Trivial Pursuit

CLASSIC EDITION

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“Fact” :

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Scalar ϕ^4 theory

$$S_E = \frac{1}{2}\int d^4x \left[\partial_\mu\phi\partial_\mu\phi + m_0^2\phi^2 + \lambda_0\phi^4\right]$$

is trivial in four dimensions:

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Scalar ϕ^4 theory

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is trivial in four dimensions:

$$\lim_{a\to 0} \lambda_{IR} = 0$$

[Wilson; Fröhlich; Lüscher; Weisz;...]

a=lattice spacing=$\frac{1}{\Lambda_{\mathrm{UV}}}$

Proofs of Quantum Triviality in 4d

Michael Aizenman

Hugo Duminil-Copin

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2019: Proof of Triviality for ϕ^4 in 4d; Fields Medal for Duminil-Copin

“Loophole” in proofs:
assume $\lambda_0 > 0$

A hint from large N Field Theory

- O(N) model provides solvable interacting field theory for $N \gg 1$
- Exact large N running coupling [2305.05678]

$$\lim_{\Lambda_{UV} \to \infty} \lambda(\mu) = \frac{(2\pi)^2}{\ln \frac{M^2}{\mu^2}};$$

- $\lambda_{IR} \neq 0$. Theory is nontrivial!

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$$\lim_{\Lambda_{UV}\to\infty} \lambda(\mu) = \frac{(2\pi)^2}{\ln \frac{M^2}{\mu^2}};$$

- $\lambda_{IR} \neq 0$. Theory is nontrivial!
- Possible because exact large N bare coupling given by

$$\lambda_0 \equiv \lambda\left(\mu = \Lambda_{\mathrm{UV}}\right) = \frac{(2\pi)}{\ln \frac{M^2}{\Lambda_{\mathrm{UV}}^2}} < 0$$

- Coupling at UV scale is **negative**

Negative Coupling Field Theory History

A Field Theory with Computable Large-Momenta Behaviour.

K. Symanzik

Deutsches Elektronen-Synchrotron DESY - Hamburg

(ricevuto il 12 Dicembre 1972)

In the current extensive discussions (*) of φ^4 theory it is usually taken for granted that the renormalized coupling constant g must be positive. As emphasized previously (2) there is no known reason, axiomatic or otherwise, for $g > 0$ to be required for a physically acceptable theory. The feeling that otherwise the theory cannot have a vacuum and particles of discrete mass is not rigorously founded as discussed near the end of this letter. The interesting feature of the theory with $g < 0$, however, appears worth pointing out: If one assumes the theory to exist, the large-momenta behaviour of its Feynman amplitudes can be computed at generic momenta to arbitrary accuracy. Besides, we find that the imaginary part of the four-point vertex function in φ^4 theory should not change sign in momentum space.

Is There More to Quantum Field Theory Than It Seems?

$\lambda_0 < 0$ $\lambda_0 > 0$

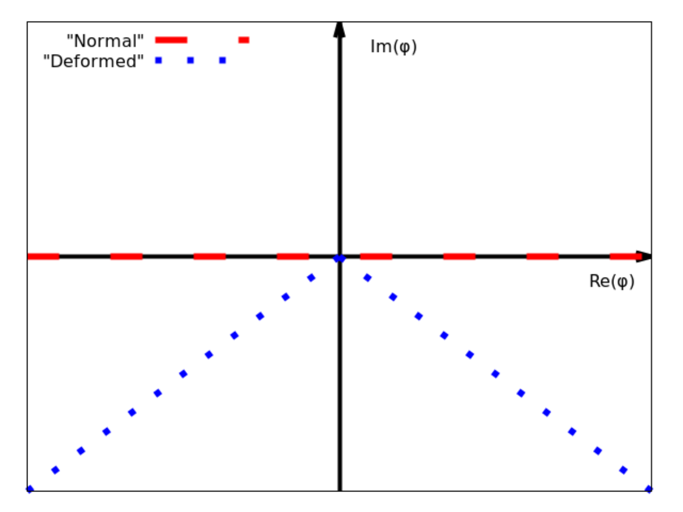
Lessons from Quantum Mechanics

- QM partition function, path integral

$$Z(\lambda) = \int \mathcal{D}\phi e^{-S_E}\,, \quad S_E = \frac{1}{2}\int_0^\beta d\tau \left[\dot{\phi}^2 + 2\lambda\phi^4\right]$$

- Can we make sense of negative coupling $Z(\lambda = -g)$?
- Yes, via analytic continuation!
- Deform integration contours into complex manifold, e.g.

$$\phi(x) = s(x)\left[e^{\frac{i\pi}{4}}\theta(-s(x)) + e^{\frac{-i\pi}{4}}\theta(s(x))\right]$$



"Normal"
"Deformed"
Im(φ)
Re(φ)

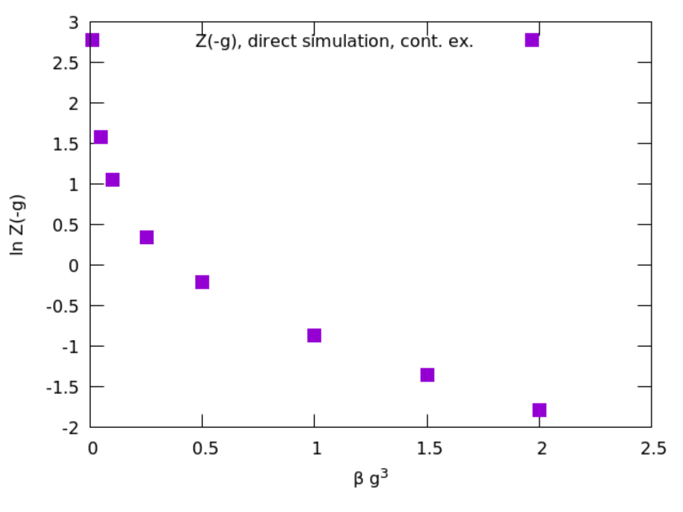
- Contour "Deformation" changes sign of quartic term:

$$-g\phi^4(x) \to +gs^4(x)$$

- Kinetic term becomes complex $\to$ **sign problem**!
- Just quantum mechanics, can beat down sign problem by brute-force numerical integration
- $\to$ We discretize action using finite-differencing and evaluate

$$Z(\lambda = -g)$$

directly on complex contour for various values of $g > 0$ [2303.01470]

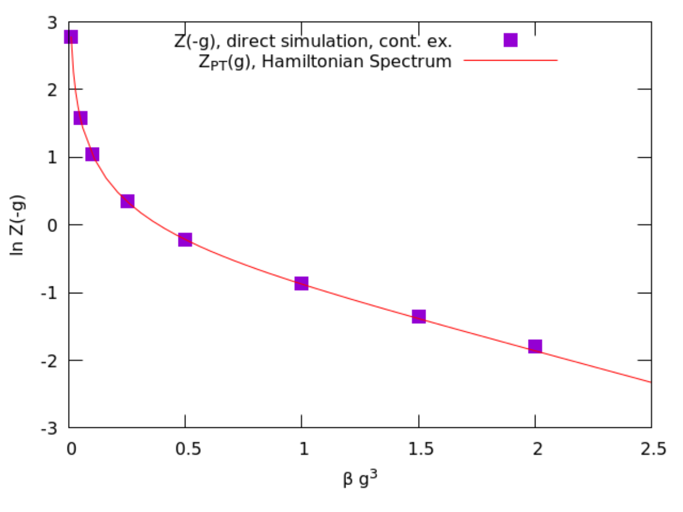


Z(-g), direct simulation, cont. ex.
ln Z(-g)
β g³
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5

- Negative coupling QM has come up in a different context before
- [Bender & Böttcher 1997] consider QM with Hamiltonian $\mathcal{H} = \frac{p^2}{2} - gx^4$
- Called 'PT-symmetric' Quantum Mechanics
- Surprisingly, spectrum of $\mathcal{H}$ is real and positive definite:
  $E_0 \simeq 0.9305460341 g^{\frac{1}{3}}$
  $E_1 \simeq 3.7818962485 g^{\frac{1}{3}}$
  $E_2 \simeq 7.4350672631 g^{\frac{1}{3}}$
  $E_{n\geq 3} \simeq 2.18507(n+0.5)^{\frac{4}{3}} g^{\frac{1}{3}}$
- Can define $\mathcal{PT}$-symmetric partition function

$$Z_{\mathcal{PT}}(g) = \sum_{n=0}^{\infty} e^{-\beta E_n}$$

- What happens if we compare this to our negative coupling Z?



Perfect match!

4d lattice ϕ^4 theory with $\lambda_0 < 0$

- Same principle as in 1d
- Write down 'usual' Euclidean field theory

$$Z = \int \mathcal{D}\phi e^{-S_E}\,, \quad S_E = \frac{1}{2}\int d^4x \left[\partial_\mu\phi\partial_\mu\phi + m_0^2\phi^2 + 2\lambda_0\phi^4\right]$$

- Discretize on hypercube
- 'Contour deform' to complex manifold

$$\phi(x) = s(x)\left[e^{\frac{i\pi}{4}}\theta(-s(x)) + e^{\frac{-i\pi}{4}}\theta(s(x))\right]\,, \quad s(x) \in \mathbb{R}$$

- On complex manifold, discretize path integral is convergent for $\lambda_0 < 0$ but has a sign problem
- Evaluate Z for $\lambda_0 < 0$ by direct numerical integration on tiny $N_\sigma^3 \times N_\tau$ lattices

What to expect – large N as a guide

- Large N coupling, identifying $\Lambda_{\rm UV} = \frac{1}{a}$

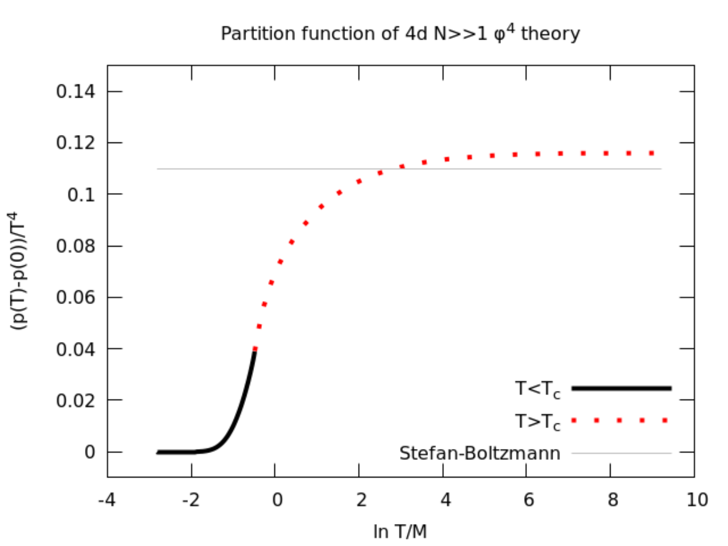
$$\lambda_0 \equiv \lambda\left(\mu = \frac{1}{a}\right) = \frac{(2\pi)^2}{\ln(M^2 a^2)}$$

- Continuum limit $a \to 0$ corresponds to $\lambda_0 \to 0^-$
- On anisotropic lattice $N_\sigma > N_\tau$,

$$T = \frac{1}{N_\tau a} = \frac{M}{N_\tau} e^{-\frac{2\pi^2}{\lambda_0}}$$

- Temperature increases as function of increasing λ_0
- Calculations at various λ_0 correspond to different values of T

What to expect – large N as a guide



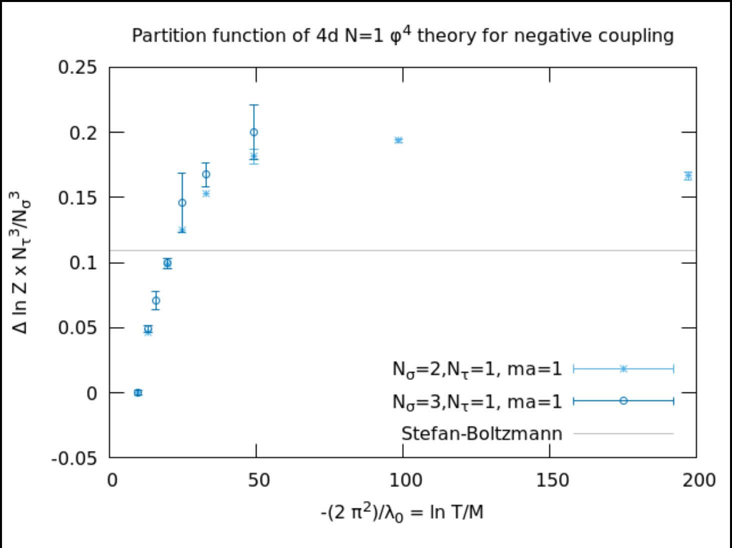
adapted from [2211.15683]

What to expect – large N as a guide

- Two phases, separated by 2^{nd} order transition
- High temperature phase overshoots SB limit and approaches from above

Direct simulation of N=1 ϕ^4 with $\lambda_0 < 0$

- Sign problem
- Brute force numerical integration of Z limited to tiny lattices
- Chose anisotropic lattices with $N_\tau = 1$
- Results for $m_0 a = 1$ for simplicity
- Normalize by low T (assume $\lambda_0 = -2$ is low enough T)
- Look at $\Delta \ln Z = \ln \frac{Z(\lambda_0)}{Z(\lambda_0 = -2)}$



adapted from [2305.05678]

Summary

- Triviality proofs for scalar ϕ^4 theory have loopholes, notably $\lambda_0 < 0$
- Negative coupling QM can be defined on the lattice on complex contours; path integral is convergent but has sign problem
- Direct numerical simulation of $\lambda_0 < 0$ QM match well-studied $\mathcal{PT}$-symmetric Quantum Mechanics
- For 4d field theory, large N results provide exciting non-perturbative guidance
- Direct numerical simulation of 4d lattice scalar field theory is possible on small lattices
- How to push calculations to larger lattices?

Bonus Material

References & Hyperlinks

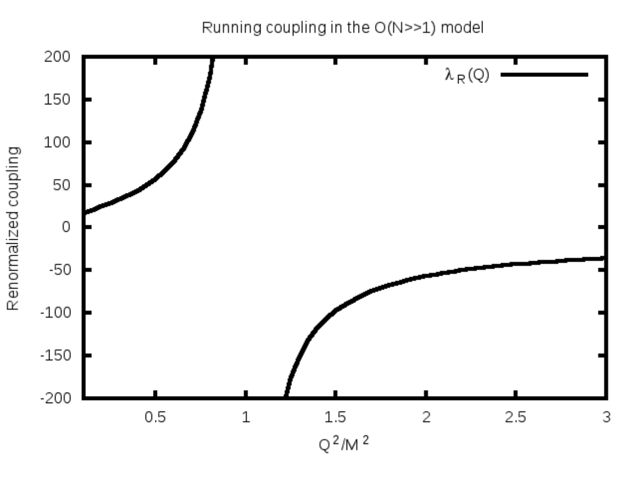
Continuum large N scalar field theory

- PR, “A solvable quantum field theory with asymptotic freedom in 3+1 dimensions”, hyperlink: [2211.15683]
- PR, “Life at the Landau pole”, [2212.03254]
- Grable and Weiner, “A Fully Solvable Model of Fermionic Interaction in 3+1d3+1d”, [2302.08603]
- PR, “What if ϕ^4 theory in 4 dimensions is non-trivial in the continuum?”, [2305.05678]

$\mathcal{PT}$-symmetric Quantum Mechanics and QFT relations

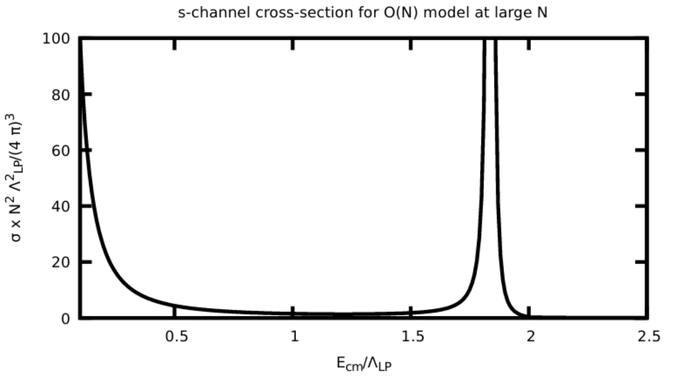
- Bender and Böttcher, “Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry”, [physics/9712001]
- Ai, Bender and Sarkar, “PT-symmetric -g ϕ^4 theory”, [2209.07897]
- Lawrence, Peterson, PR and Weller, “Instantons, analytic continuation, and PT-symmetric field theory”, [2303.01470]

Exact Running coupling in O(N) Model



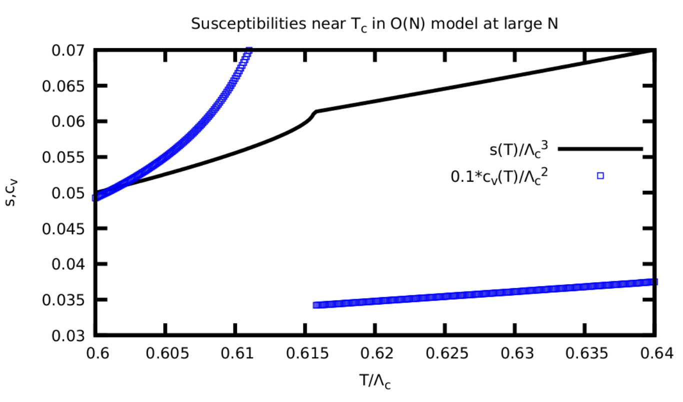
[2305.05678]

Scattering in O(N) model



[2305.05678]

Finite temperature O(N) model: susceptibilities



[2211.15683]