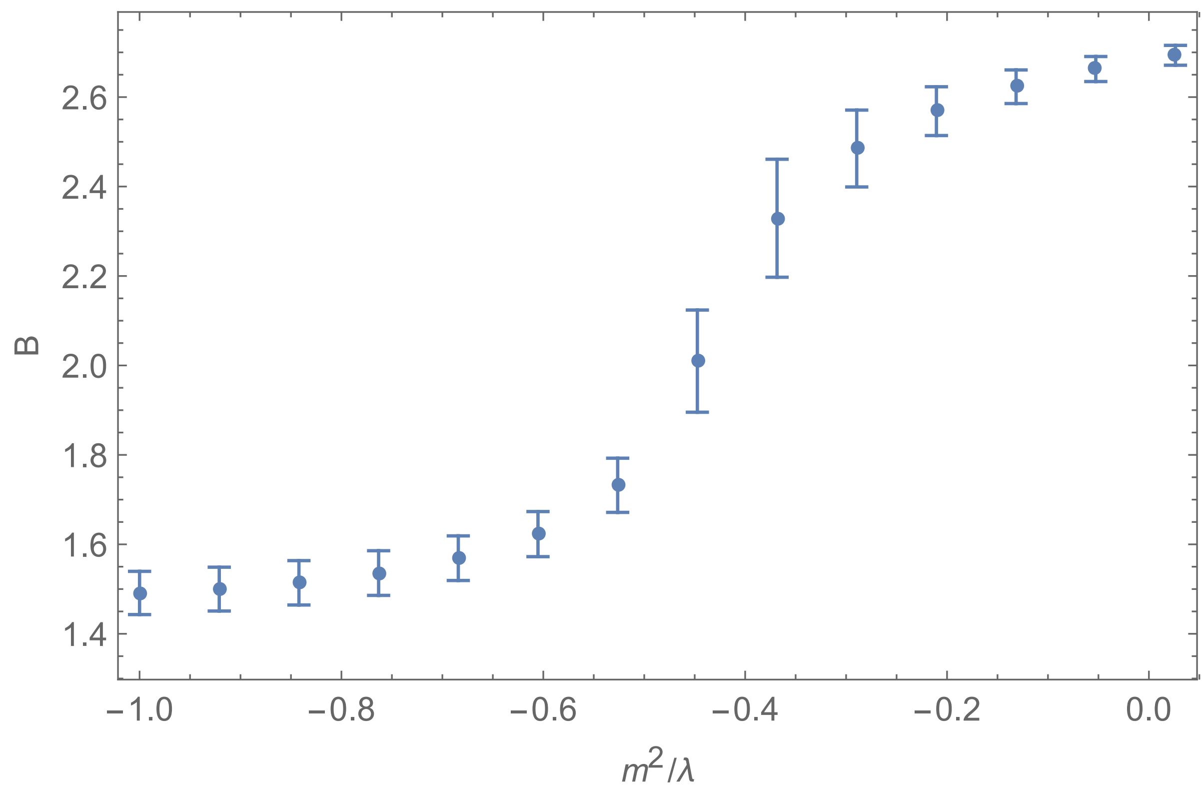

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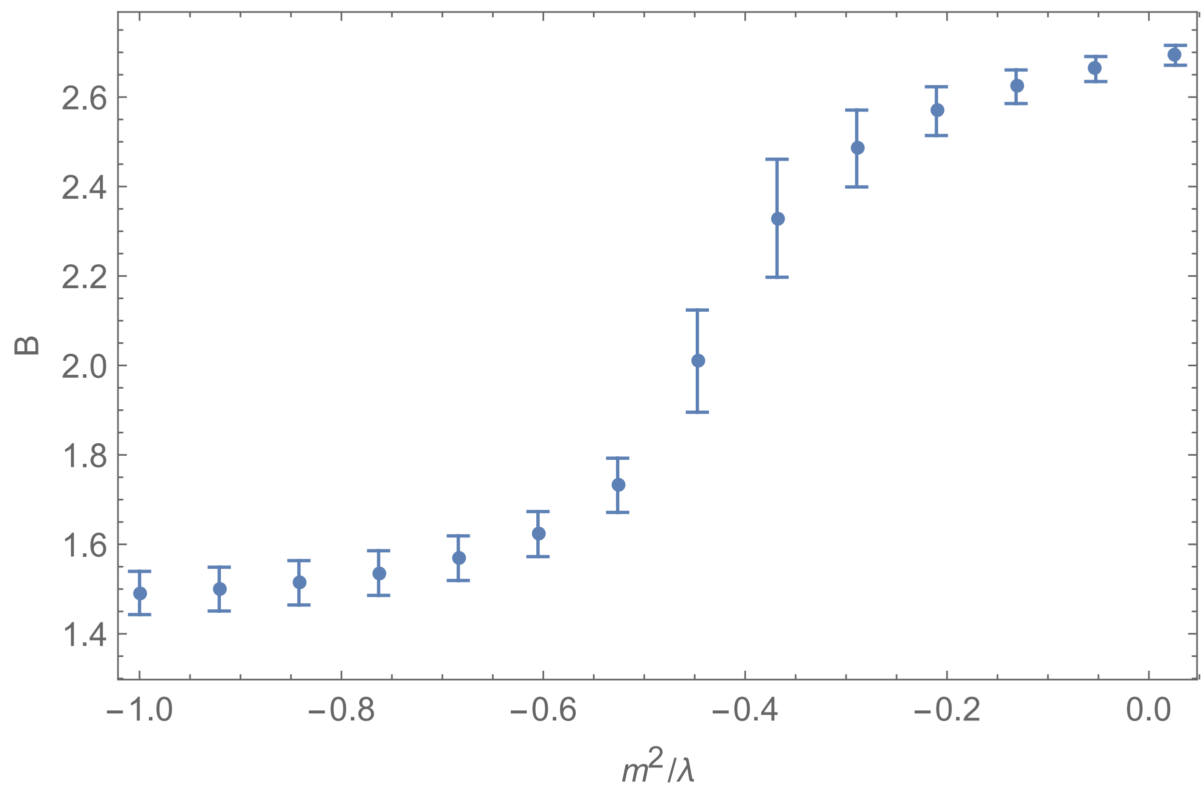

- $N \simeq 5 \times 10^5$
- Use difference of best values to estimate error

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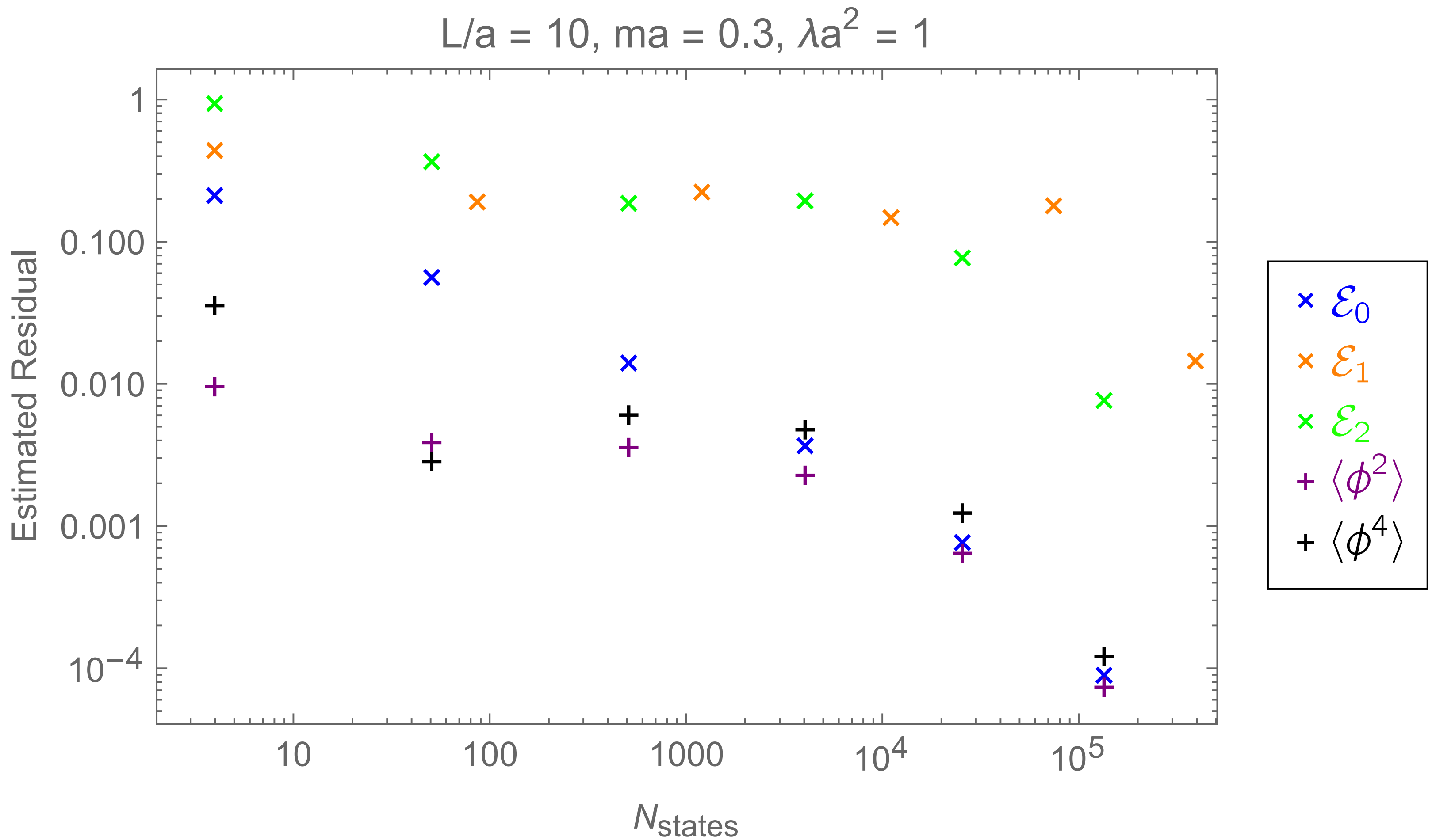
- Lots to do!

Backup Slides

Estimated Residuals



Convergence

Ground state: $|\Omega\rangle = \sum_n c_n |n\rangle \qquad c_n = \langle n|\Omega\rangle$

$$\langle\Omega|H_0^k|\Omega\rangle = \sum_{x_1,\ldots,x_k} \langle\Omega|\mathcal{H}_{0x_1}\cdots\mathcal{H}_{0x_k}|\Omega\rangle < \infty \quad \forall\, k$$

$$= \sum_n |c_n|^2 (E_n)^k \qquad \Rightarrow \quad |c_n| \sim e^{-\# E_n} \sim \frac{1}{\underbrace{N(E_n)}_{\substack{\#\text{ of states} \\ \text{with } E < E_n}}^{\#}}$$

$$1 = \left|\langle\Omega|\Omega\rangle\right|^2 = \int \underbrace{dE\,\rho(E)}_{= dN} \underbrace{\left|\langle E|\Omega\rangle\right|^2}_{\sim N^{-\#}}$$

$\Rightarrow$ convergence is power law in N:

$$|\Omega\rangle = |\Omega\rangle_N + \delta|\Omega\rangle$$

$$\delta|\Omega\rangle \sim \frac{1}{N^{\#}}$$

Continuum HT

Truncated Hilbert space:

$$\mathcal{S}_\Lambda = \left\{ |n\rangle \,\middle|\, E_n < \Lambda \right\} \qquad H_0|n\rangle = E_n|n\rangle$$

$\Lambda =$ maximum <u>total</u> energy

$\quad =$ nonlocal UV cutoff

Effective Hamiltonian exists and has power counting in Λ

Nonlocal counterterms:

$$\Delta H_{\text{eff}} = \int d^d x \left[\Lambda^d + \Lambda^{d-1} H_0 + \cdots \right] \qquad H_0 = \int d^d x \, \mathcal{H}_0$$

T. Cohen, K. Farnsworth, R. Houtz, M.L, SciPost Phys. 13 (2022)