The Critical Ising Model on a 2-Sphere

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Overview

- Background and context
- Ising model on a 2-sphere
- Conclusion and next steps

Background and context

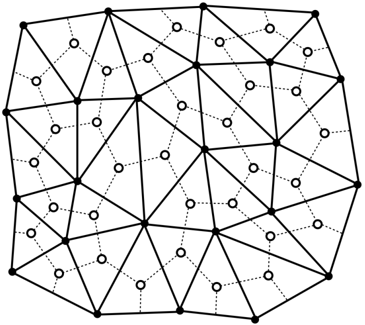
Background and context

- Conformal field theories are scale-invariant: difficult to mitigate finite volume effects.
- Traditionally, finite size scaling methods[1,2,3] are used to parametrize the dependence of the theory on the size of the lattice.
- Conformal field theories in Euclidean space ($\mathbb{R}^d$) can be mapped to e.g. a sphere (S^d) or cylinder ($\mathbb{R} \times S^{d-1}$) via a Weyl transformation to reduce or eliminate finite volume effects.
- Due to non-uniform UV quantum effects, it is challenging to define a lattice theory on a sequence of spherical or cylindrical lattices which has a valid continuum limit.
- Here,I present a method to simulate the critical 2d Ising model on a 2-sphere, and discuss how the method can be generalized for use with other theories.

Simplicial Ising Model

Simplicial Ising Model

- We can define an Ising model on a simplicial lattice and its trivalent dual lattice



$$S_{\text{tri}} = -\sum_{\langle ij \rangle} K_{ij}\sigma_i\sigma_j \qquad S_{\text{dual}} = -\sum_{\langle ij \rangle} L_{ij}\sigma_i\sigma_j$$

Simplicial Ising Model

- Using Kramers-Wannier duality[4] and a lattice Wilson-Majorana fermion action,[5,6] we derive a relationship between the lattice geometry and the critical values of the Ising coupling constants:

$$\sinh 2K_{ij} = 1/\sinh 2L_{ij} = \frac{l^*_{ij}}{l_{ij}}$$

  l_{ij} and l^*_{ij} are the edge lengths for the simplicial lattice and the dual lattice, respectively.

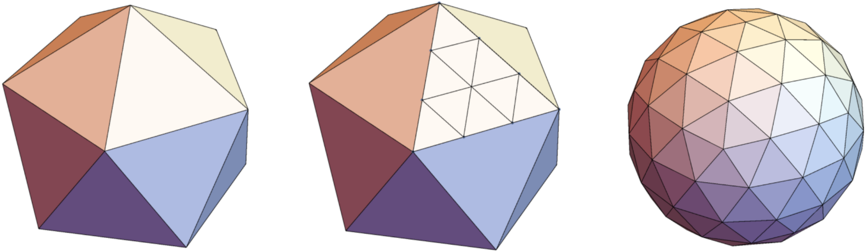
- Derivation requires several geometrical constraints:
  1. Dual lattice vertices must be at triangle circumcenters
  2. Triangles must have uniform circumradius and perimeter

Ising model on a 2-sphere

Ising model on a 2-sphere

Basic discretization of S^2:

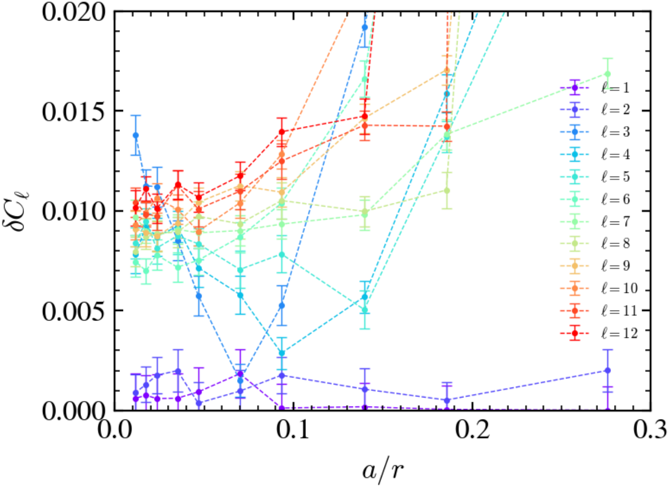
- Tessellate an octahedron or icosahedron, then project all vertices onto a unit sphere[7,8]



- Produces a non-uniform simplicial complex (triangles have non-uniform circumradius and perimeter)

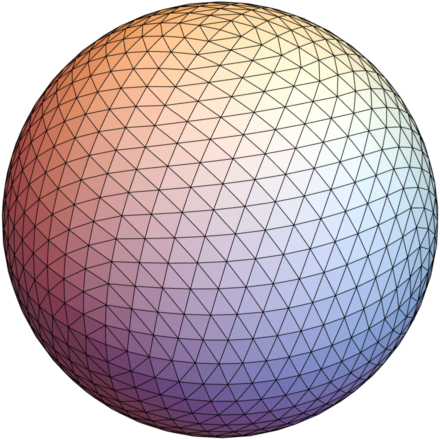
Ising model on a 2-sphere

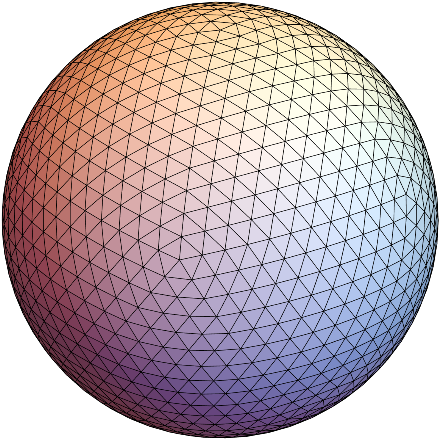
2-point function rotational symmetry is broken using basic discretization



Ising model on a 2-sphere

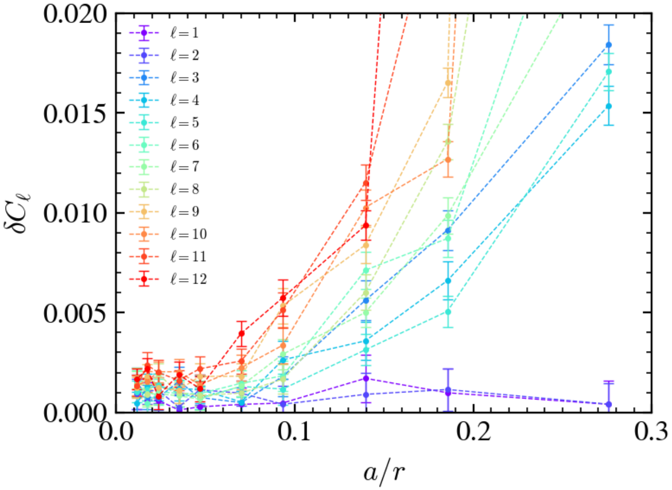
Modified octahedral and icosahedral lattices:





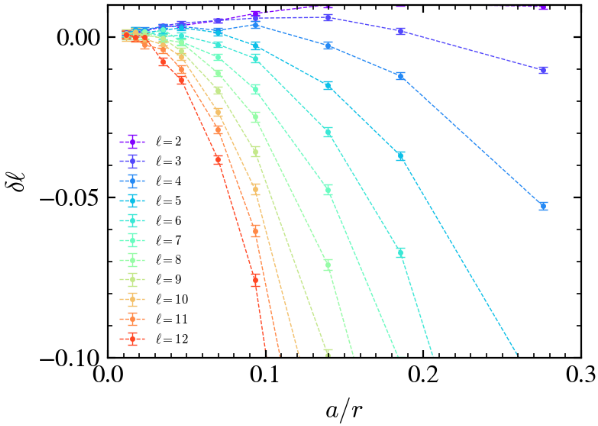
Ising model on a 2-sphere

Modified lattice, rotational symmetry restored as $a \to 0$



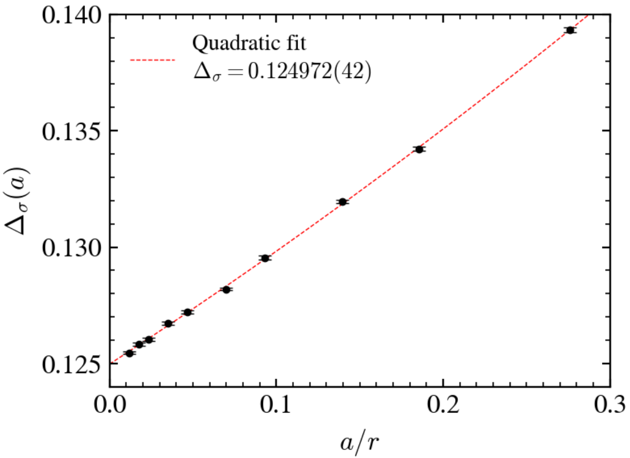
Ising model on a 2-sphere

Conformal symmetry breaking measurement:



Ising model on a 2-sphere

Direct measurement of critical exponents:



χ^2 / dof $= 1.8$

Conclusion and next steps

Conclusion and next steps

- When simulating conformal field theories on a lattice, finite volume effects can be mitigated by mapping the Euclidean plane to a sphere or cylinder.
- We derived a simplicial lattice action for the 2d Ising model which places strong constraints on lattice geometry.
- Using this action gives results consistent with the 2d Ising CFT on both an affine plane and a 2-sphere.

Conclusion and next steps

Can this method be generalized to other non-perturbative theories and other manifolds?

- Requires a map between lattice geometry and simplicial action coupling constants.
- For more complex theories, this map likely needs to be computed empirically using lattice data (like Karsch coefficients in finite temperature QCD)

Conclusion and next steps

- What geometrical constraints are required for other theories?
- It is possible to construct simplicial lattices with uniform circumradius in higher dimensions, but it is unclear if this is sufficient to ensure a valid continuum limit.
- In terms of regulating UV divergences, this may be equivalent to ensuring that we have a uniform lattice cutoff.

Conclusion and next steps

Next steps:

- Study 2d tricritical Ising model on simplicial lattices.
- Critical 3d Ising model on $\mathbb{R} \times S^2$ and S^3, compare to conformal bootstrap results.
- 3d QED on $\mathbb{R} \times S^2$, which has a critical point which is not well understood as a function of the number of fermion flavors.

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