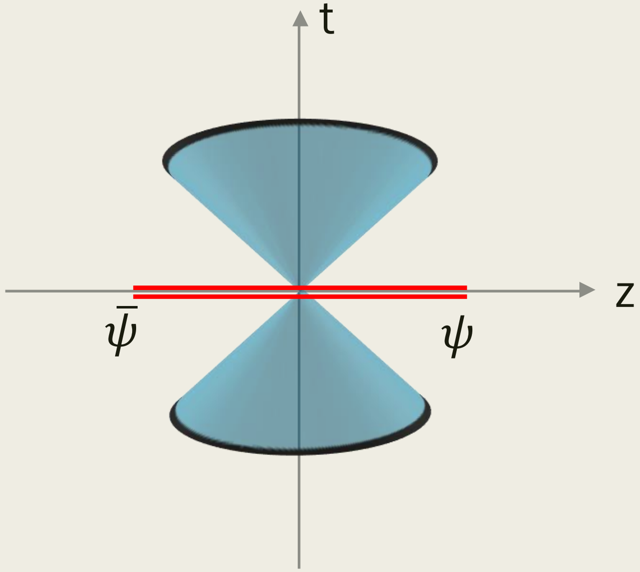
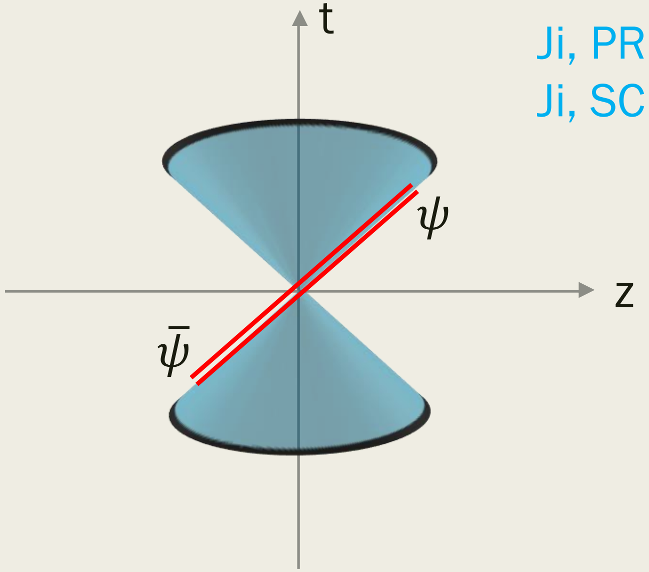
Leading Power Accuracy in Lattice Calculations of Parton Distributions

Speaker: Yushan Su

Rui Zhang, Jack Holligan, Xiangdong Ji and
Yushan Su. Phys.Lett.B 844 (2023), 138081

A brief overview





Ji, PRL 110 (2013), 262002
Ji, SCPMA 57 (2014), 1407-1412

Quasi-PDF $\tilde{f}(x, P_z) =$
$\int \frac{dz}{4\pi} e^{i\, z\, P_z\, x} \langle P_z | \bar{\psi}(z) U(z,0) \gamma^z \psi(0) | P_z \rangle$

Light cone PDF $f(y, \mu) =$
$\int \frac{d\xi^-}{4\pi} e^{-i\, \xi^- P^+ x} \langle P | \bar{\psi}(\xi^-) U(\xi^-,0) \gamma^+ \psi(0) | P \rangle$

$$\tilde{f}(x, P_z) = \int \frac{dy}{|y|} \mathcal{C}\left(\frac{x}{y}, \frac{\mu}{xP_z}\right) f(y, \mu) + \mathcal{O}\cancel{\left(\frac{\Lambda_{QCD}}{xP_z}\right)} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2}\right)$$

How to eliminate this leading power correction?

Braun, Vladimirov and Zhang, PRD 99 (2019) 1, 014013

Liu and Chen, PRD 104 (2021) 9, 094501

Jianhui Zhang, Thu 11:30 AM

Outline

1. OPE and power accuracy
2. Renormalon divergence
3. Numerical test on Pion PDF

OPE of quark bilinear operator

- Quark bilinear operator in a large momentum hadron on lattice

$$\tilde{h}(z,P_z) = \frac{1}{2P_t}\langle P_z|\bar{\psi}(z)U(z,0)\gamma^t\psi(0)|P_z\rangle$$

- OPE: a double expansion in z and α

Power expansion

Log expansion

$$\tilde{h}(z,P_z) = \boxed{\sum_{k=0} z^k}\boxed{\sum_{n=0} \alpha^n(\mu)\, c_{k,n}(\nu,\ln z^2\mu^2)}\otimes h_k(\nu\lambda,\mu)$$

Perturbative UV matching coefficients

Non-perturbative IR physics at $\mathcal{O}(\Lambda_{\mathrm{QCD}}^k)$

$\lambda = z\, P_z$

- Our goal: extract the non-perturbative parton physics for $k = 0$

Expansion accuracy

- Quark bilinear operator in a large momentum hadron on lattice

$$\tilde{h}(z,P_z)=\frac{1}{2P_t}\langle P_z|\bar{\psi}(z)U(z,0)\gamma^t\psi(0)|P_z\rangle$$

- The extraction of the parton physics $h(\lambda,\mu)$ $(\lambda=z\,P_z)$

$$\tilde{h}(z,P_z)=C(\nu,\alpha(\mu),z^2\mu^2)\otimes h(\nu\lambda,\mu)+O\left(z\,\Lambda_{\text{QCD}}\right)+O\left(z^2\Lambda_{\text{QCD}}^2\right)+\cdots$$

If the energy scale is high enough, e.g. $\mu\sim P_z\sim 20\text{ GeV}$ or $z\sim 0.01\text{ fm}$

α accuracy: $C(\nu,\alpha(\mu),z^2\mu^2)=\mathcal{O}(1)+\underset{\sim 0.15}{\mathcal{O}(\alpha(\mu))}+\underset{\sim 0.023}{\mathcal{O}(\alpha^2(\mu))}+\cdots$

power accuracy: $\underset{\sim 0.015}{\mathcal{O}\left(z\,\Lambda_{\text{QCD}}\right)}+\underset{\sim 0.00023}{\mathcal{O}\left(z^2\Lambda_{\text{QCD}}^2\right)}+\cdots$

The leading power accuracy

- Quark bilinear operator in a large momentum hadron on lattice

$$\tilde{h}(z,P_z) = \frac{1}{2P_t}\langle P_z|\bar{\psi}(z)U(z,0)\gamma^t\psi(0)|P_z\rangle$$

- The extraction of the parton physics $h(\lambda,\mu)$ $(\lambda = z\,P_z)$

$$\tilde{h}(z,P_z) = C(\nu,\alpha(\mu),z^2\mu^2)\otimes h(\nu\lambda,\mu) + O(z\,\Lambda_{\mathrm{QCD}}) + O(z^2\Lambda_{\mathrm{QCD}}^2) + \cdots$$

Under current lattice technique, $\mu\sim P_z\sim 2\ \mathrm{GeV}$ or $z\sim 0.1\ \mathrm{fm}$

α accuracy: $C(\nu,\alpha(\mu),z^2\mu^2) = \mathcal{O}(1) + \underset{\sim 0.30}{\mathcal{O}(\alpha(\mu))} + \underset{\sim 0.088}{\mathcal{O}(\alpha^2(\mu))} + \cdots$

power accuracy: $\underset{\sim 0.15}{\mathcal{O}(z\,\Lambda_{\mathrm{QCD}})} + \underset{\sim 0.023}{\mathcal{O}(z^2\Lambda_{\mathrm{QCD}}^2)} + \cdots$

- The leading power accuracy $\mathcal{O}(z\,\Lambda_{\mathrm{QCD}})$ is as important as $\mathcal{O}(\alpha(\mu))$ accuracy

A subtlety: the perturbation series is divergent

- Expand at the physical scale $1/z$

$$C = 1 + \sum_{n=1} c_n\, \alpha^n (1/z)$$

where $\alpha(\mu) = \frac{2\pi}{\beta_0 \ln\left(\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)}$

- At high orders,

$$c_n \sim \left(\frac{\beta_0}{2\pi}\right)^n n! \sim \left(\frac{\beta_0}{2\pi}\right)^n e^{n \ln n - n}$$

from bubble chain of Wilson link self energy

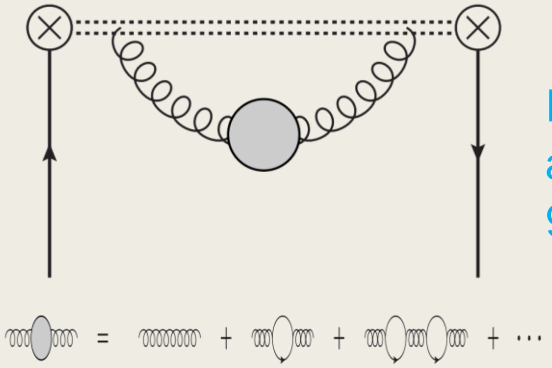
- Truncate near the minimum term

$$\frac{\mathrm{d}\left(\frac{\beta_0}{2\pi}\right)^n e^{n \ln n - n} \alpha^n}{\mathrm{d}\, n} = 0 \;\Rightarrow\; n = \frac{2\pi}{\alpha\, \beta_0}$$

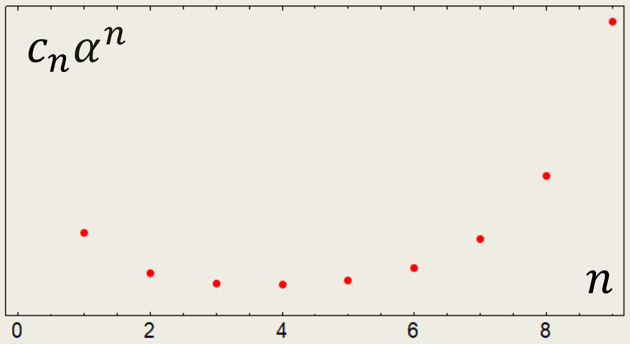
- The uncertainty of the truncation

$$\mathrm{c}_n \alpha^n|_{n \to \frac{2\pi}{\alpha\, \beta_0}} = e^{-\frac{2\pi}{\alpha\, \beta_0}} = z\, \Lambda_{\mathrm{QCD}}$$

Beneke, Phys.Rept. 317 (1999) 1-142



Braun, Vladimirov and Zhang, PRD 99 (2019) 1, 014013



Schematic diagram

The same order as the leading power correction

Introduce a parameter $m_0(\tau)$

- No clear boundary between $c_n(\nu, \ln z^2\mu^2)$ and $O(z\,\Lambda_{\text{QCD}})$

$$\tilde{h}(z, P_z) = \boxed{\sum_{n=0} \alpha^n(\mu)\, c_n(\nu, \ln z^2\mu^2)} \otimes h(\nu\lambda, \mu) + O(z\,\Lambda_{\text{QCD}})$$

Uncertainty in regulating this divergent series $\mathcal{O}(z\,\Lambda_{\text{QCD}})$

- A way to regulate the divergent series is called a summation scheme τ
- The leading power correction $m_0(\tau)z$ depends on τ

$$\tilde{h}(z, P_z) = \sum_{n=0} \alpha^n(\mu)\, c_n(\nu, \ln z^2\mu^2, \tau) \otimes h(\nu\lambda, \mu) + m_0(\tau)z$$

Ayala, Lobregat and Pineda, PRD 99 (2019) 7, 074019

- Knowing $\tilde{h}(z, P_z)$ and $c_n(\nu, \ln z^2\mu^2, \tau)$, one can extract $m_0(\tau)$
- Make predictions with $m_0(\tau)$. **Is $\mathbf{m_0(\tau)}$ a universal parameter?**

$$\tilde{h}'(z, P_z) = \sum_{n=0} \alpha^n(\mu) c_n'(\nu, \ln z^2\mu^2, \tau) \otimes h'(\nu\lambda, \mu) + m_0(\tau)z$$

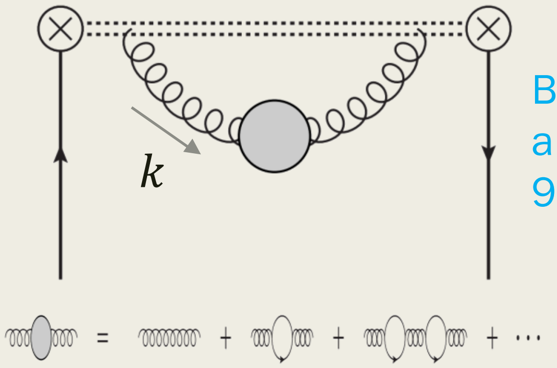
Physics of the divergent series

- Some low momentum modes in the perturbation theory lead to a divergent series,

  e.g. Wilson link self energy with n bubbles

  $$\sim \int_0 d^d k\, (-1)^n \ln^n(-k^2) \ \sim n!$$

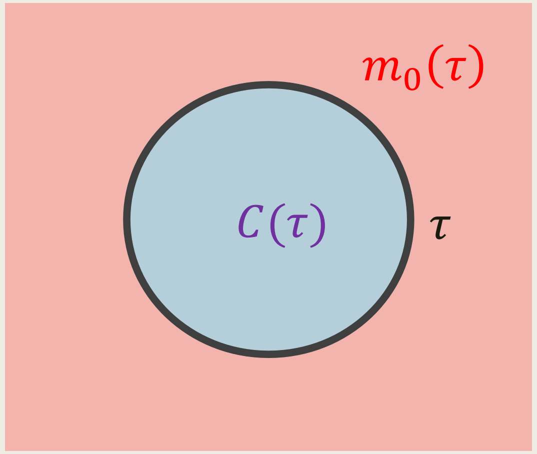
- Truncating the divergent series is to regulate the these low momentum modes



Braun, Vladimirov and Zhang, PRD 99 (2019) 1, 014013

- Coordinate space picture: long distance physics leads to a divergent series
- Truncating the divergent series is to regulate the long distance physics
- Ambiguity in choosing τ. No ambiguity in the full physics up to $\mathcal{O}(z\,\Lambda_{\mathrm{QCD}})$

Coordinate Space



τ is the summation scheme, intuitively understood as "the boundary" between UV and IR

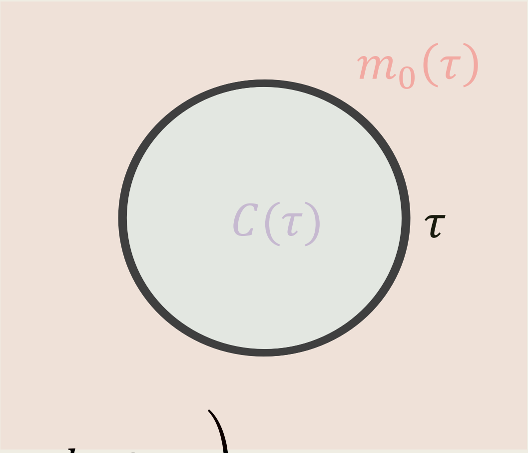
Perturbative universality

- A universality class of the perturbative τ dependence

  Linear divergence series: $\delta m(a) = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}(1/a)$

  Pole mass series: $m_{\rm OS} - m_{\overline{MS}} = \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}$

  LaMET pert for 1D objects: $\tilde{h}^R_{\rm pert} = 1 + \alpha_s c_1 + \alpha_s^2 c_2 + \alpha_s^3 c_3 + \cdots$



- Leading renormalon series Beneke, PLB 344 (1995) 341-347

$$r_n = N_m \mu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]}\left(1 + \frac{b}{n+b} b_1 + \frac{b(b-1)}{(n+b)(n+b-1)} b_2 + \cdots\right)$$

- N_m determination Bali, Bauer and Pineda, PRD 87 (2013) 094517

Linear divergence $\xrightarrow{\text{NSPT, } O[\alpha^{20}],\ n_f = 0}$ $N_m^{\rm latt} = 19.0(16),\ N_m^{\overline{\rm MS}} = \frac{N_m^{\rm latt}\Lambda_{\rm latt}}{\Lambda_{\overline{\rm MS}}} = \boxed{0.660(56)}$

Pole mass series $\xrightarrow{\text{Ana, } O[\alpha^{3}],\ n_f = 0}$ $N_m^{\overline{\rm MS}} = \boxed{0.622(23)}$

Consistent

$\xrightarrow{n_f = 3}$ $N_m^{\overline{\rm MS}} = 0.575(13)$ Pineda, JHEP 06 (2001) 022

LaMET pert $\longleftarrow$ Apply?

The leading power correction

Ishikawa, Ma, Qiu and Yoshida, arXiv:1609.02018
Chen, Ji and Zhang, NPB 915, 1 (2017)
Constantinou and Panagopoulos, arXiv:1705.11193
Alexandrou et al., NPB 923 (2017)
Chen et al., PRD 97 (2018)
Musch, Hagler, Negele and Schafer, PRD 83 (2011)
Ji, Zhang and Zhao, PRL 120 (2018)

- Ambiguity during the Mass renormalization

$$\tilde{h}^R(\lambda, P_z)_\tau \sim \tilde{h}^B(\lambda, P_z) e^{(\delta m(a) + m_0(\tau))z}$$

$$\Big\Downarrow \quad \text{Fourier transform}$$

$$\tilde{f}(x, P_z)_\tau \sim \tilde{f}(x, P_z)_{\tau'} + \mathcal{O}\left(\frac{\Delta m_0}{x P_z}\right)$$

Ji et al., NPB 964 (2021)

- Ambiguity of perturbative matching kernel

$$\mathcal{C}\left(\xi, \frac{\mu}{x P_z}\right)_\tau \sim \mathcal{C}\left(\xi, \frac{\mu}{x P_z}\right)_{\tau'} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{x P_z}\right)$$

- Under **the same scheme τ**, the leading power correction vanishes

$$\tilde{f}(x, P_z)_\tau = \int \frac{dy}{|y|} \mathcal{C}\left(\frac{x}{y}, \frac{\mu}{x P_z}\right)_\tau f(y, \mu) + \cancel{\mathcal{O}\left(\frac{\Lambda_{QCD}}{x P_z}\right)} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2}\right)$$

Lattice data and renormalization

Gao et al., PRD (2020)
Gao et al., PRD (2021)
Gao et al., PRL (2022)

- Pion valence PDF matrix element from BNL/ANL collaboration:

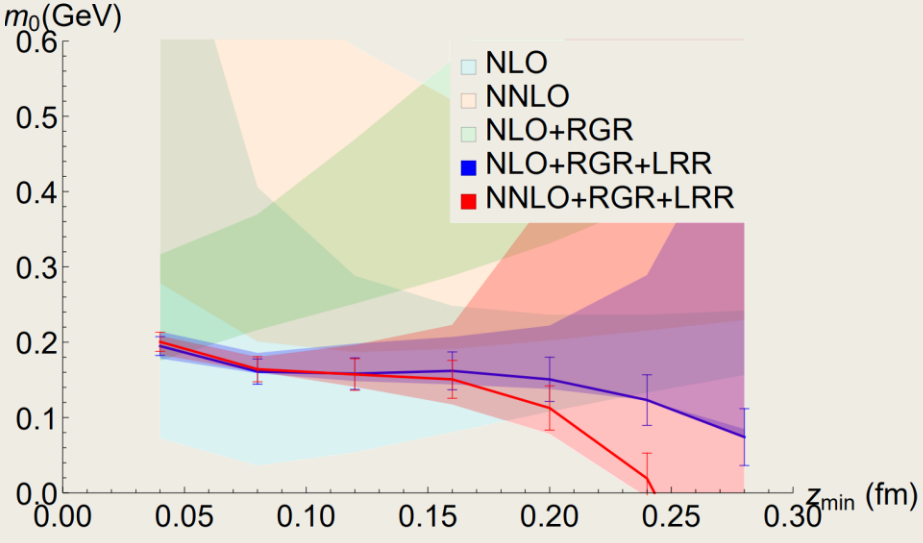
$$h^{\text{lat}}(z,a,P_z)=\langle\pi^+(P_z)|\bar{u}(z)\gamma^t U(z,0)u(0)-\bar{d}(z)\gamma^t U(z,0)d(0)|\pi^+(P_z)\rangle$$

- Hybrid renormalized matrix element:

$$\tilde{h}^R(z,P_z)=\frac{\tilde{h}^{\text{lat}}(z,a,P_z)}{\tilde{h}^{\text{lat}}(z,a,0)}\theta(z_s-|z|)+\frac{\tilde{h}^{\text{lat}}(z,a,P_z)}{Z^R(z,a,\mu)\tilde{h}^{\overline{\text{MS}}}(z_s,\mu,0)}\theta(|z|-z_s)$$

where $Z^R(z,a,\mu)\sim e^{-(\delta m(a)+m_0)z}$.

$m_0(\tau)$ is determined through requiring $\frac{\tilde{h}^{\text{lat}}(z,a,P_z=0)}{Z^R(z,a,\mu)}=\tilde{h}^{\overline{\text{MS}}}(z,\mu,P_z=0)$ for $a<z\ll 1/\Lambda_{\text{QCD}}$



Fit range: $[z_{\text{min}}, z_{\text{min}}+0.08\text{fm}]$

NLO NNLO

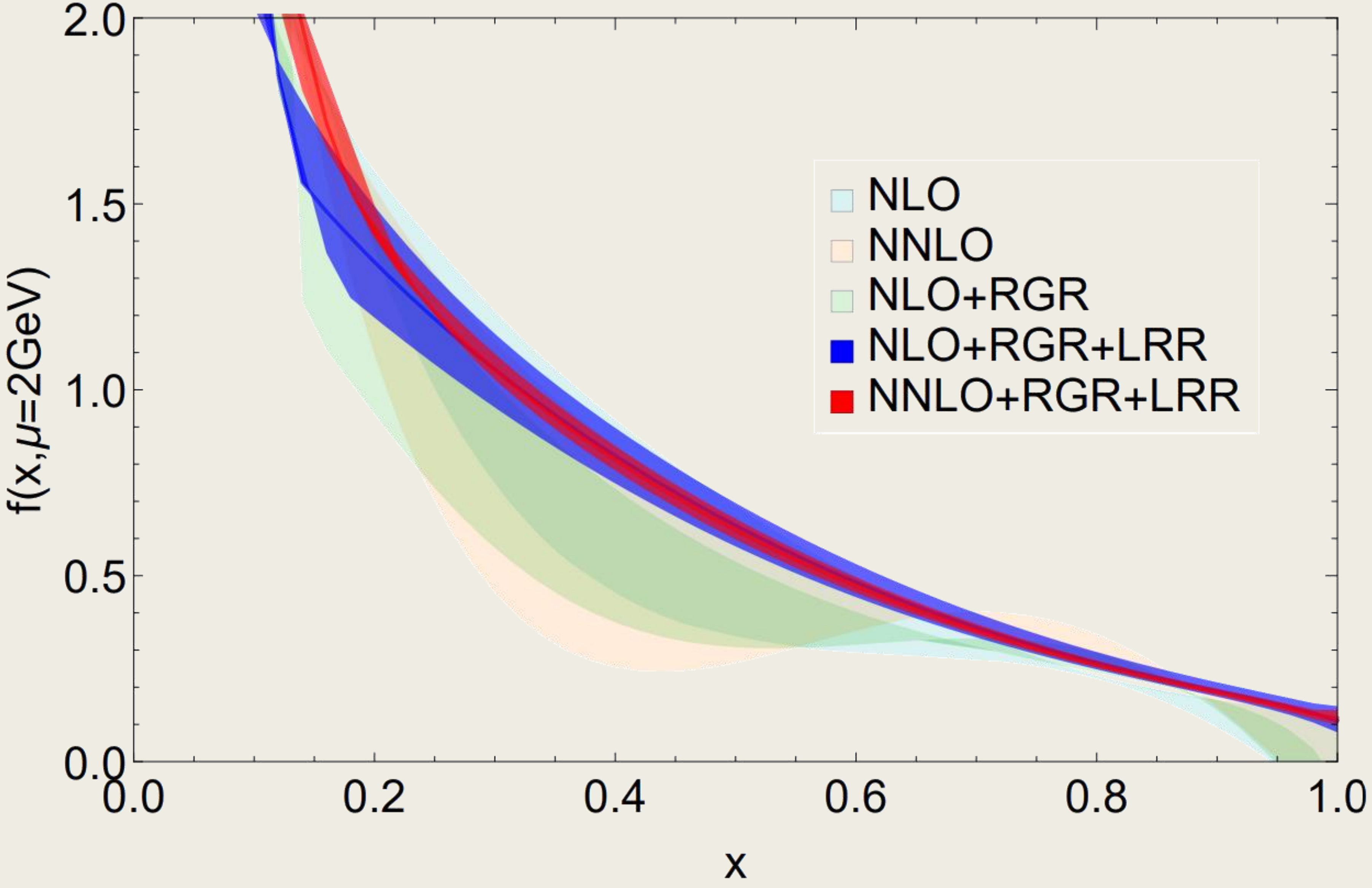
$$\tilde{h}^{\overline{\text{MS}}}=\Big(\boxed{1+\alpha_s c_1^c+\alpha_s^2 c_2^c+}\,\alpha_s^3 c_3^c+\cdots$$
$$\boxed{+\alpha_s c_1^{\text{LR}}+\alpha_s^2 c_2^{\text{LR}}+\alpha_s^3 c_3^{\text{LR}}+\cdots\Big)}\boxed{\text{Exp}\Big[\int d\alpha_s\frac{\gamma}{\beta}\Big]}$$

NLO NNLO LRR: RGR

Asymptotic Form to $O[1/n^2]$

τ scheme: Borel sum with P.V.

LRR: Matching with leading power correction



Take home messages

- Under current lattice technique (scale $\sim 2\ \mathrm{GeV}$), the leading power accuracy $\mathcal{O}\left(z\,\Lambda_{\mathrm{QCD}}\right)$ is as important as $\mathcal{O}(\alpha(\mu))$

- The definition of the leading power correction depends on the summation scheme τ of the lead twist perturbation series

- After the leading power accuracy is controlled, the precisions of PDF are dramatically improved. The methods in our paper can be generalized to other LaMET 1D objects, such as GPD and DA.

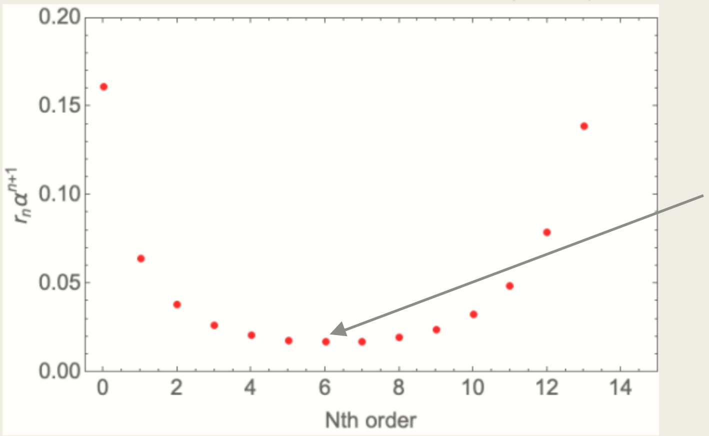
Appendix

Renormalon divergence

- $R = \sum_{n=0} r_n\, \alpha_s^{n+1}$
- At high orders,

  $r_n \sim n!,$

  series is divergent for any α_s

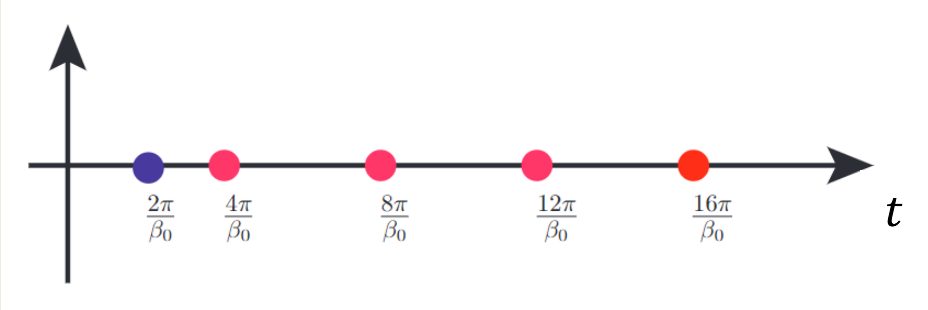


At minimum term: $n \sim 1/\alpha_s$, which comes from $\frac{\mathrm{d}\, n!\alpha_s^{n+1}}{\mathrm{d}\, n} = 0$

Data from Bali, et al. PRD(2013)

- Borel transformation, $B[t] = \sum_n \frac{r_n}{n!} t^n$

  Renormalon divergences correspond to singularity poles on Borel Plane



M. Beneke, Phys.Rept. 317 (1999) 1-142

Summation Schemes and Renormalon Ambiguity

- Borel Sum

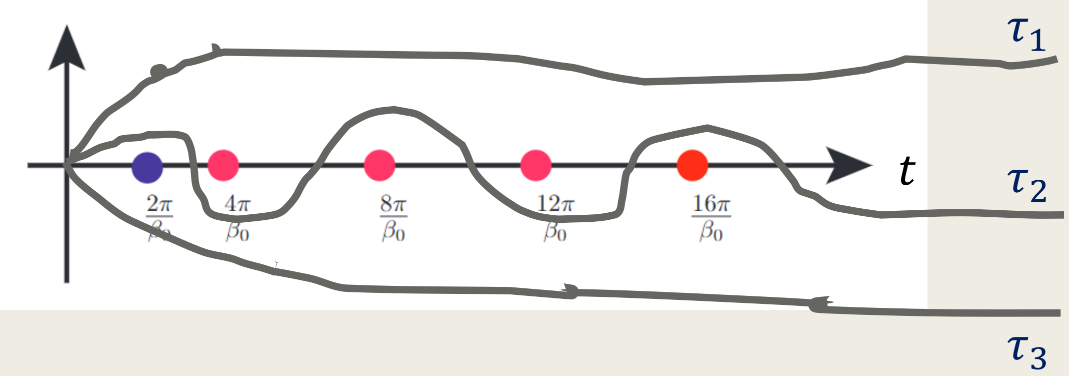
$$\tilde{R} = \int_0^{+\infty} dt\, e^{-\frac{t}{\alpha_s}} B[t]$$

which depends on the integral path

- To regularize the renormalon divergence, just choose an integral path that leads to a finite result

- Each integral path in Borel sum corresponds to a summation scheme τ

M. Beneke, Phys.Rept. 317 (1999) 1-142



- The resulting difference between summation schemes is called renormalon ambiguity, which can be estimated as the residues of the poles on Borel plane.
- e.g. the leading pole:

$$\Delta[\delta m(a)] \sim \mathcal{O}(\Lambda_{\mathrm{QCD}})$$
$$\Delta[Z(v, z^2\mu^2)] \sim \mathcal{O}(z\Lambda_{\mathrm{QCD}})$$
$$\Delta\left[\mathcal{C}\left(\xi, \frac{\mu}{xP_z}\right)\right] \sim \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{xP_z}\right)$$

Physical observables are renormalon free

Ayala et al., PRD 99 (2019) 7, 074019

- The QCD physical observables (Suppose we measure them in a hadron state) should be independent of summation scheme τ

$$\langle O_1(Q^2)\rangle = C_1(Q^2,\tau) + \frac{\langle O_G\rangle(\tau)}{Q^2} + \cdots$$

Pert Nonpert

- The universality of summation scheme dependence

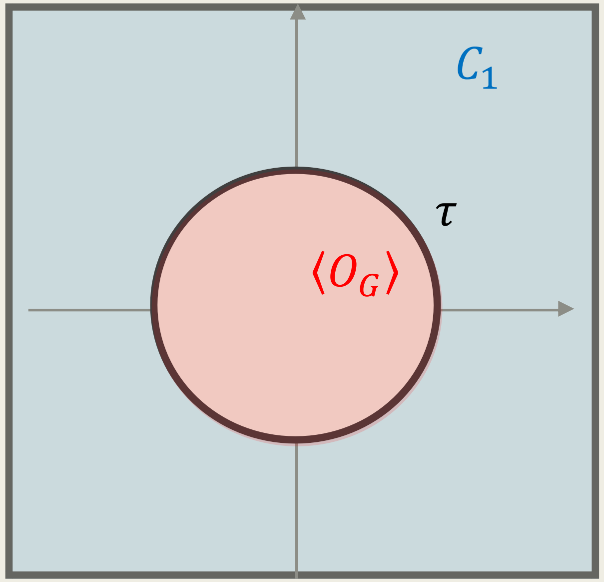
$$\langle O_1(Q^2)\rangle = C_1(Q^2,\tau) + \frac{\langle O_G\rangle(\tau)}{Q^2} + \cdots$$

$$\langle O_2(Q^2)\rangle = C_2(Q^2,\tau) + \frac{\langle O_G\rangle(\tau)}{Q^2} + \cdots$$

$$\langle O_3(Q^2)\rangle = C_3(Q^2,\tau) + \frac{\langle O_G\rangle(\tau)}{Q^2} + \cdots$$

…

Momentum Space

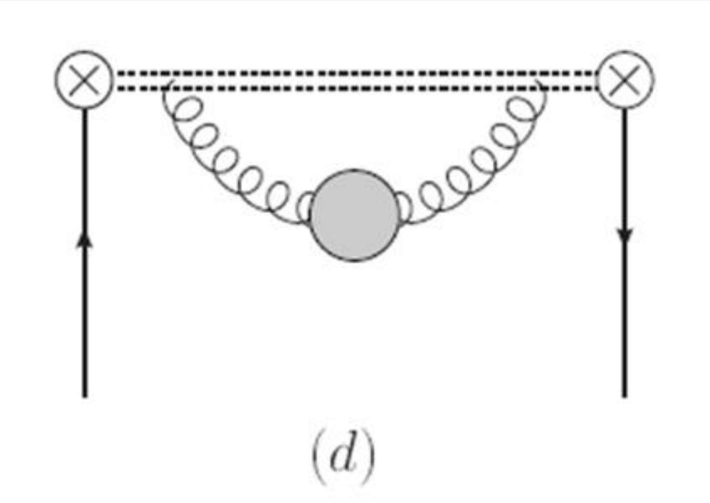


The origin of the leading renormalon in LaMET

- The non-local quark bilinear operator with a Wilson link

$$\hat{O}(z) = \bar{\psi}_1(z)U(z,0)\Gamma\psi_2(0)$$

- The leading renormalon comes from the self energy of the Wilson link (or as a subdiagram)



- During the renormalization

$$\widetilde{H}^R(\lambda, P_z) \sim \widetilde{H}^B(\lambda, P_z)e^{(\delta m - m_0)z}$$

where m_0 is a nonperturbative mass which may be mixed with the leading renormalon ambiguity of δm and $\widetilde{H}^R_{\text{pert}}$

$$\Delta[\delta m(a)] \sim \mathcal{O}(\Lambda_{\text{QCD}})$$

$$\Delta\left[\widetilde{H}^R_{\text{pert}}\right] \sim \mathcal{O}\left(z\Lambda_{\text{QCD}}\right)$$

G. Bali et al., PRD 87 (2013) 094517

- In the matching kernel

$$\mathcal{C}\left(\xi, \frac{\mu}{xP_z}\right) \sim \mathcal{C}\left(\xi, \frac{\mu}{xP_z}\right)_\tau + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right)$$

where $\mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right)$ denotes the leading renormalon ambiguity of the matching kernel

V. Braun et al., PRD 99 (2019) 1, 014013

Twist-three accuracy

- The bare matrix element $\widetilde{H}^B(z, P_z, a) = \langle P_z | \bar{\psi}(z) U(z,0) \gamma^t \psi(0) | P_z \rangle$
- The renormalized matrix element

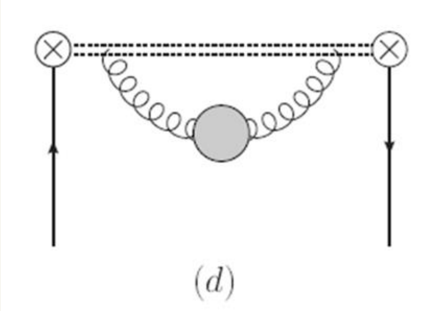
$$\widetilde{H}^R(z, P_z, \mu, \tau) = e^{-m_0(\tau)\, z} \sum_{k=0} C_k(\alpha(\mu), z^2\mu^2, \tau) \frac{\lambda^k}{k!} \langle x^k \rangle(\mu) + O(z^2\Lambda_{\mathrm{QCD}}^2)$$

$$= \sum_{k=0} [C_k(\alpha(\mu), z^2\mu^2, \tau) - m_0(\tau)\, z] \frac{\lambda^k}{k!} \langle x^k \rangle(\mu) + O(z\Lambda_{\mathrm{QCD}}\alpha(\mu), z^2\Lambda_{\mathrm{QCD}}^2)$$

- $m_0(\tau)$ is a twist-three non-perturbative parameter during the renormalization

  The leading renormalon ambiguity in the $C_k(\alpha(\mu), z^2\mu^2, \tau)$ is independent of k

The Leading Power Accuracy



Regularize the leading renormalon under the same summation scheme τ for:

Perturbative Matrix Element: $\widetilde{H}^R_{\text{pert}}(\lambda, P_z)_\tau$

Perturbative Matching Kernel: $\mathcal{C}\left(\xi, \frac{\mu}{xP_z}\right)_\tau$

$$\frac{\text{F. T.}\,[\widetilde{H}^R_{\text{pert}}(\lambda, P_z)_\tau]}{\mathcal{C}\left(\xi, \frac{\mu}{xP_z}\right)_\tau} \sim \mathcal{O}\xcancel{\left(\frac{\Lambda_{QCD}}{xP_z}\right)}$$

$\Big\Downarrow$ In our strategy, τ dependence is perturbative

$$\frac{\text{F. T.}\,[\widetilde{H}^R(\lambda, P_z)_\tau]}{\mathcal{C}\left(\xi, \frac{\mu}{xP_z}\right)_\tau} \sim \mathcal{O}\xcancel{\left(\frac{\Lambda_{QCD}}{xP_z}\right)}$$

Renormalization:

$\widetilde{H}^R(\lambda, P_z)_\tau \sim \widetilde{H}^B(\lambda, P_z)e^{(\delta m + m_0)z}$

Determine m_0 through requiring $\widetilde{H}^R(\lambda, P_z)_\tau = \widetilde{H}^R_{\text{pert}}(\lambda, P_z)_\tau$ for $a < z \ll 1/\Lambda_{\text{QCD}}$

Matching:

$$\tilde{f}(x, P_z)_\tau = \int \frac{dy}{|y|}\mathcal{C}\left(\frac{x}{y}, \frac{\mu}{xP_z}\right)_\tau f(y, \mu)$$

Use the matching kernel under summation scheme τ

Decomposition of LaMET perturbation series

- The LaMET perturbation series (e.g. Wilson coefficient and Matching kernel)

$$\widetilde{H}^{R}_{\text{pert}} = 1 + \alpha_s c_1 + \alpha_s^2 c_2 + \alpha_s^3 c_3 + \cdots$$

$$= \boxed{1 + \alpha_s c_1^c + \alpha_s^2 c_2^c + \alpha_s^3 c_3^c + \cdots}$$

Convergent Series keep

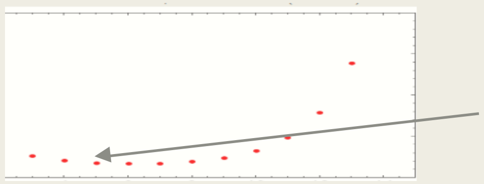
$$\boxed{+ \alpha_s c_1^{\text{LR}} + \alpha_s^2 c_2^{\text{LR}} + \alpha_s^3 c_3^{\text{LR}} + \cdots}$$

Leading Renormalon Series Borel sum τ

$$\boxed{+ \alpha_s c_1^{\text{NLR}} + \alpha_s^2 c_2^{\text{NLR}} + \alpha_s^3 c_3^{\text{NLR}} + \cdots}$$

Higher Power Renormalon Series ignore

- Conjecture I: the leading renormalon is important at the initial several orders in LaMET perturbation series. We expect good convergence after Borel suming the leading renormalon series.



NLO,
NNLO

Hint 1: the minimum term of the leading renormalon

is at the order $\mathrm{n} \sim \frac{1}{\beta_0 \alpha_s[\mu=2\mathrm{GeV}]} \sim 2$

Hint 2: the pole mass series $m_{\mathrm{OS}} = m_{\overline{MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}$

has the similar scales ($m_c = 1.2\mathrm{GeV}, m_b = 4.2\mathrm{GeV}$) as LaMET

$\tilde{r}_n = r_n/m_{\overline{\text{MS}}}$	$\tilde{r}_0$	$\tilde{r}_1$	$\tilde{r}_2$
exact ($n_f = 3$)	0.424413	1.04556	3.75086
Eq. (12) ($n_f = 3$)	0.617148	0.977493	3.76832

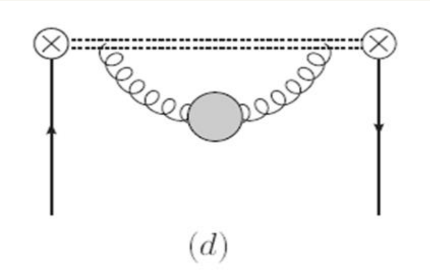
A. Pineda, JHEP 06 (2001) 022

Full Series

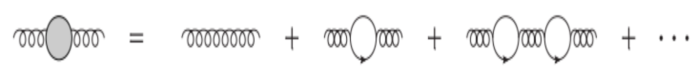
Leading Renormalon Series

Large β_0 Approximation

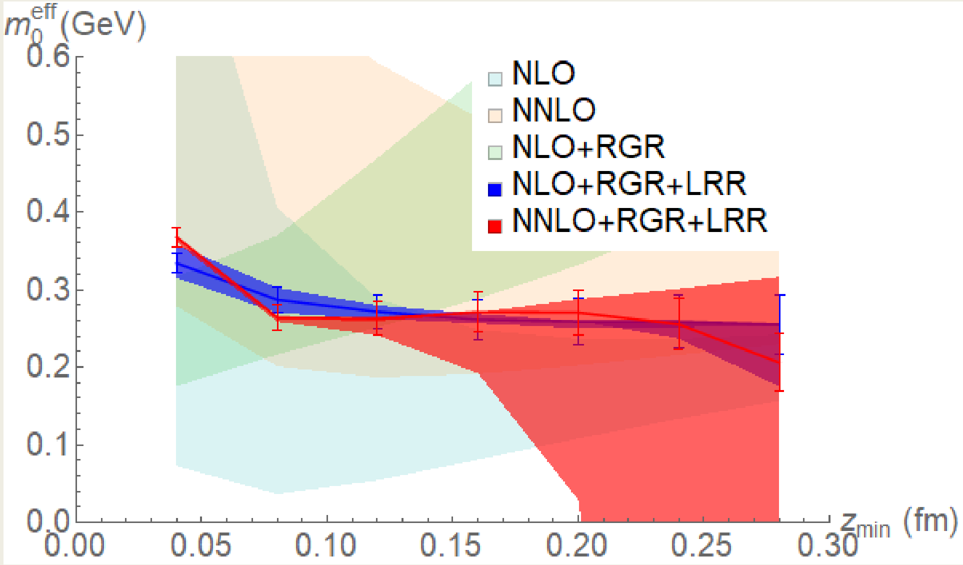
- To Borel sum the leading renormalon series, we need to know it up to all orders
- In large β_0 approximation, only the bubble chain diagram of self energy contributes to the leading renormalon



V. Braun et al., PRD 99 (2019) 1, 014013

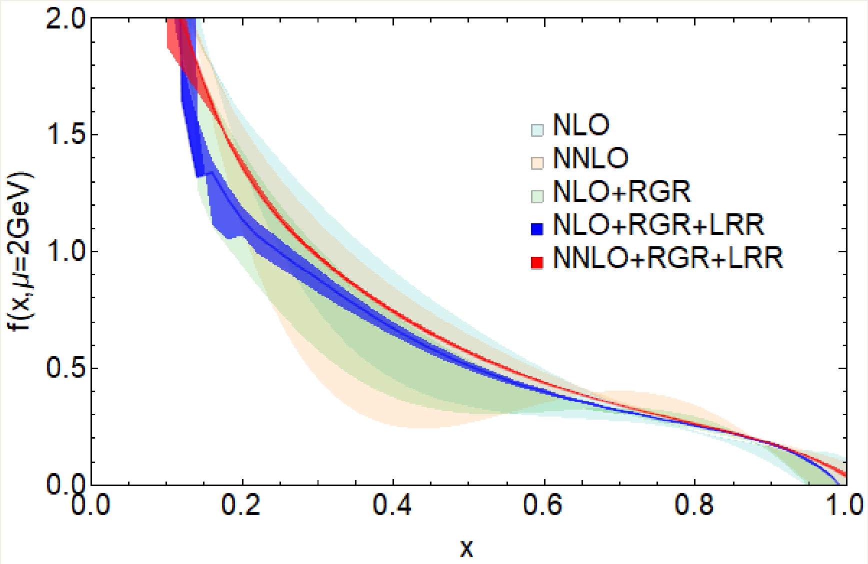


- m_0 from fitting lattice data to perturbation series at short distance



Data from BNL/ANL collaboration

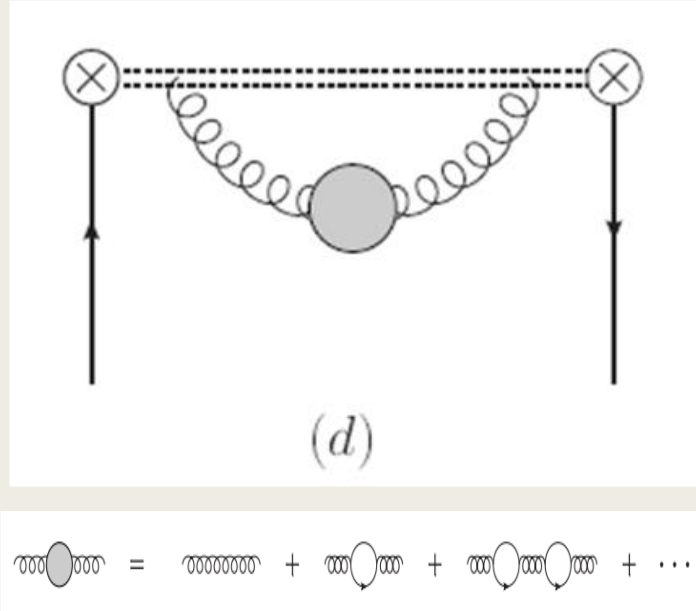
- Pion PDF under LaMET with different perturbation series



How to go beyond large β_0?

Large β_0 approximation

Bubble Chain of one loop β function



How to improve the accuracy?

Unsolved in literature

Studied in literature. Applied to LaMET.

Higher Order corrections: β_1, β_2, β_3 …

Method I:
Calculate high loop Feynman diagrams. Find a class of diagrams that contribute to the leading renormalon.

Method II:
Guess the renormalon series based on physics requirements. Verify the guess with data.

Beyond Large β_0

- Conjecture II: the asymptotic form of the leading renormalon should satisfy the following conditions:
  1) It contains a pole at $t = 2\pi/\beta_0$ on Borel plane;
  2) The leading renormalon ambiguity is renormalization scale independent;
  3) The leading renormalon ambiguity is renormalization scheme independent
- These conditions determine a unique asymptotic form except for an overall normalization factor in Borel plane $B[t] = \sum_{n=0} \frac{r_n}{n!} t^n$

$$B[u] = N_m \mu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right)$$

M. Beneke, PLB 344 (1995) 341-347

where $u = \frac{\beta_0 t}{4\pi}$, $\mathrm{b} = \frac{\beta_1}{2\,\beta_0^2}$, $\mathrm{c}_1 = \frac{1}{4\,\mathrm{b}\,\beta_0^3}\left(\frac{\beta_1^2}{\beta_0} - \beta_2\right)$ …

- The asymptotic form for the coefficients of the renormalon series $R = \sum_{n=0} r_n\, \alpha_s^{n+1}$

$$r_n = N_m \mu \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]} \left(1 + \frac{b}{n+b} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right)$$

Verified in the linear divergence perturbation series $\delta m = \frac{1}{a} \sum_{n=0}^{\infty} c_n \alpha^{n+1}(1/a)$ to $O[\alpha^{20}]$ in lattice scheme for $n_f = 0$

G. Bali et al., PRD 87 (2013) 094517

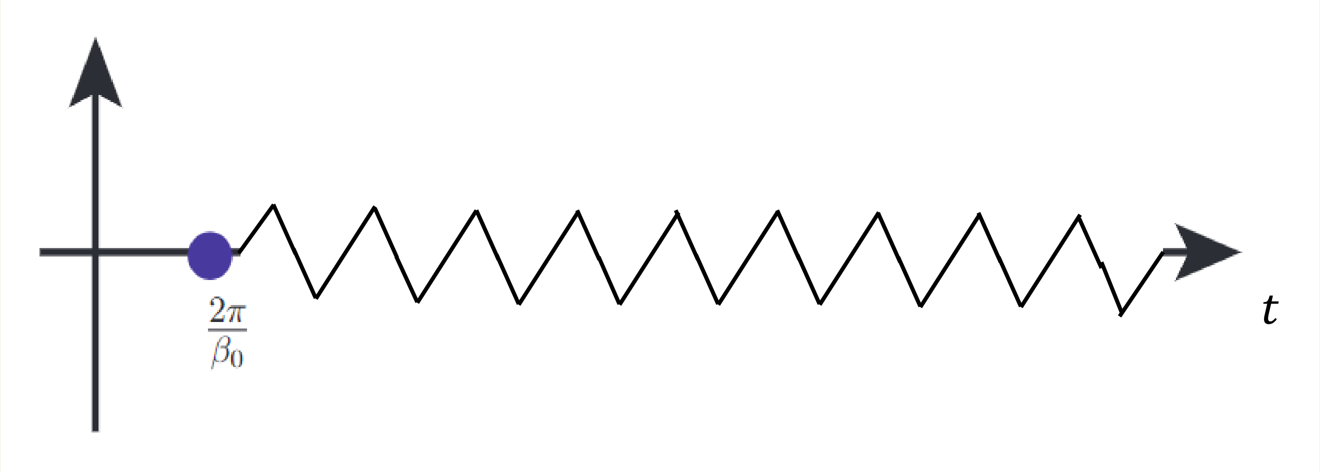
The asymptotic form with the leading pole

- 1) It contains a pole at $\mathbf{t = 2\pi/\beta_0}$ on Borel plane ($\mathbf{u = \frac{\beta_0 t}{4\pi}}$):

$$B[u] = N_m \mu \frac{1}{(1-2u)^{1+b}}\left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right)$$

where N_m, b, c_1, c_2 … are parameters to be determined

The renormalon ambiguity: $\mathrm{Im}[\tilde{R}] = \mathrm{Im}\left[\frac{4\pi}{\beta_0}\int_0^{+\infty} du\, e^{-\frac{4\pi\, u}{\beta_0 \alpha_s}} B[u + i\epsilon]\right]$



M. Beneke, PLB 344 (1995) 341-347
G. Cvetic, PRD 67 (2003) 074022

The renormalization scale independence

$$B[u] = N_m \mu \frac{1}{(1-2u)^{1+b}}(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots)$$

- 2) **The leading renormalon ambiguity is renormalization scale independent.**

The renormalon ambiguity (set $M = \frac{2\pi}{\beta_0 \alpha_s}$)

$$\mathrm{Im}[\tilde{R}] = \mathrm{Im}\left[\frac{4\pi}{\beta_0}\int_0^{+\infty} du\, e^{-\frac{4\pi\, u}{\beta_0 \alpha_s}} B[u+i\epsilon]\right]$$

$$= N_m \frac{2\pi}{\beta_0}\frac{\Gamma[1-b]\mathrm{Sin}[\pi b]}{b}\mu\, e^{-M} M^b \left(1 + \frac{c_1 b}{M} + \frac{c_2 b(b-1)}{M^2} + \cdots\right)$$

The RG invariant scale (b', c_1', c_2' are known) F. Karbstein, JHEP09(2014)114

$$\Lambda_{\mathrm{QCD}} = \mu\, \mathrm{Exp}\left[\int_{\alpha_s[\mu]}^{+\infty} \frac{d\alpha'}{\beta[\alpha']}\right] = \mu\, e^{-M}(2M)^{b'}\left(1 + \frac{c_1{}'b'}{M} + \frac{c_2{}'b'(b'-1)}{M^2} + \cdots\right)$$

If we choose $b = b',\ c_1 = c_1',\ c_2 = c_2'$ …, the renormalon ambiguity is scale independent:

$$\mathrm{Im}[\tilde{R}] = N_m \frac{2\pi}{\beta_0}\frac{\Gamma[1-b]\mathrm{Sin}[\pi b]}{b\, 2^b}\Lambda_{\mathrm{QCD}}$$

And we only have one choice for $\{b, c_1, c_2 \ldots\}$ since they are coefficients for different kinds of scale dependences

The renormalization scheme independence

$$B[u] = N_m \mu \frac{1}{(1-2u)^{1+b}} \left(1 + c_1(1-2u) + c_2(1-2u)^2 + \cdots\right)$$

- **3) The leading renormalon ambiguity is renormalization scheme independent.**

The renormalon ambiguity

$$\mathrm{Im}[\tilde{R}] = N_m \frac{2\pi}{\beta_0} \frac{\Gamma[1-b]\mathrm{Sin}[\pi b]}{b\ 2^b} \Lambda_{\mathrm{QCD}}$$

where $b = \frac{\beta_1}{2\beta_0^2}$.

Since β_0, β_1 are renormalization scheme independent, $N_m \Lambda_{\mathrm{QCD}}$ is renormalization scheme independent.

Λ_{QCD} is different for different schemes (e.g. $\Lambda_{\mathrm{lat}} \sim 10\mathrm{MeV}, \Lambda_{\overline{\mathrm{MS}}} \sim 300\mathrm{MeV}$). So the overall normalization factor is scheme dependent:

$$N_m^X \Lambda_{\mathrm{X}} = N_m^R \Lambda_R$$

Verify the asymptotic form

$$r_n = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]}\left(1 + \frac{b}{n+b}c_1 + \frac{b(b-1)}{(n+b)(n+b-1)}c_2 + \cdots\right)$$

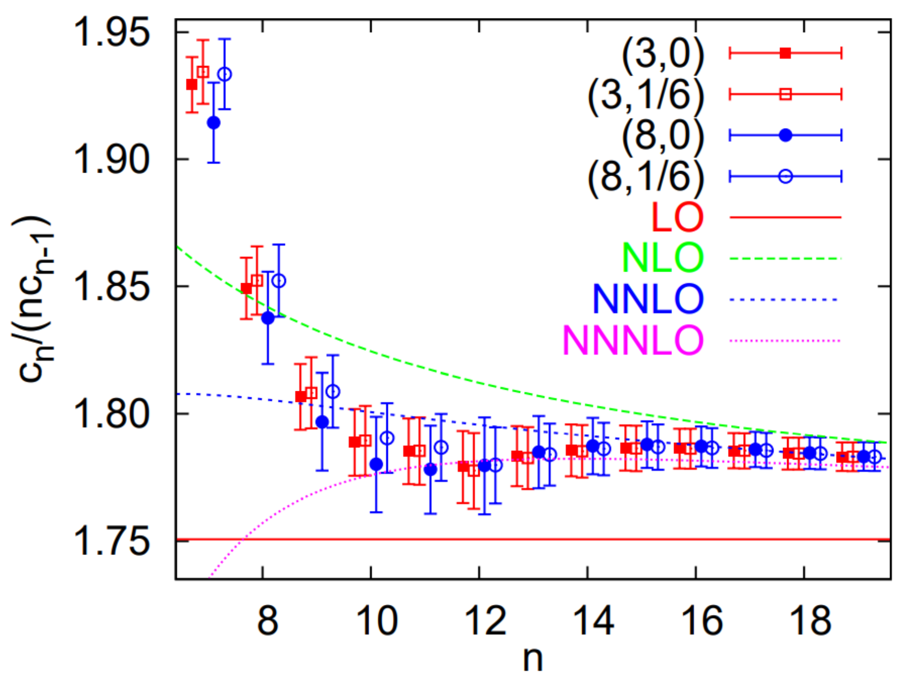
- The linear divergence perturbation series $\delta m = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}(1/a)$ to $O[\alpha^{20}]$ in lattice scheme for $n_f = 0$

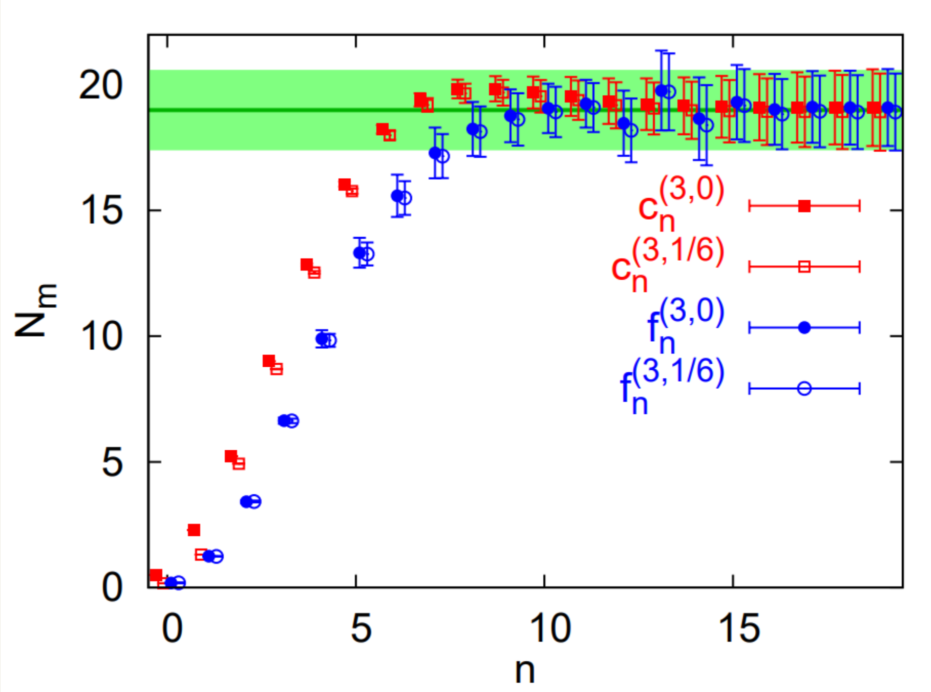
- Determine the N_m

$$N_m = \lim_{n\to\infty} \frac{c_n}{r_n/N_m}$$

NSPT

Asymp





G. Bali et al., PRD 87 (2013) 094517

Verify the asymptotic form

$$r_n = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]} \left(1 + \frac{b}{n+b} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right)$$

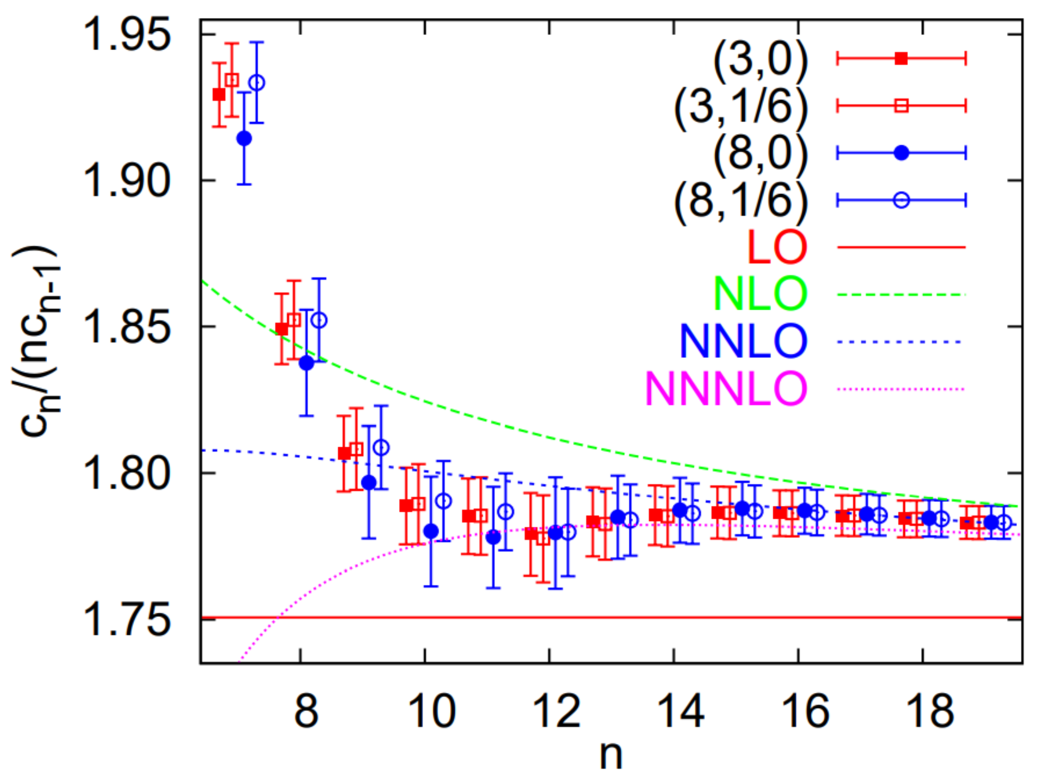
- Calculate the linear divergence perturbation series up to $O[\alpha^{20}]$ in lattice scheme through numerical stochastic perturbation theory (NSPT) for $n_f = 0$:

$\delta m = \frac{1}{a}\sum_{n=0}^{\infty} c_n^{(3,\rho)} \alpha^{n+1}(1/a)$ (fundamental)

$\delta m_{\tilde{g}} = \frac{1}{a}\sum_{n=0}^{\infty} c_n^{(8,\rho)} \alpha^{n+1}(1/a)$ (adjoint)

- Verify the asymptotic form through the ratio

$$\underbrace{\frac{c_n}{n\, c_{n-1}}}_{\text{NSPT}} = \underbrace{\frac{r_n}{n\, r_{n-1}}}_{\text{Asymp}}$$



G. Bali et al., PRD 87 (2013) 094517

Determine the overall factor N_m

$$r_n = N_m \left(\frac{\beta_0}{2\pi}\right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]} \left(1 + \frac{b}{n+b} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \cdots\right)$$

- Determine the N_m

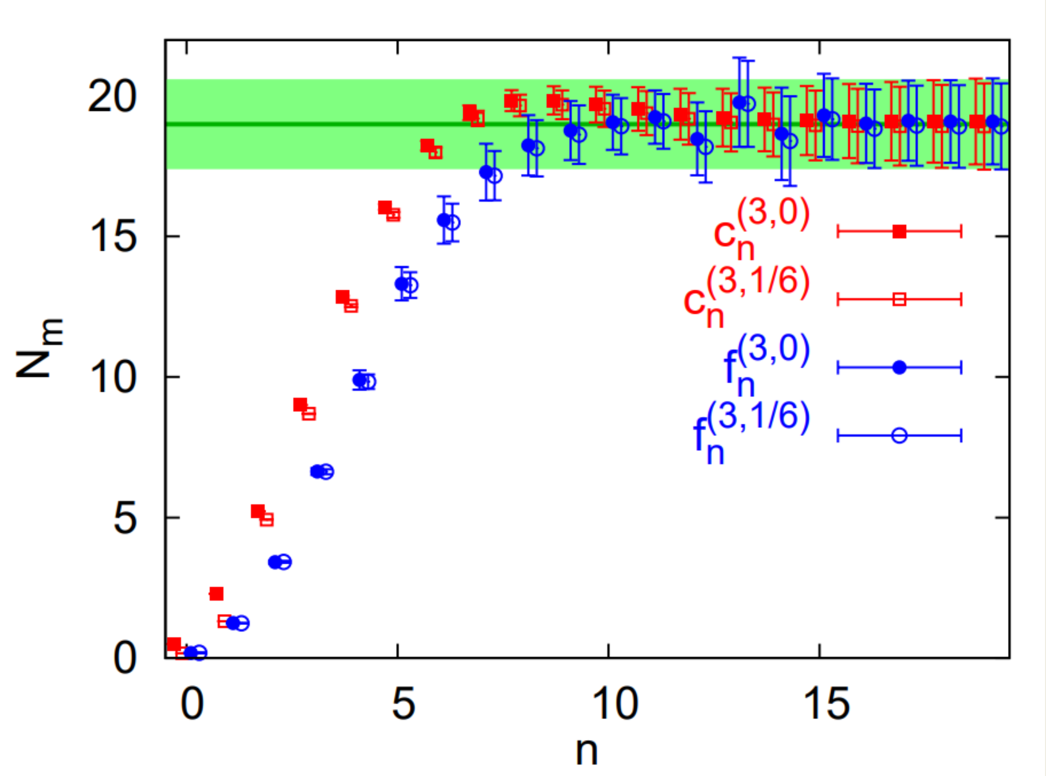
$$N_m = \lim_{n\to\infty} \frac{c_n}{r_n/N_m}$$

NSPT

Asymp

- N_m is the same for HYP smeared and unsmeared cases for $n_f = 0$:

$$N_m^{\mathrm{latt}}(\rho = 0) = 19.1(15)$$
$$N_m^{\mathrm{latt}}(\rho = 1/6) = 18.9(15)$$



G. Bali et al., PRD 87 (2013) 094517

A universality class for the leading renomalon

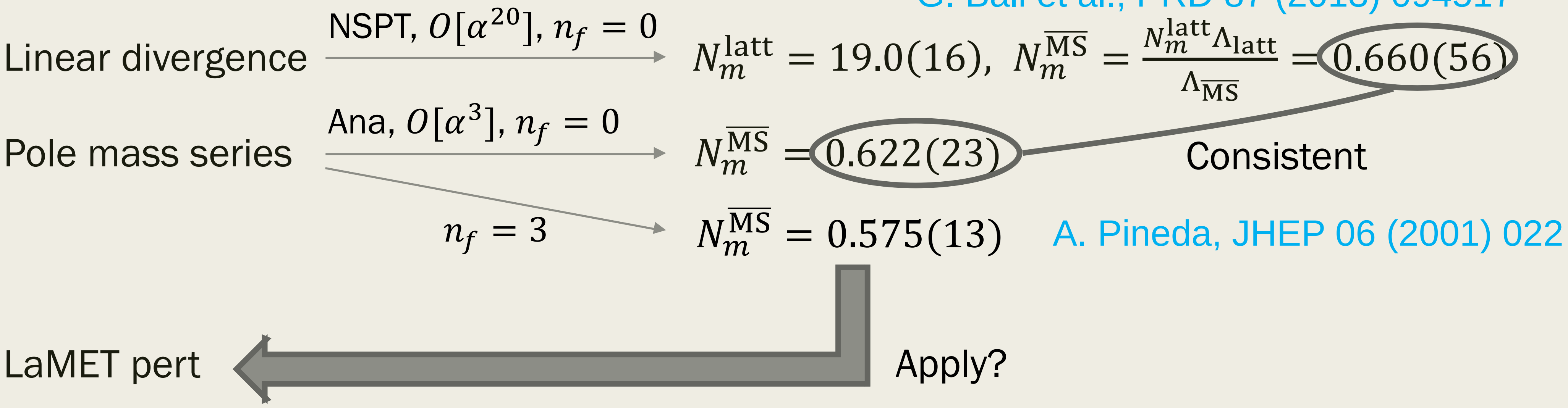
- Conjecture III: the following perturbation series share the same leading renormalon series (even the same overall factor N_m):

  Linear divergence series: $\delta m = \frac{1}{a}\sum_{n=0}^{\infty} c_n \alpha^{n+1}(1/a)$

  Pole mass series: $m_{\rm OS} - m_{\overline{MS}} = \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}$

  LaMET pert for 1D objects: $\widetilde{H}^R_{\rm pert} = 1 + \alpha_s c_1 + \alpha_s^2 c_2 + \alpha_s^3 c_3 + \cdots$

- N_m determination



G. Bali et al., PRD 87 (2013) 094517

Linear divergence $\xrightarrow{\text{NSPT, } O[\alpha^{20}],\ n_f = 0}$ $N_m^{\rm latt} = 19.0(16),\ N_m^{\overline{\rm MS}} = \frac{N_m^{\rm latt}\Lambda_{\rm latt}}{\Lambda_{\overline{\rm MS}}} = 0.660(56)$

Pole mass series $\xrightarrow{\text{Ana, } O[\alpha^{3}],\ n_f = 0}$ $N_m^{\overline{\rm MS}} = 0.622(23)$

Consistent

$\xrightarrow{n_f = 3}$ $N_m^{\overline{\rm MS}} = 0.575(13)$

A. Pineda, JHEP 06 (2001) 022

LaMET pert $\longleftarrow$ Apply?

Non-perturbative universality

- $m_0(\tau)$ may depend on the hadron species

  e.g. pion and proton have different long distance physics

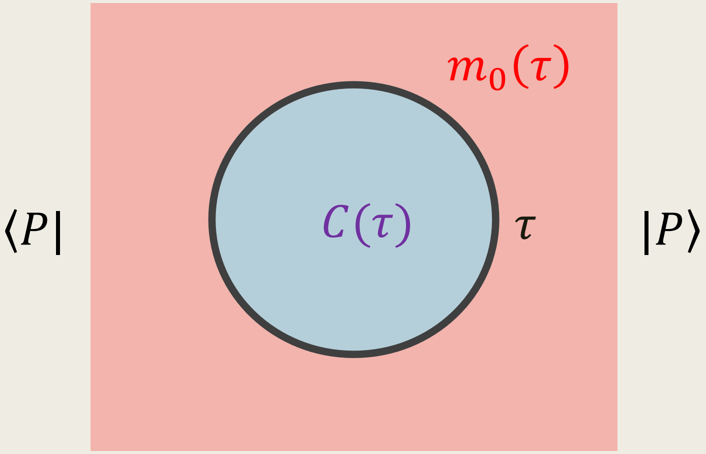
  Thus they may have different $m_0(\tau)$

- $m_0(\tau)$ is momentum P independent

  $n_z = (0,0,0,1)$ the direction of the gauge link

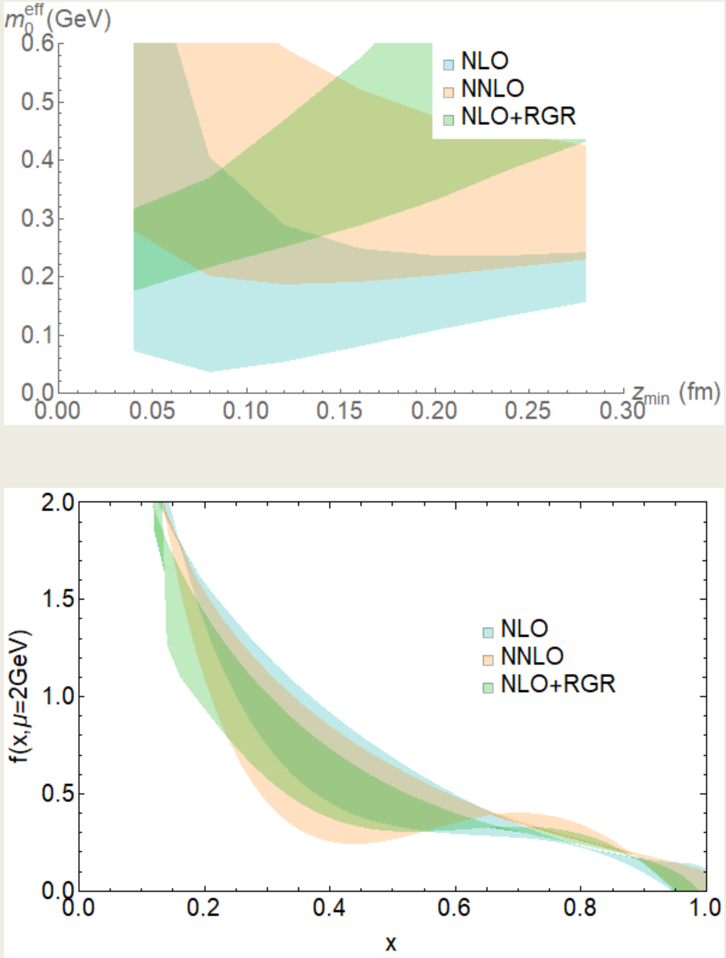
  OPE with $n_z.z$

$$\underbrace{\boxed{\tilde{h}(n_z.z, z.P)}}_{\text{Lorentz invariant}} = \sum_{n=0} \alpha^n(\mu)\, c_n(\nu, \ln[(n_z.z)^2\mu^2], \tau) \otimes h(\nu\lambda, \mu) + \underbrace{\boxed{m_0(\tau)}}_{\text{Lorentz invariant}}\underbrace{\boxed{n_z.z}}_{\text{Lorentz invariant}} + \cdots$$

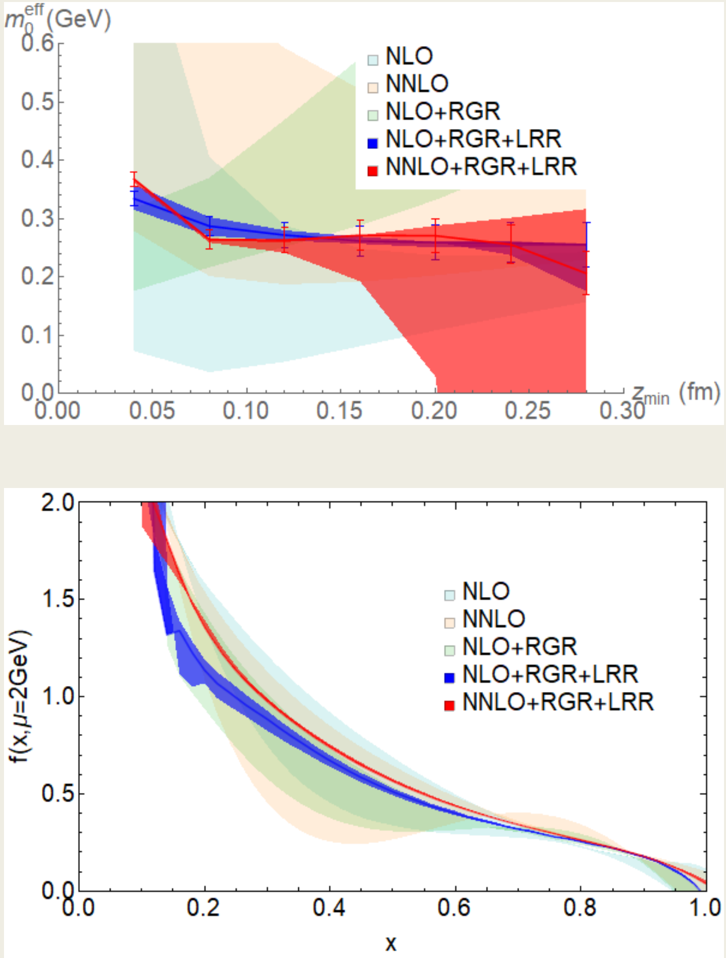


Review

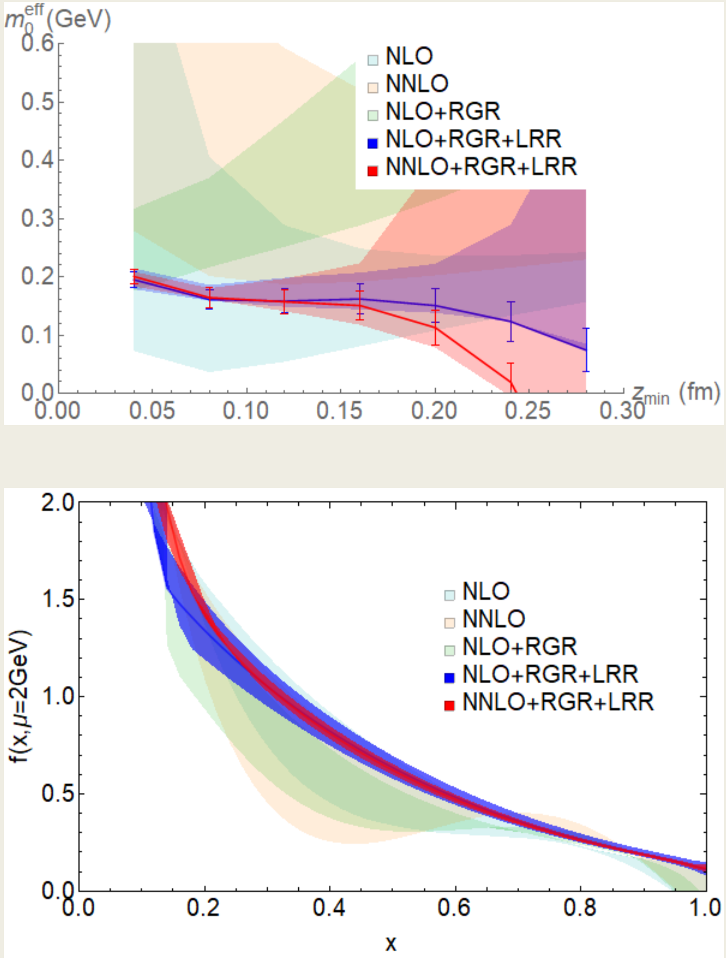
No control on renormalon



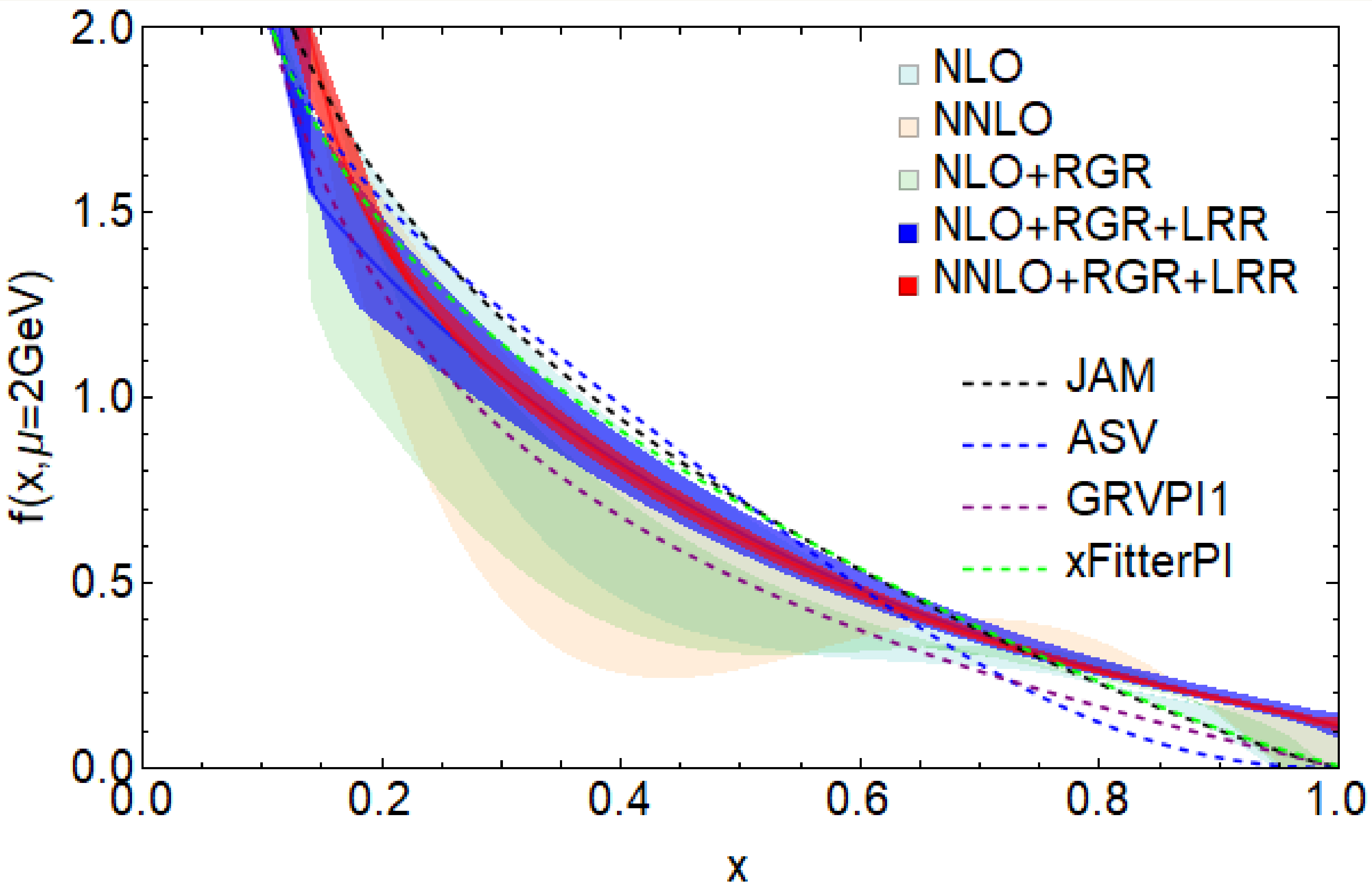
Large β_0 approximation
Fine-Tuning



Asymptotic form
beyond large β_0



Appendix


2.0
1.5
1.0
0.5
0.0
0.0
0.2
0.4
0.6
0.8
1.0
x
f(x,μ=2GeV)
NLO
NNLO
NLO+RGR
NLO+RGR+LRR
NNLO+RGR+LRR
JAM
ASV
GRVPI1
xFitterPI