Measurement-based quantum simulation of Abelian lattice gauge theories

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Plan of the talk

- Motivations and background
- Review of relevant materials
- Proposal: measurement-based quantum simulation
- Other aspects and generalizations

Motivations and background

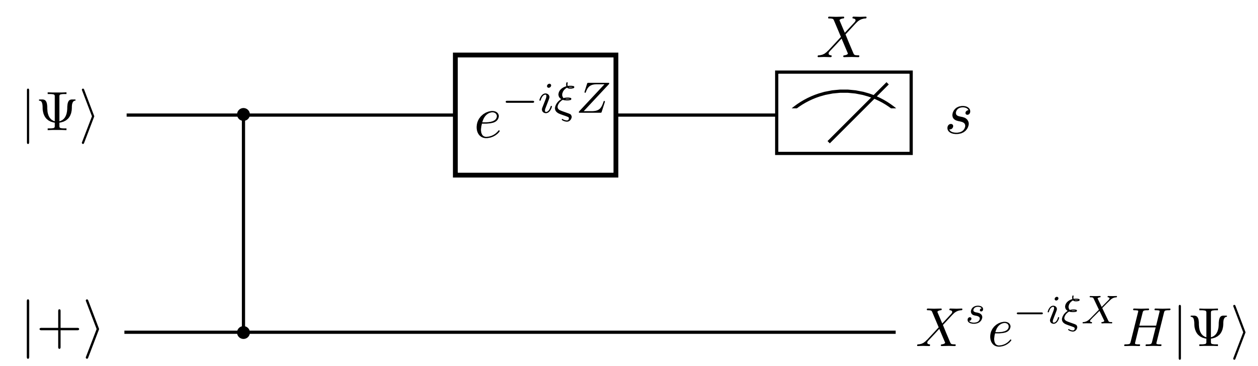
- Quantum simulation of lattice gauge theories is worth investigating.
- It's still too early to decide which simulation schemes will be the most efficient, and different schemes should be investigated.
- Simulation schemes can be roughly divided into digital and analog quantum simulations. I focus on digital schemes.

- Digital simulation applies quantum gates to realize the discrete time evolution using the Suzuki-Trotter approximation.
- So far, most efforts have focused on circuit-based methods.
- In quantum computation, there are alternative quantum computation (QC) schemes: measurement-based QC, adiabatic QC, etc.

Review: measurement-based quantum computation (MBQC)

- Introduced by Raussendorf and Briegel (2001).
- Also called one-way quantum computation.
- An alternative computational scheme that replaces circuit-based computation.
- Uses quantum teleportation and adaptive measurements on a resource (cluster) state.

Gate teleportation



- X-eigenstate $X|\pm\rangle = \pm|\pm\rangle$
- $|\Psi\rangle$ is an arbitrary 1-qubit state
- Entangle $|\Psi\rangle$ and $|+\rangle$ by a controlled-Z gate CZ.
- Measure the first qubit in bases $\{e^{i\xi Z}|\pm\rangle\}$. The measurement outcome is $s = 0{,}1$ corresponding to $\pm 1 = (-1)^s$.
- The state on the second qubit becomes

$$X^s e^{-i\xi X} H|\Psi\rangle.$$

Up to X^s and H, the state and the unitary transformation $e^{-i\xi X}$ are teleported. X^s is an example of a **byproduct operator**.

Adaptive measurement

- Suppose that an earlier measurement in a bigger circuit had produced the state $|\Psi\rangle = X^t H |\Phi\rangle$, where $t = 0,1$ is the **known** measurement outcome. Suppose also that we wish to obtain $e^{-i\alpha X}|\Phi\rangle$.

- Substituting this to the teleportation formula $X^s e^{-i\xi X} H |\Psi\rangle$, we get $X^s e^{-i\xi X} H X^t H |\Phi\rangle = X^s Z^t e^{-i(-1)^t \xi X}|\Phi\rangle$.

- To get the desired state $e^{-i\alpha X}|\Phi\rangle$ (up to byproducts), we need to set ξ to $\xi = (-1)^t \alpha$. $\Rightarrow$ We need to adjust the measurement angle ξ adaptively according to earlier measurement outcomes.

Resource state

- Measurement based quantum computation is performed by adaptive one-qubit measurements on a **resource state**.

- As a resource state, one usually considers a graph state

$$\bigotimes_{\text{edge}} CZ_{\text{edge}} |+\rangle^{\otimes \text{vertices}}.$$

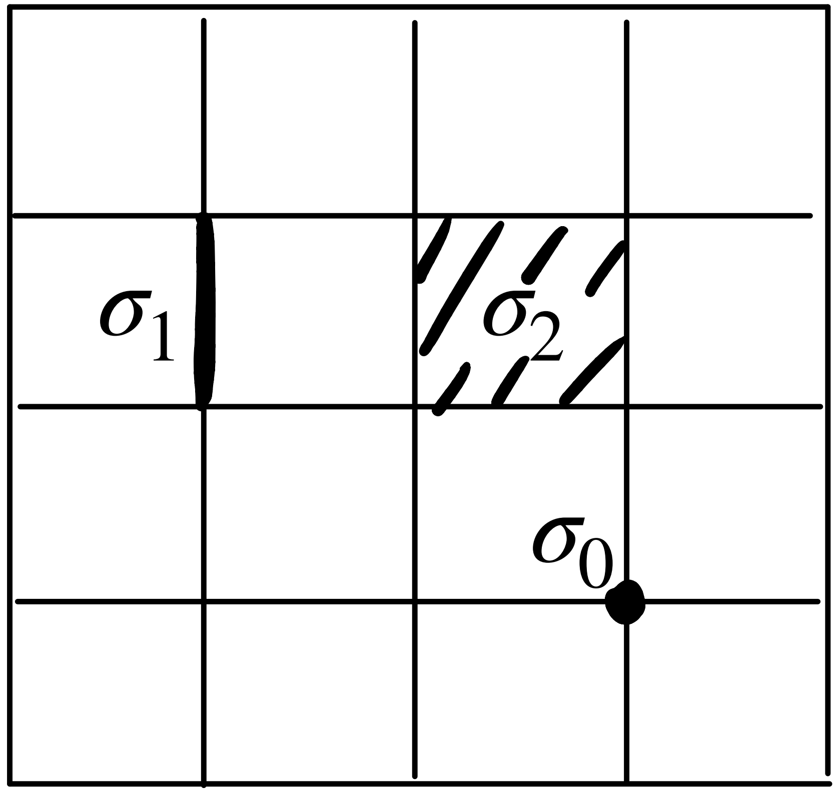
- For a large graph with a repeated pattern, the graph state is called a **cluster state**.

- Graph states and cluster states can be characterized by stabilizers.

- Measurement-based quantum computation is universal: it can reproduce any unitary operation over an arbitrary number of qubits.
- There exist versions of MBQC and cluster states with discrete and continuous-variable qudits.
- Large-scale $\mathcal{O}(10^4)$ (continuous-variable) optical cluster states have been experimentally generated.

Review: Hamiltonian lattice gauge theory in 2+1 dimensions

- Cell complex for a square lattice.
  - 0-cells $\sigma_0 \in \Delta_0$
  - 1-cells $\sigma_1 \in \Delta_1$
  - 2-cells $\sigma_2 \in \Delta_2$
- Degrees of freedom (qubits) are on 1-cells (edges) $\sigma_1 \in \Delta_1$.



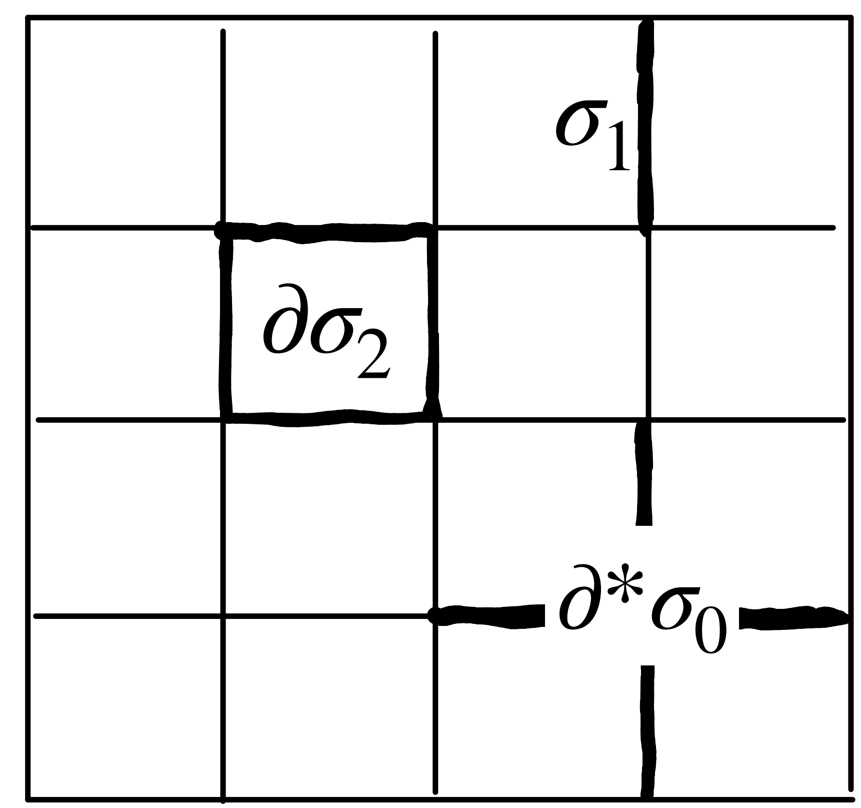
- Hamiltonian: $H = -\sum_{\sigma_1 \in \Delta_1} X(\sigma_1) - \lambda \sum_{\sigma_2 \in \Delta_2} Z(\partial\sigma_2)$ with $Z(\partial\sigma_2) = \prod_{\sigma_1 \subset \partial\sigma_2} Z(\sigma_1)$.

- Gauss law constraint: for any $\sigma_0 \in \Delta_0$,

$$X(\partial^*\sigma_0)\,|\psi_{\text{phys}}\rangle = |\psi_{\text{phys}}\rangle.$$

- The $\lambda \to 0$ limit is Kitaev's toric code.

- Generalization: $\mathbb{Z}_2$ gauge theory in 2+1 dimensions = $M_{(3,2)}$ $\Rightarrow$ **Wegner's model** $M_{(d,n)}$: higher-form gauge theory in d dimensions. The $n = 1$ case is the Ising model.



Trotterization

- Ideally we want to implement the continuous time evolution e^{-iHt} for any t. Decompose $H = H_1 + H_2$. $H_1 = -\sum_{\sigma_1 \in \Delta_1} X(\sigma_1)$ and $H_2 = -\lambda \sum_{\sigma_2 \in \Delta_2} Z(\partial \sigma_2)$ do not commute.
- In digital quantum simulation (such as by quantum circuits), we implement $e^{-iH_1 t}$ and $e^{-iH_2 t}$ separately.
- Suzuki-Trotter approximation: $e^{-Ht} \simeq \left(e^{-iH_1 t/n} e^{-iH_2 t/n}\right)^n$.
- We want to realize $e^{-iH_1 \delta t} = \prod_{\sigma_1 \in \Delta_1} e^{i\delta t X(\sigma_1)}$ and $e^{-iH_2 \delta t} = \prod_{\sigma_2 \in \Delta_2} e^{i\lambda \delta t Z(\partial \sigma_2)}$.

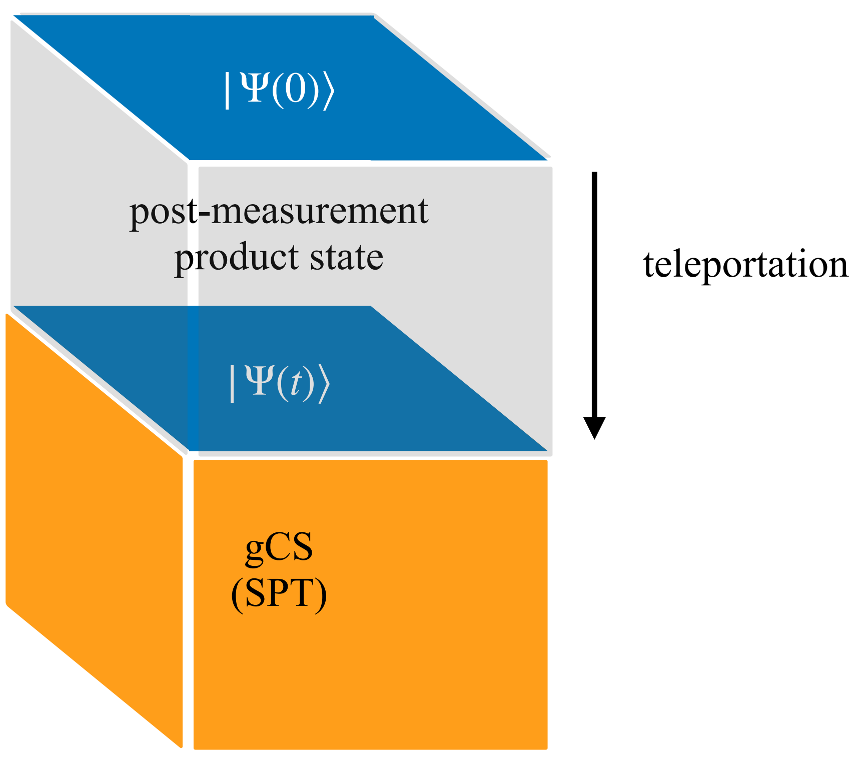
Proposal: measurement-based quantum simulation of abelian lattice gauge theories

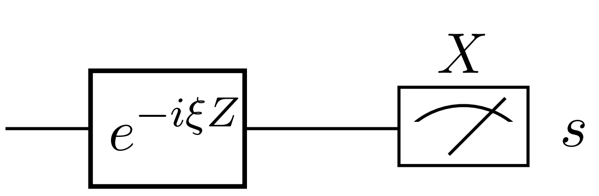
- Claim: we can implement the Trotterized time evolution $\left(e^{-iH_1 t/n} e^{-iH_2 t/n}\right)^n$ by

  1. preparing a generalized cluster state that reflects the spacetime structure of the gauge theory

  and then by

  2. performing adaptive single-qubit measurements adaptively in a prescribed **pattern**.

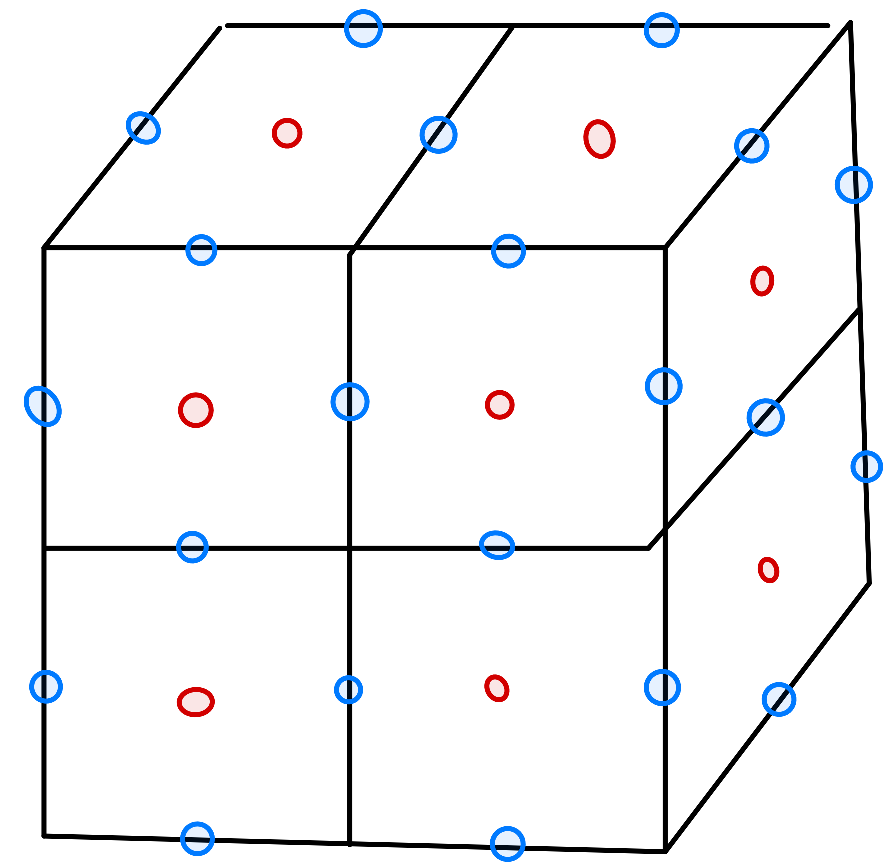




Resource state for $\mathbb{Z}_2$ lattice gauge theory in 2+1 dimensions

- Place one qubit on each 1-cell $\boldsymbol{\sigma}_1 \in \boldsymbol{\Delta}_1$ and 2-cell $\boldsymbol{\sigma}_2 \in \boldsymbol{\Delta}_2$ on a 3d cubic lattice.

- Entangle the neighboring 1-cells and 2-cells by controlled-Z gates.
$$|\mathrm{gCS}\rangle = \prod_{\boldsymbol{\sigma}_1 \subset \partial \boldsymbol{\sigma}_2} CZ_{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2} |+\rangle^{\otimes \boldsymbol{\Delta}_1 \cup \boldsymbol{\Delta}_2}$$



- A version of three-dimensional cluster state.

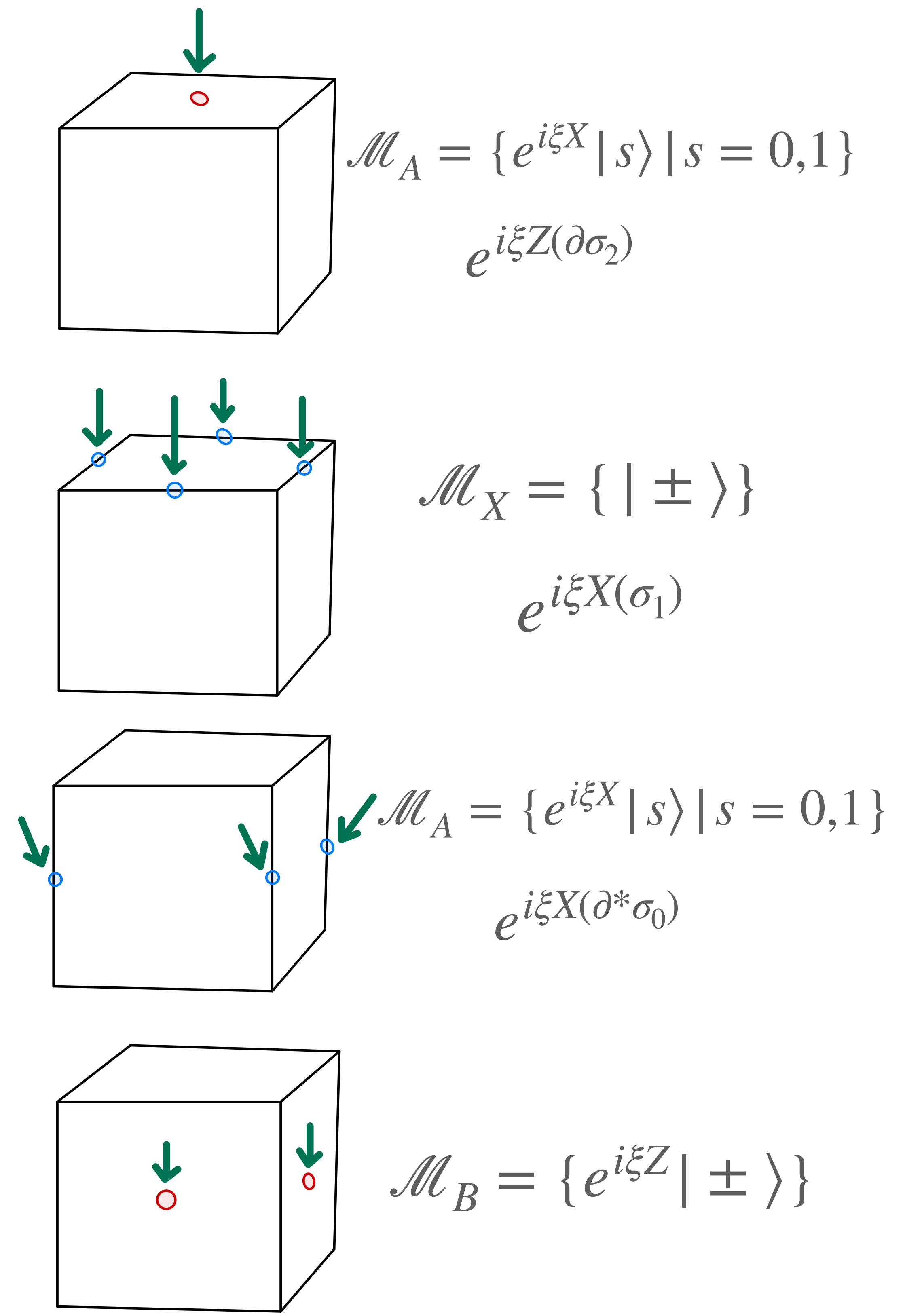
- Stabilizers $K(\boldsymbol{\sigma}_2) = X(\boldsymbol{\sigma}_2) X(\partial \boldsymbol{\sigma}_2)$ and $K(\boldsymbol{\sigma}_1) = X(\boldsymbol{\sigma}_1) X(\partial^* \boldsymbol{\sigma}_1)$.

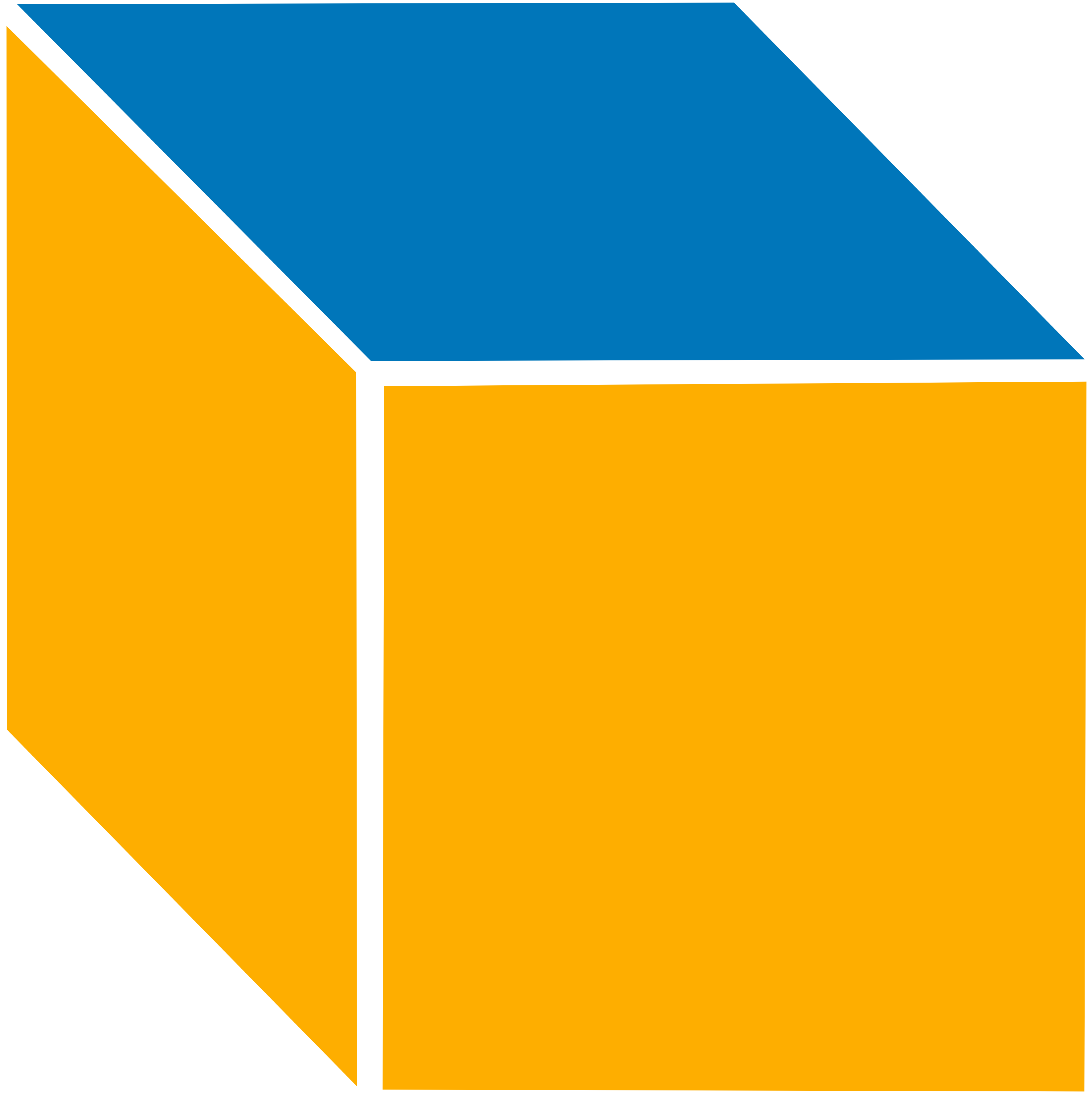
$$K(\boldsymbol{\sigma}_1)|\mathrm{gCS}\rangle = K(\boldsymbol{\sigma}_2)|\mathrm{gCS}\rangle = |\mathrm{gCS}\rangle$$

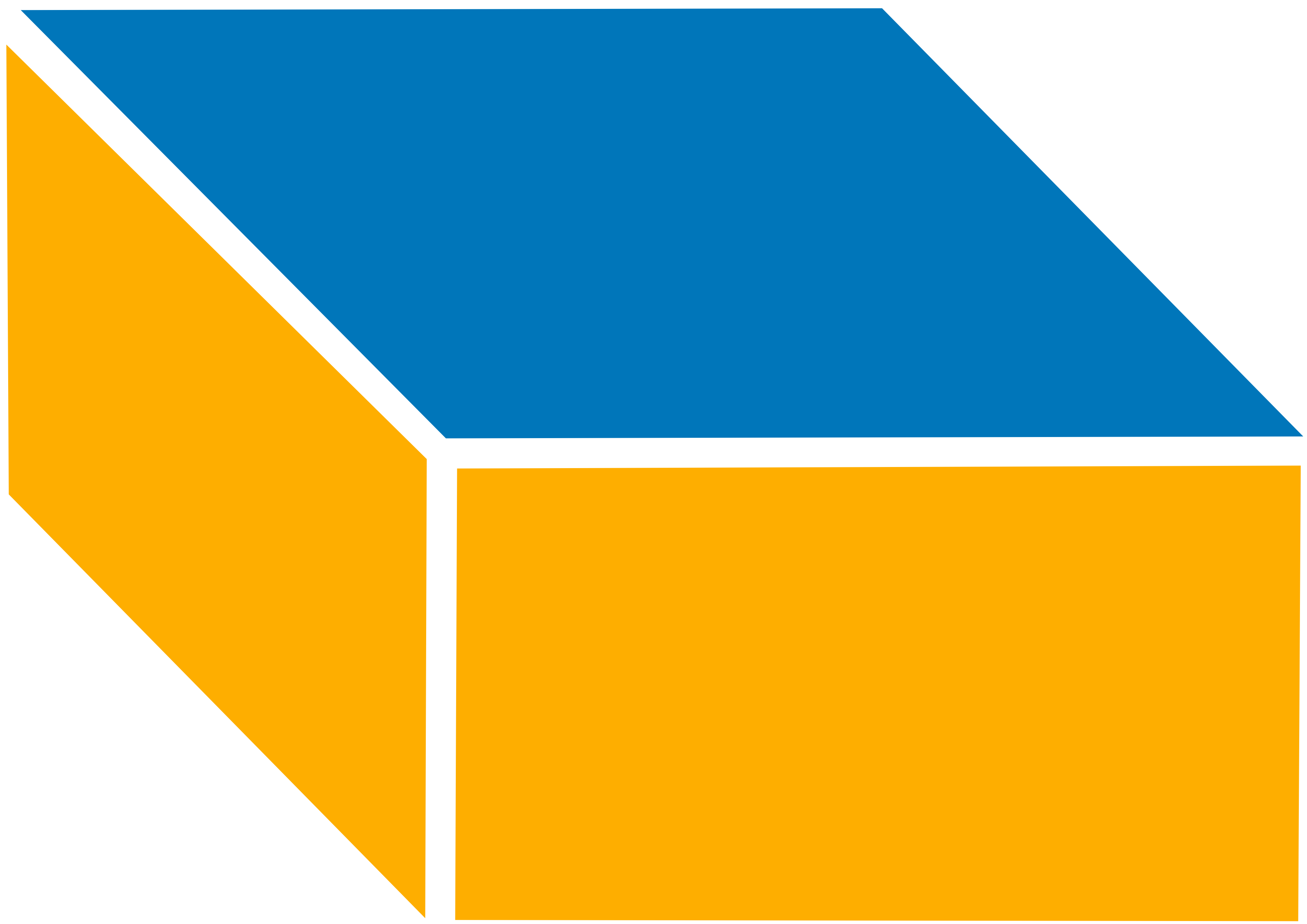
Measurement pattern = simulation protocol

- Trotterized time evolution is implemented by the measurement pattern and adaptive choices of the measurement angles ξ to absorb minus signs $(-1)^s$.

- Main result of the paper. The resource state reflects the spacetime structure of the simulated gauge theory.







Other aspects and generalizations

- The resource state is a cluster state tailored for the simulation of a given abelian lattice gauge theory. In our paper, we showed that it is an SPT (symmetry-protected topological) state protected by higher-form symmetries.

- There is an anomaly inflow between the simulated theory and the resource state (work in progress).

- Generalizations to $\mathbb{Z}_N$ gauge groups and the Kitaev Majorana chain are given in the paper. (Other generalizations in progress.)

- One can derive a correspondence between the resource state and the classical partition function.

- Possible experimental realizations may use (continuous-variable) cluster states created optically.

Backup slides

SPT order of the resource state

- Often, the computational power of a resource state can be attributed to the symmetry-protected topological (SPT) order. Examples: AKLT state and 1d cluster state protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- Claim: the natural resource state (qubits on n- and $(n-1)$-cells) for simulating Wegner's model $M_{(d,n)}$ is protected by global $\mathbb{Z}_2$ $(n-1)$- and $\mathbb{Z}_2$ $(d-n)$-form symmetries. (For $d=3$, $n=2$, shown by Yoshida.)

- For the $\mathbb{Z}_2$ gauge theory in $2+1$ dimensions $M_{(3,2)}$, they are both one-form symmetries generated by membrane (surface) operators $\prod_{\boldsymbol{\sigma}_2 \subset z_2} X(\boldsymbol{\sigma}_2)$ with 2-cycle z_2 ($\partial z_2 = 0$) and $\prod_{\boldsymbol{\sigma}_1 \subset z_2^*} X(\boldsymbol{\sigma}_1)$ with dual 2-cycle z_2^* ($\partial^* z_2^* = 0$).

- The SPT order of the resource state for $M_{(d,n)}$ can be demonstrated by showing that “gauging” the symmetries of the resource state and the product state give rise to distinct topological orders. [Levin-Gu, Yoshida]

- Other evidence for the SPT order includes
  - appearance of a projective representation on the boundary
  - appearance of a projective representation in the tensor network representation of the resource state

- Anomaly inflow between the resource state and the simulated theory. (In progress with Mana and Sukeno.)

Toward experimental realization

- The measurement-based approach requires only simple interactions (such as Ising interactions) between qubits because interactions are only used to create the resource state.
- Since the resource state includes the time direction, the measurement-based approach requires more qubits than the circuit-based approach.
- Possible experimental platforms:
    - Lattices formed by cold atoms
    - Continuous-variable cluster states created optically

Comparison with circuit-based simulation

- Simulation time is linear in the number of Trotter steps in both schemes.
  - $T_{\mathrm{MB}} \sim (\text{\#Trotter steps}) \times T_{\text{meas}}$
  - $T_{\mathrm{CB}} \sim (\text{\#Trotter steps}) \times T_{CZ}$
- In the measurement-based scheme, the resource state is created by a finite-depth circuit consisting of CZ. The number of necessary qubits grows linearly in the number of Trotter steps.

Comparison with classical simulation

- Exact diagonalization is only possible for up to tens of sites.
- Using tensor network methods, low-entanglement states are accessible for up to thousands of sites.
- In MBQS, the number of required qubits scales linearly with the number of Trotter steps.
- MBQS may have an advantage for problems with high-entanglement states if there are sufficiently many $\mathcal{O}(10^4)$ qubits of good quality.

Future directions

- More general gauge theories: non-abelian gauge groups, fracton models.
- More general fermions.
- Relate SPT order to computational power.
- Experimental realizations.
- Quantum simulation on cloud quantum computers with (adaptive) mid-circuit measurement capabilities.