Simulating the lattice $SU(2)$ Hamiltonian with discrete manifolds

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Introduction and theoretical background

Hamiltonian simulations

Schrödinger equation $\rightarrow$ evolution of a system:

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle\,, \tag{1.1}$$

but the Hilbert space is often infinite dimensional...

- Truncation of the Hilbert space to a vector space $\mathcal{V}$ of size N
- Operators as matrices on $\mathcal{V}$
- Limit recovered when $N \rightarrow \infty$

Formalism suited for: **tensor networks, quantum devices**

Lattice formulation

Gauge invariance:

$$U_\mu(x) \to V(x) U_\mu(x) V^{-1}(x+\hat{\mu})\,. \qquad (1.2)$$

In the $A_0 = 0$ gauge we find:

$$H = \frac{g^2}{2}\sum_{x,i,a}(L_i)_a^2(x) - \frac{1}{4g^2}\sum_{x,i>j}\mathrm{Tr}[U_{ij}(x) + U_{ij}^\dagger(x)]\,,$$

$$L_a L_a = R_a R_a \qquad [L_a, U] = -\tau_a U \qquad [R_a, L_b] = U\tau_a$$

Representing the Hilbert space (I)

A basis for the Hilbert space are the Lie algebra irreps (**electric basis**):

$$|j, m, \mu\rangle \ , \ j \in \mathbb{N}/2 \ \ |m|, |\mu| < j$$

Clebsh-Gordan expansion

$$\begin{aligned} U^{(\alpha,\beta)}|J,m,\mu\rangle = \sum_{j\in\mathbb{N}/2} \sqrt{\frac{2J+1}{2j+1}} \langle J, m; \frac{1}{2}, \alpha|j, m+\alpha\rangle \\ \langle J, \mu; \frac{1}{2}, \beta|j, \mu+\beta\rangle \, |j, m+\alpha, \mu+\beta\rangle \, . \end{aligned} \tag{1.3}$$

This is all fine in an infinite dimensional space, but…

In a **finite** Hilbert space we have to give up something 🙁:

$$\mathrm{tr}[A, B] = 0$$

Representing the Hilbert space (II)

Clebsh-Gordan truncation

- Commutation relations ✅
- Gauss law invariance: $[H, G_a] = 0$ ✅

- Non unitary links ❌
- Need penalty term for $G_a|\psi\rangle = 0$ ❌

Unitary links (our approach)

- Commutation relations ❌
- Gauss law breaking: $[H, G_a] \neq 0$ ❌

- Unitary links ✅
- U as gates $\to$ initial state s.t. $G_a|\psi\rangle = 0$ ✅

Unitary links in the magnetic basis

- $(x, \mu) \to$ group manifold $\mathcal{M} = \{p_1, \ldots, p_N\}$. Diagonal links:

$$U = \text{diag}\left(\mathcal{U}(p_1), \ldots, \mathcal{U}(p_1)\right)\ ,\ p_i \in \mathcal{M}$$

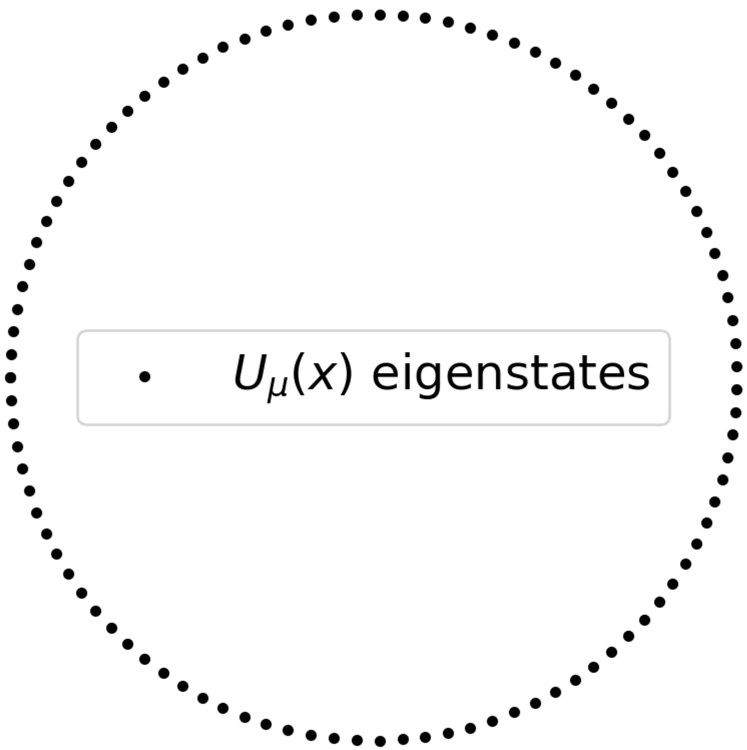
- Canonical momenta are Lie derivatives:

$$L_a f(U) = -i \frac{d}{d\epsilon} f(e^{i\epsilon\tau_a} U)\ ,\ R_a f(U) = -i \frac{d}{d\epsilon} f(U e^{i\epsilon\tau_a})$$

- **Note**: this is just like $p = -i\frac{d}{dx}$ in NRQM! $|\psi(x)\rangle$

U(1) theory

Continuum limit on the manifold



Points on C are a basis:

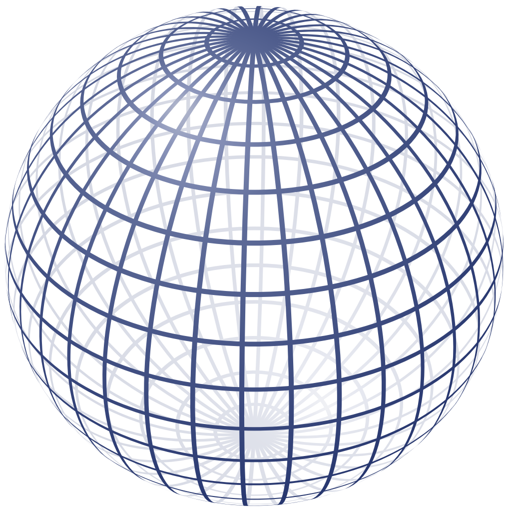
$$U|\alpha\rangle = e^{i\alpha}|\alpha\rangle$$

The momenta are simply (abelian group):

- $L_a = -i\frac{d}{d\omega}$
- $R_a = +i\frac{d}{d\omega}$

$SU(2)$ theory

Derivatives on S_3



Eigenfunctions on S_3 (Wigner D-functions):

$$D(\theta, \phi, \psi) = e^{im\phi} d^j_{m,\mu}(\theta) e^{i\mu\psi}$$

$$L_\pm = e^{\mp i\phi}\left[\pm\frac{1}{\sin\theta}\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\theta} \mp \cot\theta\frac{\partial}{\partial\phi}\right]$$

$$L_3 = -i\frac{\partial}{\partial\phi}$$

$su(2)$ irreducible representations

$$\left(\sum_a R_a^2\right)|j,m,\mu\rangle = \left(\sum_a L_a^2\right)|j,m,\mu\rangle = j(j+1)|j,m,\mu\rangle$$

$$L_3|j,m,\mu\rangle = m|j,m,\mu\rangle$$

$$R_3|j,m,\mu\rangle = -\mu|j,m,\mu\rangle$$

$$(L_1 \pm iL_2)|j,m,\mu\rangle = \sqrt{j(j+1)-m(m\pm 1)}|j,m\pm 1,\mu\rangle$$

$$(R_1 \mp iR_2)|j,m,\mu\rangle = -\sqrt{j(j+1)-\mu(\mu\pm 1)}|j,m,\mu\pm 1\rangle$$

Now fix a truncation: $j \leq q$. We have N_q states:

$$N_q = \sum_{j\leq q}(2j+1)^2 = \frac{1}{6}(4q+3)(2q+2)(2q+1) \sim O(q^3)$$

Question: How many eigenstates of U can I reproduce in the discrete space?

Truncated $su(2)$ irreps (example)

$$
L_3^{\text{el.}} = \sum_{j=0}^{q} \sum_{|m|\leq j} |j,m\rangle m \langle j,m|
$$

$$
\doteq \begin{bmatrix} & \cdots & \cdots & 0 \\ \vdots & \ddots & & \\ 0 & \cdots & \begin{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}_{\mu=1/2} & 0 \\ 0 & \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}_{\mu=-1/2} \end{bmatrix}_{j=1/2} & 0 \\ 0 & \cdots & 0 & [0]_{j=0} \end{bmatrix},
$$

Question: How many of these survive after discretizing the S_3?

Spoiler alert ⚠: It depends on the discretization (see e.g. M. Garofalo - Canonical Momenta in Digitized SU(2) Lattice Gauge Theory)

Frequencies on S_3

S_3 is a non-abelian manifold $\rightarrow N_\alpha$ points cannot sample N_α Fourier modes! (c.f. *Shannon-Nyquist theorem*)

$N_\alpha > N_q$ 😤

$$N_\alpha \geq \begin{cases} (q+1/2)(4q+1)^2 & q \, \text{half integer} \\ (q+1)(4q+1)^2 & q \, \text{integer} \end{cases}$$

Physical consequence:

- $U^\dagger U = UU^\dagger = 1 \implies \nexists$ square matrix V of change of basis between electric and magnetic basis.

Canonical momenta on S_3 partitionings (I)

- V is at most rectangular $\rightarrow$ enlarging the space of the first N_q $su(2)$ irreps.
- Presence of extra "garbage states" 🗑️

What is the form of V?

$$f(\vec{\alpha}_k) = f(\theta, \phi, \psi) = \sum_{j=0}^{q} \sum_{m,\mu=-j}^{j} V^j_{m,\mu}(\vec{\alpha}_k)\hat{f}(j, m, \mu)$$

Discrete Jacobi Transform

$$V^j_{m,\mu}(\vec{\alpha}_k) = (j+1/2)^{1/2}\sqrt{\frac{w_s}{N_\phi N_\psi}}D^j_{m,\mu}(\vec{\alpha}_k) \quad (1.4)$$

- w_s Gaussian weights of Legendre polynomials

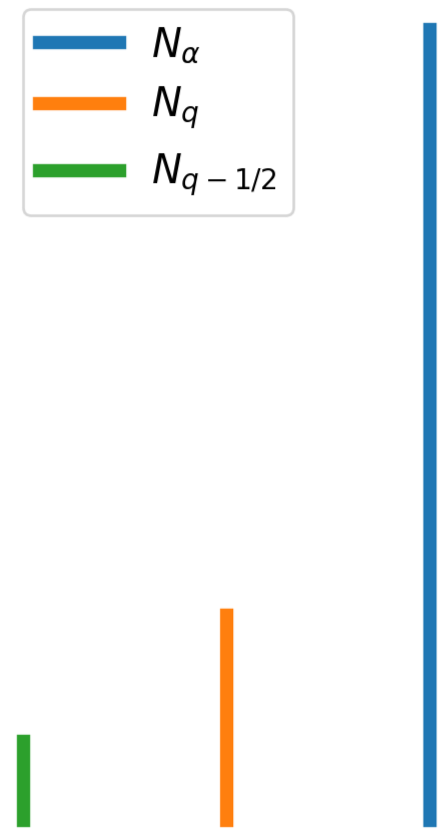
- V of size $N_\alpha \times N_q$
- $V^\dagger V = 1_{N_q \times N_q}$ (but not $VV^\dagger = 1_{N_\alpha \times N_\alpha}$)
- $\dim[\ker(V^\dagger)] = N_\alpha - N_q$

Properties of the discrete momenta

$$L_a = V\hat{L}_a V^\dagger \ , \ R_a = V\hat{R}_a V^\dagger$$

Properties:

- Exact Lie algebra: $i f_{abc}$ ✅
- First N_q eigenstates $|j, m, \mu\rangle$ reproduced exactly ✅
- Commutation relations fulfilled for the first $N_{q'} = N_{q-1/2}$ irreps ✅

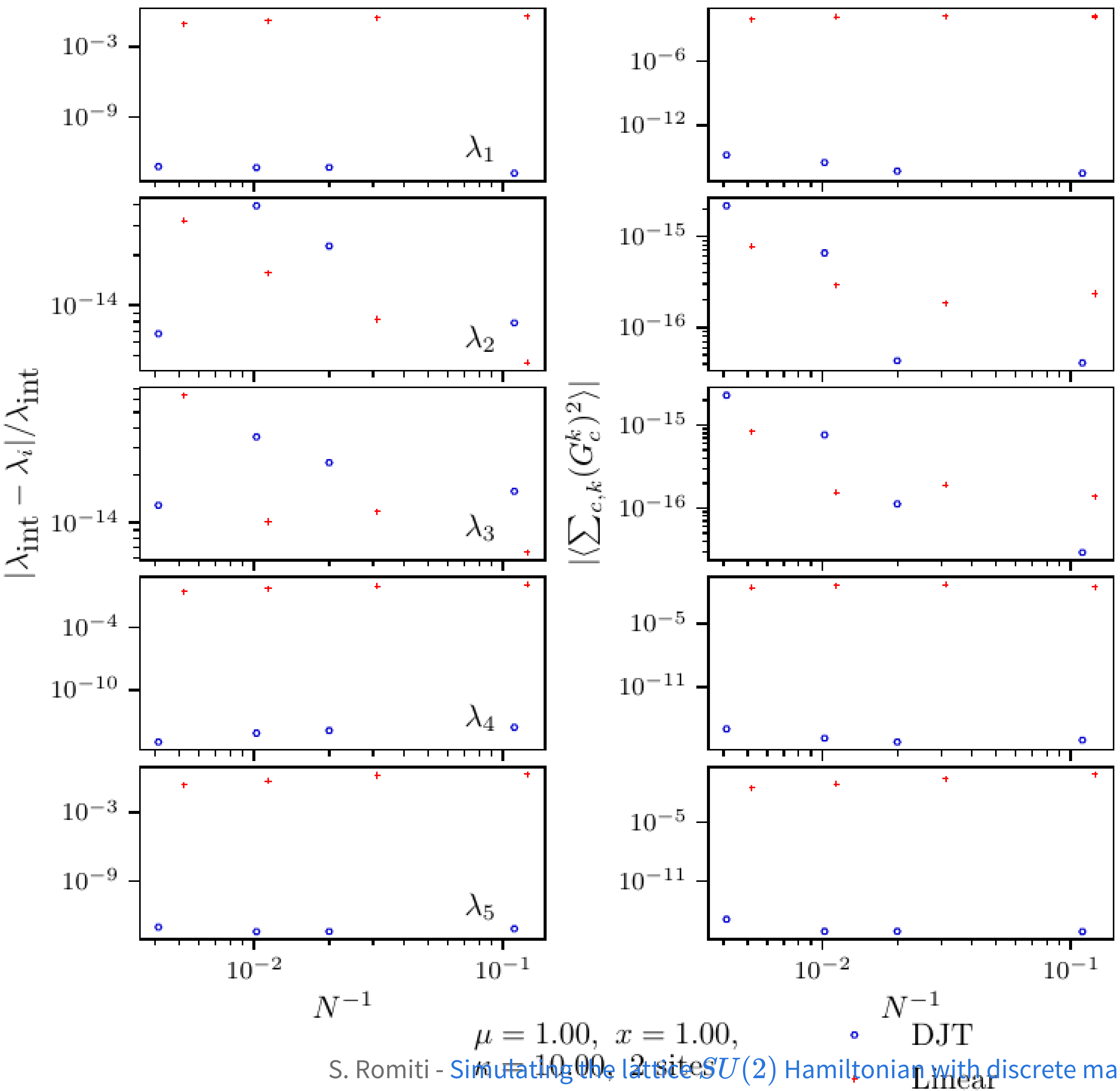


Vacuum and Gauss Law

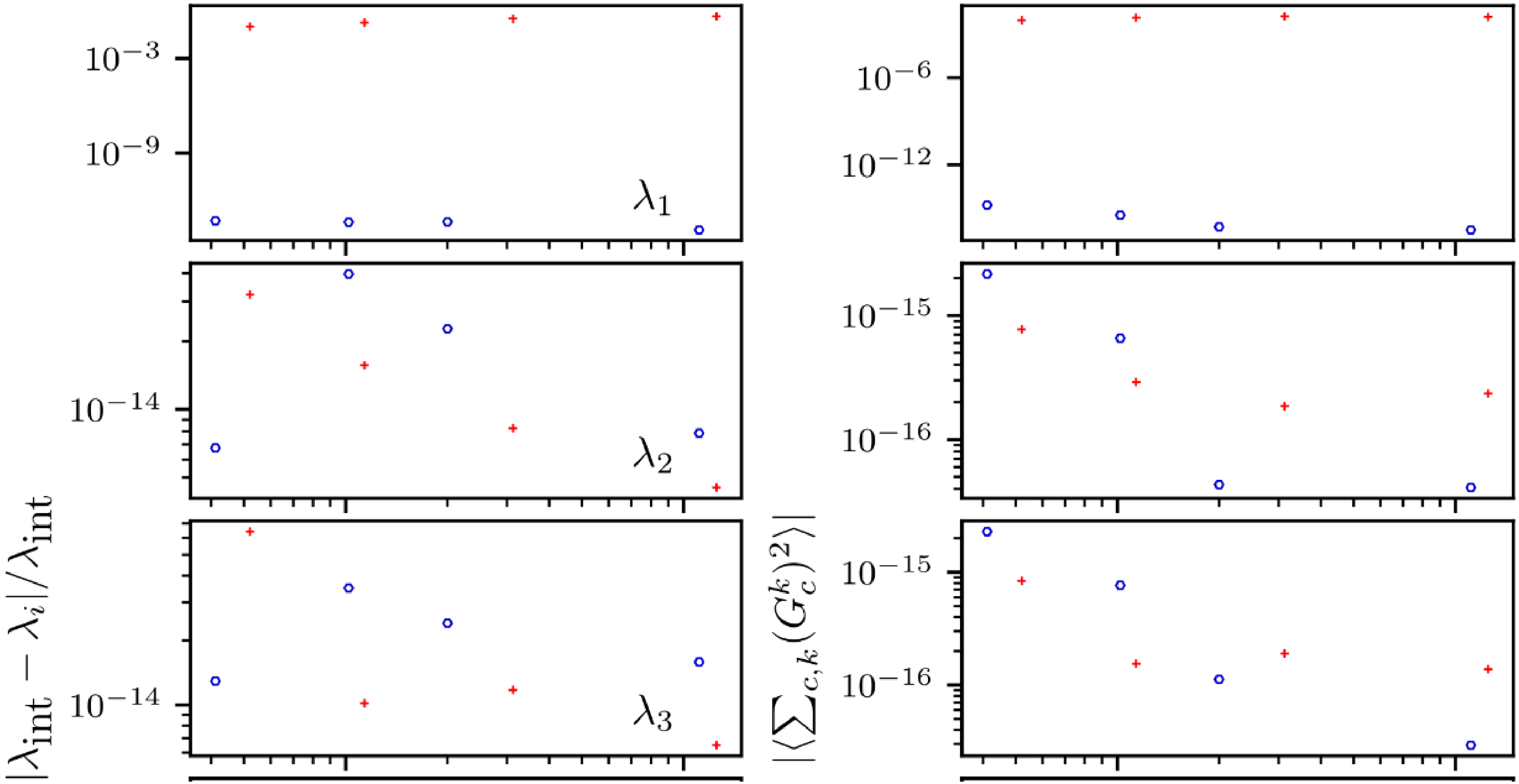
- Dense matrices for the momenta (local for $q \to \infty$) ❌
- $N_\alpha - N_q$ states degenerate with the electric vacuum ❌

  $\to$ lift with projector $P_{j>q} \to$ decoupled ✅
- $[G_a, H] \neq \vec{0}$ on $N_\alpha - N_{q'}$ states. ❌

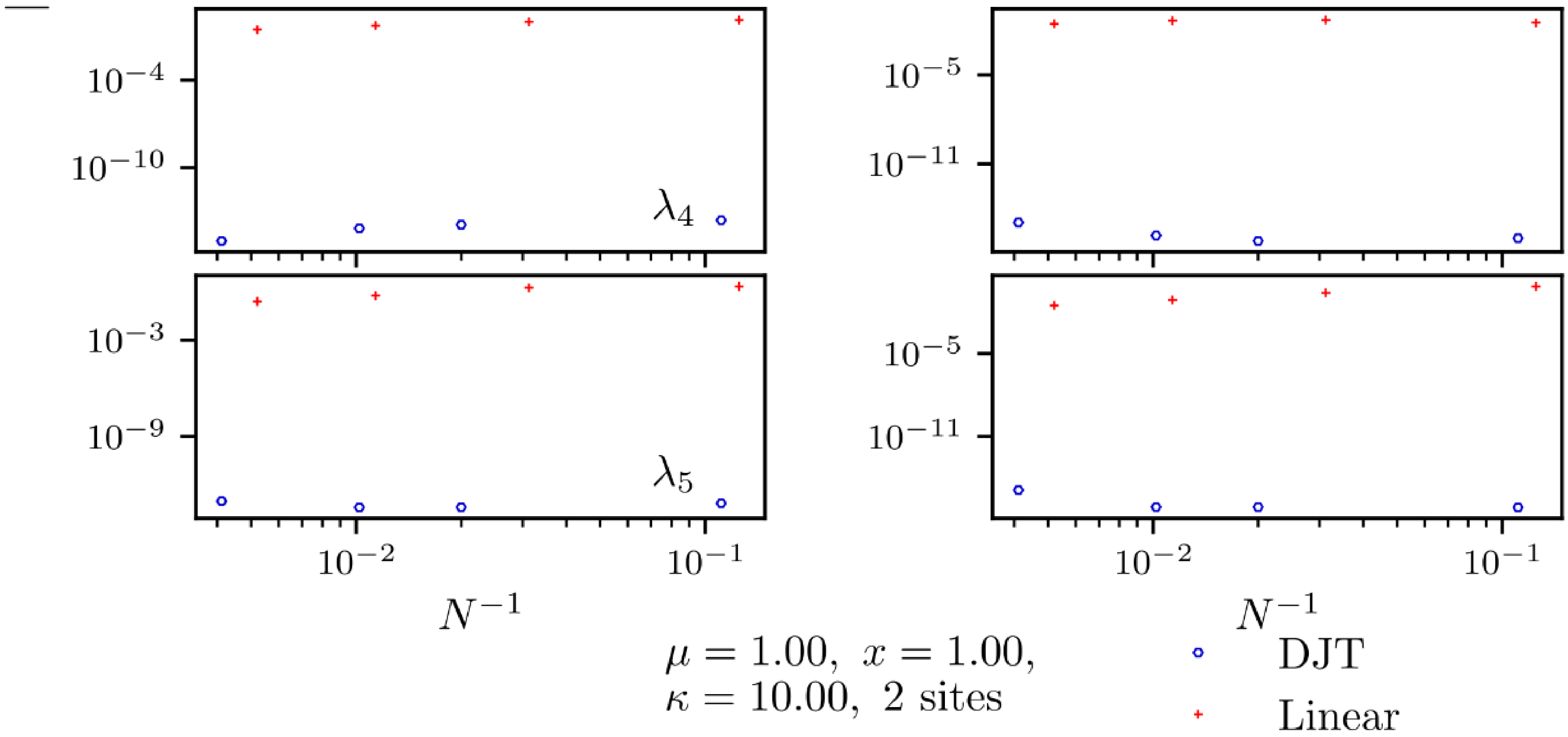
Preliminary results: $SU(2)$ in 1+1 dimensions



$|\lambda_{\rm int} - \lambda_i|/\lambda_{\rm int}$
$|\langle \sum_{c,k} (G_c^k)^2 \rangle|$
λ_1
λ_2
λ_3
λ_4
λ_5
N^{-1}
N^{-1}
$\mu = 1.00, \ x = 1.00,$
$\kappa = 10.00, \ 2$ sites
DJT
Linear



10^{-3}
10^{-9}
λ_1
10^{-14}
λ_2
10^{-14}
λ_3
$|\lambda_{\text{int}} - \lambda_i|/\lambda_{\text{int}}$
10^{-6}
10^{-12}
10^{-15}
10^{-16}
10^{-15}
10^{-16}
$|\langle\sum_{c,k}(G_c^k)^2\rangle|$



10^{-4}
10^{-10}
λ_4
10^{-3}
10^{-9}
λ_5
10^{-2}
10^{-1}
N^{-1}
10^{-5}
10^{-11}
10^{-5}
10^{-11}
10^{-2}
10^{-1}
N^{-1}
$\mu = 1.00,\ x = 1.00,$
$\kappa = 10.00,\ 2$ sites
DJT
Linear

Conclusion

- We can’t have both unitary links and exact commutation relations on all states:

$$[L_a, U] \text{ ✅ } \implies UU^\dagger = U^\dagger U = 1 \text{ ❌}$$

$$UU^\dagger = U^\dagger U = 1 \text{ ✅ } \implies [L_a, U] \text{ ❌}$$

- Both unitary and non-unitay links formulations deserve to be considered
- Unitary links limit the number of faithful represented electric eigenstates

- *Desirable feature*: being able to reduce the dimensionality of the space, e.g. constraining the values of the plaquette close to 1.

Thank you for your attention!

Backup

Truncated Clebsh-Gordan expansion when $a \to 0$ (I)

Approaching $a \to 0\ldots$

Truncated Clebsh-Gordan exapansion: U don't resemble group elements anymore

- Need to check $\exists$ critical point
- It is the same as the continuum theory?
  - what are the residual symmetries?
  - can we exclude nasty operators?

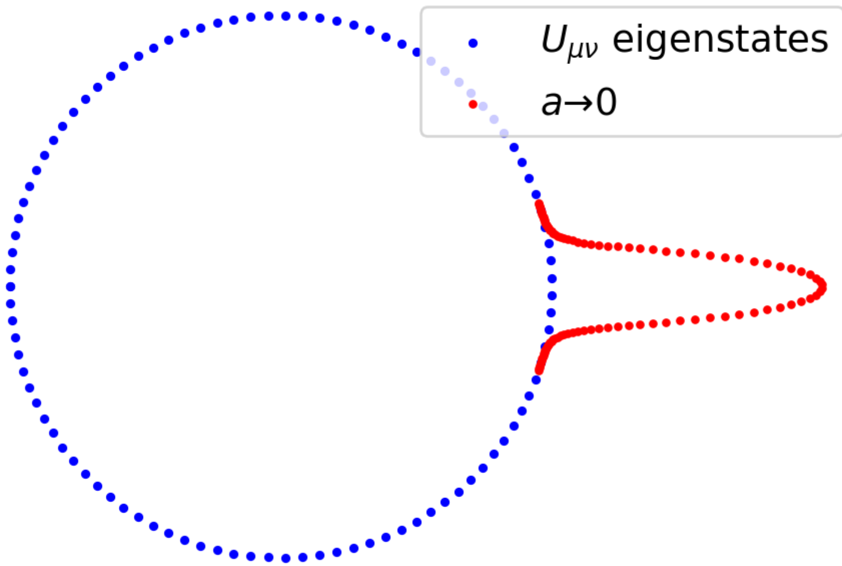
Discrete manifolds when $a \to 0$ (II)

Approaching $a \to 0 \ldots$

Unitary links: U take values in the manifold

- Same as Lagrangian simulation (finite machine precision $\to$ not exactly $SU(2)$)
- Need to check $\exists$ 2nd order phase transition at finite N (Monte Carlo with same partitioning)
- Check that it has the same step scaling function

$U(1)$ theory: $a \to 0$



$$U_{\mu\nu}|\alpha_{1\to4}\rangle = e^{i\alpha}|\alpha_{1\to4}\rangle$$

In the continuum limit $a \to 0$ the plaquette approaches 1:

$$U_{\mu\nu} = e^{ia^2 F_{\mu\nu}+O(a^3)} = 1 + ia^2 F$$

$\to$ restrict to the corresponding eigenstates gives an effective theory for fine lattices (*if correlation length fits the lattice*)