The lattice extraction of the TMD soft function using the auxiliary field representation of the Wilson line

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with

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TMD factorization

Drell-Yan factorization:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\,\mathrm{d}Y\,\mathrm{d}^2\vec{q}_\perp} = \sum_{i,j} H_{ij}(Q,\mu) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} B_i\left(x_a, \vec{b}_\perp, \mu, \frac{\zeta_a}{\nu^2}\right) B_j\left(x_b, \vec{b}_\perp, \mu, \frac{\zeta_b}{\nu^2}\right)$$
$$\times S_{ij}(b_\perp, \mu, \nu)\left[1 + \mathcal{O}\left(\frac{q_\perp^2}{Q^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{Q^2}\right)\right]$$

Soft function absorbed into definition of TMDPDF:

$$f_q^{\mathrm{TMD}}(x, \vec{b}_\perp, \mu, \zeta)$$
$$= \lim_{\substack{y_A \to +\infty \\ y_B \to -\infty}} Z_{\mathrm{UV}} B_q\left(x, \vec{b}_\perp, \epsilon, y_P - y_B\right) \sqrt{\frac{S(b_\perp, \epsilon, y_A - y_n)}{S(b_\perp, \epsilon, y_A - y_B)\, S(b_\perp, \epsilon, y_n - y_B)}}$$

[Collins 2011]

Collins-Soper kernel:

$$\gamma_\zeta^q(\mu, b_\perp) = 2\frac{\mathrm{d}\ln f_q^{\mathrm{TMD}}}{\mathrm{d}\ln\zeta} = \frac{\mathrm{d}\ln B_q}{\mathrm{d}y_P} = \frac{\mathrm{d}\ln S}{\mathrm{d}y_n}, \qquad y_n \to \text{rapidity parameter}$$

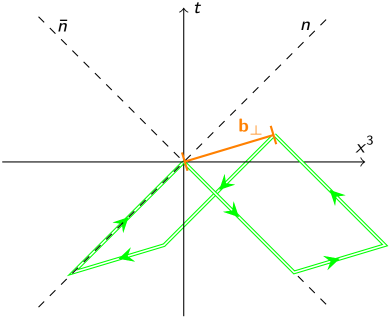
Soft function

Naive soft function:

$$S\left(b_\perp,\epsilon\right) = \frac{1}{N_c}\left\langle 0\right|\mathrm{Tr}\, S_n^\dagger(\vec{b}_\perp)S_{\bar{n}}(\vec{b}_\perp)S_\perp(-\infty\bar{n};\vec{b}_\perp,\vec{0}_\perp)S_{\bar{n}}^\dagger(\vec{0}_\perp)S_n(\vec{0}_\perp)S_\perp^\dagger(-\infty n;\vec{b}_\perp,\vec{0}_\perp)\left|0\right\rangle$$

Soft Wilson line:

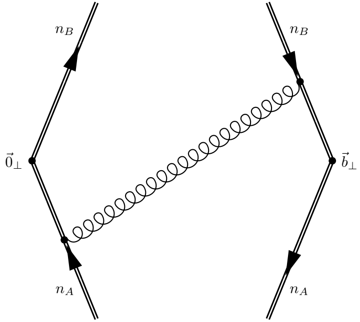
$$S_n\left(x\right) = P\exp\left\{-ig\int_{-\infty}^{0}\mathrm{d}s n^\mu A_\mu(x+sn)\right\}$$



Lightlike vectors:

$$n=(1,0,0,1),\qquad \bar{n}=(1,0,0,-1)$$

$$n^2=0,\qquad \bar{n}^2=0,\qquad n\cdot\bar{n}=2$$



Rapidity divergence:

$$\int \frac{\mathrm{d}k^+ \mathrm{d}k^-}{(2\pi)^2} \frac{1}{k^+k^- - k_\perp^2 + i0} \frac{1}{n\cdot k - i0} \frac{1}{\bar{n}\cdot k + i0}$$
$$= \int \frac{\mathrm{d}\alpha}{(2\pi)^2} \frac{1}{\alpha - k_\perp^2 + i0} \frac{1}{\alpha - i0} \int_{-\infty}^{\infty} \mathrm{d}y$$

$$k^- = n\cdot k, \qquad k^+ = \bar{n}\cdot k, \qquad k^\pm = k^0 \pm k^3$$

$$\alpha = k^+k^-, \quad y = \frac{1}{2}\ln\left(\frac{k^-}{k^+}\right)$$

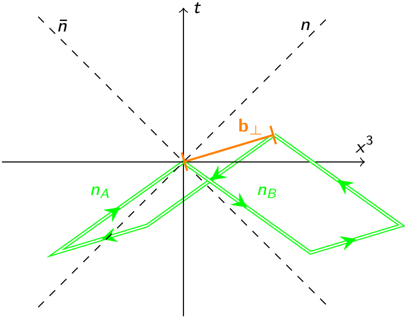
Divergence associated with rapidity

$$y \to \pm\infty$$

Off lightcone regulator

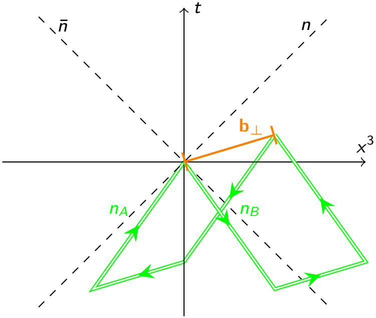
Spacelike Wilson lines:

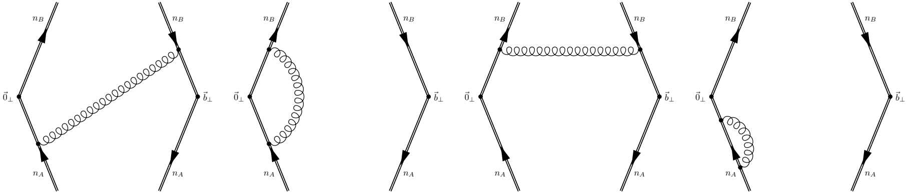
$$n_A \equiv n - e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} - e^{+y_B} n$$



Timelike Wilson lines:

$$n_A \equiv n + e^{-y_A}\bar{n},$$
$$n_B \equiv \bar{n} + e^{+y_B} n$$





A B C D

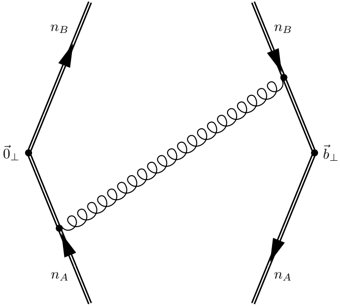
Diagram A:

$$S_{\mathrm{A}}(b_\perp, \epsilon, y_A, y_B) = g^2 C_F (n_A \cdot n_B) \mu_0^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-i\vec{b}_\perp \cdot \vec{k}_\perp}}{k^2 + i0} \frac{1}{n_A \cdot k - i0} \frac{1}{n_B \cdot k + i0}$$

$$= \frac{\alpha_s C_F}{2\pi} (y_A - y_B) \frac{1 + e^{2(y_B - y_A)}}{1 - e^{2(y_B - y_A)}} \left(-\frac{1}{\epsilon} - \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right) \right)$$

One loop result:

$$S(b_\perp, \epsilon, y_A, y_B) = 1 + \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right) \right) (2 - 2|y_A - y_B|) + \mathcal{O}(\alpha_s^2)$$



Define Euclidean space Wilson line directions as:

$$\tilde{n}_A = (ia_0, \vec{0}_\perp, a_3), \qquad \tilde{n}_B = (ib_0, \vec{0}_\perp, -b_3)$$

$$r_a \equiv \frac{a_3}{a_0}, \qquad r_b \equiv \frac{b_3}{b_0}$$

$$S_A^E(b_\perp, \epsilon, r_a, r_b) = g^2 C_F(\tilde{n}_A \cdot \tilde{n}_B) \int_{-\infty}^{0} \mathrm{d}s \int_{-\infty}^{0} \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2}$$

Try to write down Wilson line propagators:

$$\int_{-\infty}^{0} \mathrm{d}s e^{sa_0 k_4 - isa_3 k_3} = \left. \frac{e^{sa_0 k_4 - isa_3 k_3}}{a_0 k_4 - ia_3 k_3} \right|_{s=-\infty}^{0} \xrightarrow{k_4<0} \quad ?$$

Coordinate space

Perform integration in coordinate space:

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2} = \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A - t\tilde{n}_B)} \int_0^\infty \mathrm{d}u\, e^{-uk^2}$$

$$= \int_0^\infty \mathrm{d}u \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-uk^2} e^{-(b+s\tilde{n}_A - t\tilde{n}_B)^2/4u}$$

$$= \frac{\Gamma(d/2-1)}{(4\pi)^{d/2}} \frac{1}{((b+s\tilde{n}_A - t\tilde{n}_B)^2/4)^{d/2-1}}$$

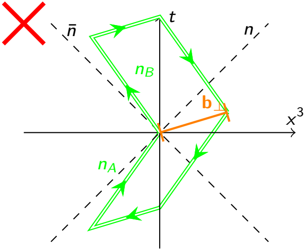
'u' integral only valid for

$$(s\tilde{n}_A - t\tilde{n}_B)^2 = s^2(a_3^2 - a_0^2) + t^2(b_3^2 - b_0^2) + st(a_3 b_3 + a_0 b_0) > 0$$

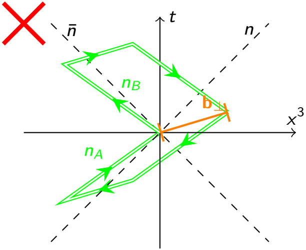
Euclidean space integral only finite for spacelike Wilson line directions:

$$a_3 > a_0, \quad b_3 > b_0, \quad a_3 b_3 + a_0 b_0 > 0 \qquad \rightarrow \qquad r_a > 1, \qquad r_b > 1$$

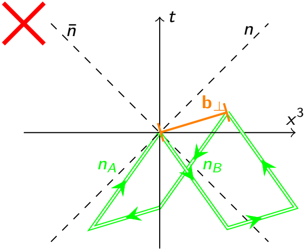
Wilson line directions



$a_3 < a_0\,, \quad b_3 < b_0\,, \quad a_3 b_3 + a_0 b_0 < 0$



$a_3 > a_0\,, \quad b_3 > b_0\,, \quad a_3 b_3 + a_0 b_0 < 0$



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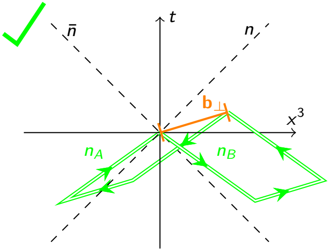

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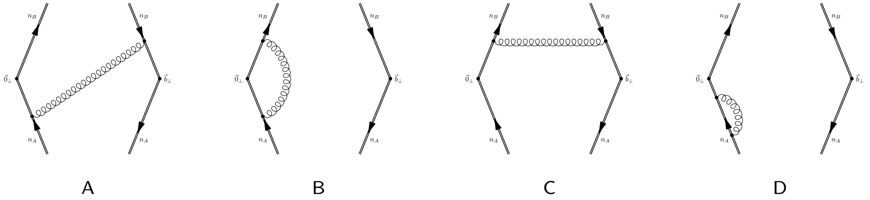

Calculation in configuration space at one loop ($r_a > 1, r_b > 1$):

$$S^{(1)}\left(b_\perp, \epsilon, r_a = \frac{a_3}{a_0}, r_b = \frac{b_3}{b_0}\right)$$
$$= \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 - \log\left(\left|\frac{(r_a-1)(r_b-1)}{(r_a+1)(r_b+1)}\right|\right)\frac{r_a r_b + 1}{r_a + r_b}\right\}$$
$$= \frac{\alpha_s C_F}{2\pi}\left(\frac{1}{\epsilon} + \ln\left(\pi b_\perp^2 \mu_0^2 e^{\gamma_E}\right)\right)\left\{2 - 2|y_A - y_B|\frac{1 + e^{2(y_B - y_A)}}{1 - e^{2(y_B - y_A)}}\right\}$$

Result is the same as the Collins soft function in Minkowski space, with:

$$e^{2y_A} = \left|\frac{r_a + 1}{r_a - 1}\right|, \quad e^{2y_B} = \left|\frac{r_b - 1}{r_b + 1}\right|$$

Auxiliary field definition of the Wilson line

Write Wilson line in terms of one dimensional 'fermions' that live along the path:

$$P \exp\left\{-ig \int_{s_i}^{s_f} \mathrm{d}s n^\mu A_\mu(y(s))\right\}$$
$$= Z_\psi^{-1} \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \psi\bar{\psi} \exp\left\{ig \int_{s_i}^{s_f} \mathrm{d}s \bar{\psi} i \partial_s \psi - \bar{\psi} n \cdot A \psi\right\}$$

[Gervais, Nevau 1980], [Aref'eva 1980]

Free auxiliary field propagator:

$$in \cdot \partial H_n(y) = \delta(y) \xrightarrow{\text{Euclidean space}} -i\tilde{n} \cdot \partial H_{\tilde{n}}(y) = \delta(y), \qquad \tilde{n} = (in_0, \vec{n})$$

[Mandula, Ogilvie 1992], [Aglietti, *et. al.* 1992]

For timelike n, we get the moving heavy quark effective theory (HQET).

(see [Ji, Liu, Liu 2020]).

Conclusion and outlook

- Euclidean space calculation of soft function can be analytically continued to Minkowski space result.
- Continuation only valid for spacelike Wilson lines that are both future pointing or both past pointing.
- Need matching relation to connect lattice calculation to continuum result.
- Currently setting up lattice computation.

Thank you!

Group members

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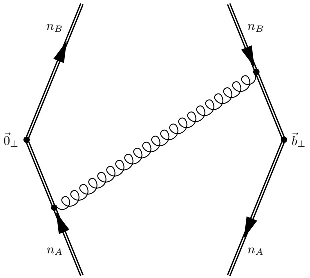
Backup slides

Momentum space



Define Euclidean space Wilson line directions as:

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$$r_a \equiv \frac{a_3}{a_0}, \qquad r_b \equiv \frac{b_3}{b_0}$$

Using finite length Wilson lines:

$$\int_{-L}^{0} \mathrm{d}s e^{-is\tilde{n}_A \cdot k} = \frac{1 - e^{L(a_0 k_4 - i a_3 k_3)}}{a_0 k_4 - i a_3 k_3}$$

Momentum space

$$
\begin{aligned}
&S_A^E(b_\perp, \epsilon, r_a, r_b) \\
&= g^2 C_F(\tilde{n}_A \cdot \tilde{n}_B) \int_{-L}^{0} \mathrm{d}s \int_{-L}^{0} \mathrm{d}t \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-ik(b+s\tilde{n}_A - t\tilde{n}_B)} \frac{1}{k^2} \\
&= g^2 C_F(\tilde{n}_A \cdot \tilde{n}_B) \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{-i\vec{k}_\perp \cdot \vec{b}_\perp} \frac{1 - e^{L(a_0 k_4 - i a_3 k_3)}}{a_0 k_4 - i a_3 k_3} \frac{1 - e^{L(-b_0 k_4 - i b_3 k_3)}}{b_0 k_4 + i b_3 k_3} \frac{1}{k_3^2 + k_4^2 + k_\perp^2}
\end{aligned}
$$

Peform contour integral on $k_3 = -i\sqrt{k_4^2 + k_\perp^2}$, lower half plane:

$$
\begin{aligned}
&-\frac{1}{2} g^2 C_F(\tilde{n}_A \cdot \tilde{n}_B) \int \frac{\mathrm{d}k_4 \mathrm{d}^{d-2} k_\perp}{(2\pi)^{d-1}} e^{-i\vec{k}_\perp \cdot \vec{b}_\perp} \\
&\times \frac{1 - e^{L(a_0 k_4 - a_3\sqrt{k_4^2 + k_\perp^2})}}{a_0 k_4 - a_3\sqrt{k_4^2 + k_\perp^2}} \frac{1 - e^{L(-b_0 k_4 - b_3\sqrt{k_4^2 + k_\perp^2})}}{b_0 k_4 + b_3\sqrt{k_4^2 + k_\perp^2}} \frac{1}{\sqrt{k_4^2 + k_\perp^2}}
\end{aligned}
$$

$L \to \infty \implies a_3 > a_0,\ b_3 > b_0,\ \text{and } a_3, a_0, b_3, b_0 > 0.$

Momentum space

Taking $L \to \infty$ (with $a_3 > a_0,\ b_3 > b_0$):

$$-\frac{1}{2}g^2 C_F(\tilde{n}_A \cdot \tilde{n}_B) \int \frac{\mathrm{d}k_4 \mathrm{d}^{d-2}k_\perp}{(2\pi)^{d-1}} e^{-i\vec{k}_\perp \cdot \vec{b}_\perp}$$
$$\times \frac{1}{a_0 k_4 - a_3\sqrt{k_4^2 + k_\perp^2}} \frac{1}{b_0 k_4 + b_3\sqrt{k_4^2 + k_\perp^2}} \frac{1}{\sqrt{k_4^2 + k_\perp^2}}$$

Solve with trigonometric substitution:

$$S_A^E(b_\perp, \epsilon, r_a, r_b) = -\frac{g^2 C_F}{4\pi^{2-\epsilon}} \frac{\left(b_T^2\right)^\epsilon \Gamma(1-\epsilon)}{2\epsilon} \frac{1}{2} \log\left(\frac{(r_a-1)(r_b-1)}{(r_a+1)(r_b+1)}\right) \frac{r_a r_b + 1}{(r_a + r_b)}$$