Measurement-based quantum simulation of Abelian lattice gauge theories

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Plan of the talk

- Motivations and background
- Review of relevant materials
- Proposal: measurement-based quantum simulation
- Other aspects and generalizations
Motivations and background

• Quantum simulation of lattice gauge theories is worth investigating.

• It's still too early to decide which simulation schemes will be the most efficient, and different schemes should be investigated.

• Simulation schemes can be roughly divided into digital and analog quantum simulations. I focus on digital schemes.
• Digital simulation applies quantum gates to realize the discrete time evolution using the Suzuki-Trotter approximation.

• So far, most efforts have focused on circuit-based methods.

• In quantum computation, there are alternative quantum computation (QC) schemes: measurement-based QC, adiabatic QC, etc.
• Introduced by Raussendorf and Briegel (2001).

• Also called one-way quantum computation.

• An alternative computational scheme that replaces circuit-based computation.

• Uses quantum teleportation and adaptive measurements on a resource (cluster) state.
Gate teleportation

- X-eigenstate $X|\pm\rangle = \pm |\pm\rangle$
- $|\Psi\rangle$ is an arbitrary 1-qubit state
- Entangle $|\Psi\rangle$ and $|+\rangle$ by a controlled-Z gate $CZ$.
- Measure the first qubit in bases $\{e^{i\xi Z}|\pm\rangle\}$. The measurement outcome is $s = 0,1$ corresponding to $\pm 1 = (-1)^s$.
- The state on the second qubit becomes

$$X^s e^{-i\xi X} H |\Psi\rangle.$$ 

Up to $X^s$ and $H$, the state and the unitary transformation $e^{-i\xi X}$ are teleported. $X^s$ is an example of a byproduct operator.
Adaptive measurement

- Suppose that an earlier measurement in a bigger circuit had produced the state $|\Psi\rangle = X^t H |\Phi\rangle$, where $t = 0, 1$ is the known measurement outcome. Suppose also that we wish to obtain $e^{-i\alpha X} |\Phi\rangle$.

- Substituting this to the teleportation formula $X^s e^{-i\xi X} H |\Psi\rangle$, we get $X^s e^{-i\xi X}HX^t H |\Phi\rangle = X^s Z^t e^{-i(-1)^t \xi X} |\Phi\rangle$.

- To get the desired state $e^{-i\alpha X} |\Phi\rangle$ (up to byproducts), we need to set $\xi$ to $\xi = (-1)^t \alpha$. $\Rightarrow$ We need to adjust the measurement angle $\xi$ adaptively according to earlier measurement outcomes.
Resource state

• Measurement based quantum computation is performed by adaptive one-qubit measurements on a resource state.

• As a resource state, one usually considers a graph state

\[ \bigotimes_{\text{edge}} CZ_{\text{edge}} | + \rangle ^{\otimes \text{vertices}}. \]

• For a large graph with a repeated pattern, the graph state is called a cluster state.

• Graph states and cluster states can be characterized by stabilizers.
• Measurement-based quantum computation is universal: it can reproduce any unitary operation over an arbitrary number of qubits.

• There exist versions of MBQC and cluster states with discrete and continuous-variable qudits.

• Large-scale Ọ(10^4) (continuous-variable) optical cluster states have been experimentally generated.
Review: Hamiltonian lattice gauge theory in 2+1 dimensions

• Cell complex for a square lattice.
  • 0-cells $\sigma_0 \in \Delta_0$ •
  • 1-cells $\sigma_1 \in \Delta_1$
  • 2-cells $\sigma_2 \in \Delta_2$

• Degrees of freedom (qubits) are on 1-cells (edges) $\sigma_1 \in \Delta_1$. 
• Hamiltonian: \( H = - \sum_{\sigma_1 \in \Delta_1} X(\sigma_1) - \lambda \sum_{\sigma_2 \in \Delta_2} Z(\partial \sigma_2) \) with
\[
Z(\partial \sigma_2) = \prod_{\sigma_1 \subset \partial \sigma_2} Z(\sigma_1).
\]

• Gauss law constraint: for any \( \sigma_0 \in \Delta_0 \),
\[
X(\partial^* \sigma_0) |\psi_{\text{phys}}\rangle = |\psi_{\text{phys}}\rangle.
\]

• The \( \lambda \to 0 \) limit is Kitaev’s toric code.

• Generalization: \( \mathbb{Z}_2 \) gauge theory in 2+1 dimensions = \( M_{(3,2)} \) \Rightarrow \textbf{Wegner’s model} \( M_{(d,n)} \): higher-form gauge theory in \( d \) dimensions. The \( n = 1 \) case is the Ising model.
Trotterization

• Ideally we want to implement the continuous time evolution $e^{-iHt}$ for any $t$. Decompose $H = H_1 + H_2$. $H_1 = -\sum_{\sigma_1 \in \Delta_1} X(\sigma_1)$ and $H_2 = -\lambda \sum_{\sigma_2 \in \Delta_2} Z(\partial \sigma_2)$ do not commute.

• In digital quantum simulation (such as by quantum circuits), we implement $e^{-iH_1t}$ and $e^{-iH_2t}$ separately.

• Suzuki-Trotter approximation: $e^{-Ht} \simeq \left( e^{-iH_1t/n} e^{-iH_2t/n} \right)^n$.

  We want to realize $e^{-iH_1\delta t} = \prod_{\sigma_1 \in \Delta_1} e^{i\delta t X(\sigma_1)}$ and $e^{-iH_2\delta t} = \prod_{\sigma_2 \in \Delta_2} e^{i\lambda \delta t Z(\partial \sigma_2)}$. 
Proposal: measurement-based quantum simulation of abelian lattice gauge theories

• Claim: we can implement the Trotterized time evolution \( (e^{-iH_1t/n} e^{-iH_2t/n})^n \) by

1. preparing a generalized cluster state that reflects the spacetime structure of the gauge theory

and then by

2. performing adaptive single-qubit measurements adaptively in a prescribed pattern.
Resource state for $\mathbb{Z}_2$ lattice gauge theory in 2+1 dimensions

- Place one qubit on each 1-cell $\sigma_1 \in \Delta_1$ and 2-cell $\sigma_2 \in \Delta_2$ on a 3d cubic lattice.

- Entangle the neighboring 1-cells and 2-cells by controlled-Z gates.

$$|gCS\rangle = \prod_{\sigma_1 \subset \partial \sigma_2} CZ_{\sigma_1,\sigma_2} |+\rangle \otimes \Delta_1 \cup \Delta_2$$

- A version of three-dimensional cluster state.

- Stabilizers $K(\sigma_2) = X(\sigma_2)X(\partial \sigma_2)$ and $K(\sigma_1) = X(\sigma_1)X(\partial^* \sigma_1)$.

$$K(\sigma_1) |gCS\rangle = K(\sigma_2) |gCS\rangle = |gCS\rangle$$
Measurement pattern = simulation protocol

- Trotterized time evolution is implemented by the measurement pattern and adaptive choices of the measurement angles $\xi$ to absorb minus signs $(-1)^s$.

- Main result of the paper. The resource state reflects the spacetime structure of the simulated gauge theory.

\[
\mathcal{M}_A = \{ e^{i\xi X} | s \rangle | s = 0,1 \}
\]
\[
e^{i\xi Z(\partial \sigma_2)}
\]
\[
\mathcal{M}_X = \{ | \pm \rangle \}
\]
\[
e^{i\xi X(\sigma_1)}
\]
\[
\mathcal{M}_A = \{ e^{i\xi X} | s \rangle | s = 0,1 \}
\]
\[
e^{i\xi X(\partial^* \sigma_0)}
\]
\[
\mathcal{M}_B = \{ e^{i\xi Z} | \pm \rangle \}
\]
Other aspects and generalizations

• The resource state is a cluster state tailored for the simulation of a given abelian lattice gauge theory. In our paper, we showed that it is an SPT (symmetry-protected topological) state protected by higher-form symmetries.

• There is an anomaly inflow between the simulated theory and the resource state (work in progress).

• Generalizations to $\mathbb{Z}_N$ gauge groups and the Kitaev Majorana chain are given in the paper. (Other generalizations in progress.)

• One can derive a correspondence between the resource state and the classical partition function.

• Possible experimental realizations may use (continuous-variable) cluster states created optically.
Backup slides
**SPT order of the resource state**

- Often, the computational power of a resource state can be attributed to the symmetry-protected topological (SPT) order. Examples: AKLT state and 1d cluster state protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- Claim: the natural resource state (qubits on $n$- and $(n - 1)$-cells) for simulating Wegner’s model $M_{(d,n)}$ is protected by global $\mathbb{Z}_2 (n - 1)$- and $\mathbb{Z}_2 (d - n)$-form symmetries. (For $d = 3$, $n = 2$, shown by Yoshida.)

- For the $\mathbb{Z}_2$ gauge theory in $2 + 1$ dimensions $M_{(3,2)}$, they are both one-form symmetries generated by membrane (surface) operators \[
\prod_{\sigma_2 \subset \mathbb{Z}_2} X(\sigma_2) \text{ with 2-cycle } \mathbb{Z}_2 (\partial \mathbb{Z}_2 = 0) \text{ and } \prod_{\sigma_1 \subset \mathbb{Z}_2^*} X(\sigma_1) \text{ with dual 2-cycle } \mathbb{Z}_2^* (\partial^* \mathbb{Z}_2^* = 0).
\]
• The SPT order of the resource state for $M_{(d,n)}$ can be demonstrated by showing that “gauging” the symmetries of the resource state and the product state give rise to distinct topological orders. [Levin-Gu, Yoshida]

• Other evidence for the SPT order includes
  
  • appearance of a projective representation on the boundary
  
  • appearance of a projective representation in the tensor network representation of the resource state
  
• Anomaly inflow between the resource state and the simulated theory. (In progress with Mana and Sukeno.)
Toward experimental realization

• The measurement-based approach requires only simple interactions (such as Ising interactions) between qubits because interactions are only used to create the resource state.

• Since the resource state includes the time direction, the measurement-based approach requires more qubits than the circuit-based approach.

• Possible experimental platforms:
  • Lattices formed by cold atoms
  • Continuous-variable cluster states created optically
Comparison with circuit-based simulation

- Simulation time is linear in the number of Trotter steps in both schemes.
  - \( T_{\text{MB}} \sim (\#\text{Trotter steps}) \times T_{\text{meas}} \)
  - \( T_{\text{CB}} \sim (\#\text{Trotter steps}) \times T_{\text{CZ}} \)
- In the measurement-based scheme, the resource state is created by a finite-depth circuit consisting of CZ. The number of necessary qubits grows linearly in the number of Trotter steps.
Comparison with classical simulation

• Exact diagonalization is only possible for up to tens of sites.

• Using tensor network methods, low-entanglement states are accessible for up to thousands of sites.

• In MBQS, the number of required qubits scales linearly with the number of Trotter steps.

• MBQS may have an advantage for problems with high-entanglement states if there are sufficiently many $\mathcal{O}(10^4)$ qubits of good quality.
Future directions

• More general gauge theories: non-abelian gauge groups, fracton models.

• More general fermions.

• Relate SPT order to computational power.

• Experimental realizations.

• Quantum simulation on cloud quantum computers with (adaptive) mid-circuit measurement capabilities.