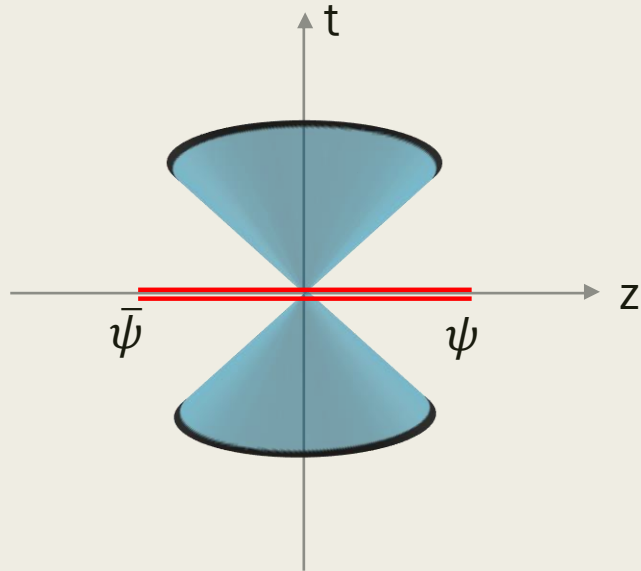


# Leading Power Accuracy in Lattice Calculations of Parton Distributions

Speaker: Yushan Su

Rui Zhang, Jack Holligan, Xiangdong Ji and  
Yushan Su. Phys.Lett.B 844 (2023), 138081

# A brief overview

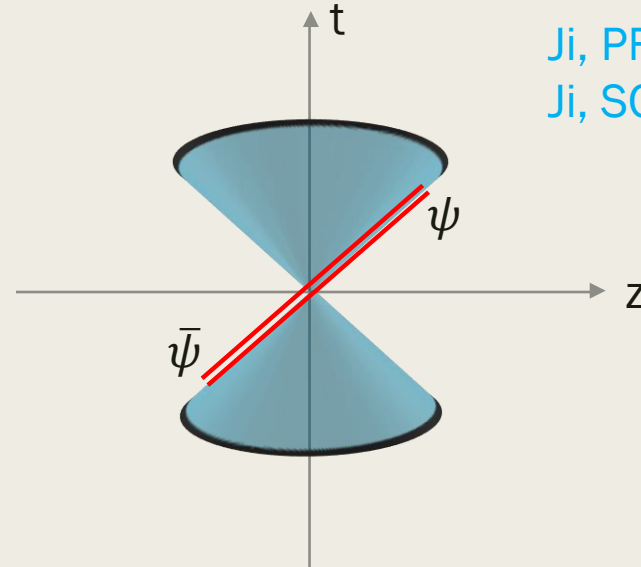


Quasi-PDF  $\tilde{f}(x, P_z) =$   

$$\int \frac{dz}{4\pi} e^{i z P_z x} \langle P_z | \bar{\psi}(z) U(z, 0) \gamma^z \psi(0) | P_z \rangle$$

$$\tilde{f}(x, P_z) = \int \frac{dy}{|y|} \mathcal{C} \left( \frac{x}{y}, \frac{\mu}{x P_z} \right) f(y, \mu) + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}^2}{x P_z} \right) + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2} \right)$$

How to eliminate this leading power correction?



Light cone PDF  $f(y, \mu) =$   

$$\int \frac{d\xi^-}{4\pi} e^{-i \xi^- P^+ x} \langle P | \bar{\psi}(\xi^-) U(\xi^-, 0) \gamma^+ \psi(0) | P \rangle$$

Ji, PRL 110 (2013), 262002  
 Ji, SCPMA 57 (2014), 1407-1412

Braun, Vladimirov and Zhang,  
 PRD 99 (2019) 1, 014013  
 Liu and Chen, PRD 104 (2021)  
 9, 094501

Jianhui Zhang, Thu 11:30 AM

# Outline

1

OPE and power accuracy

2

Renormalon divergence

3

Numerical test on Pion PDF

# OPE of quark bilinear operator

- Quark bilinear operator in a large momentum hadron on lattice

$$\tilde{h}(z, P_z) = \frac{1}{2P_t} \langle P_z | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | P_z \rangle$$

- OPE: a double expansion in  $z$  and  $\alpha$

$$\tilde{h}(z, P_z) = \underbrace{\sum_{k=0} z^k}_{\text{Power expansion}} \underbrace{\sum_{n=0} \alpha^n(\mu) c_{k,n}(v, \ln z^2 \mu^2)}_{\text{Log expansion}} \otimes h_k(v\lambda, \mu)$$

Perturbative UV matching coefficients
Non-perturbative IR physics at  $\mathcal{O}(\Lambda_{\text{QCD}}^k)$   
 $\lambda = z P_z$

- Our goal: extract the **non-perturbative parton physics for  $k = 0$**

# Expansion accuracy

- Quark bilinear operator in a large momentum hadron on lattice

$$\tilde{h}(z, P_z) = \frac{1}{2P_t} \langle P_z | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | P_z \rangle$$

- The extraction of the **parton physics**  $h(\lambda, \mu)$  ( $\lambda = z P_z$ )

$$\tilde{h}(z, P_z) = C(\nu, \alpha(\mu), z^2 \mu^2) \otimes h(\nu \lambda, \mu) + \mathcal{O}(z \Lambda_{\text{QCD}}) + \mathcal{O}(z^2 \Lambda_{\text{QCD}}^2) + \dots$$

If the energy scale is high enough, e.g.  $\mu \sim P_z \sim 20$  GeV or  $z \sim 0.01$  fm

$$\alpha \text{ accuracy: } C(\nu, \alpha(\mu), z^2 \mu^2) = \mathcal{O}(1) + \mathcal{O}(\alpha(\mu)) + \mathcal{O}(\alpha^2(\mu)) + \dots$$

$\sim 0.15$ 
 $\sim 0.023$

$$\text{power accuracy: } \mathcal{O}(z \Lambda_{\text{QCD}}) + \mathcal{O}(z^2 \Lambda_{\text{QCD}}^2) + \dots$$

$\sim 0.015$ 
 $\sim 0.00023$

# The leading power accuracy

- Quark bilinear operator in a large momentum hadron on lattice

$$\tilde{h}(z, P_z) = \frac{1}{2P_t} \langle P_z | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | P_z \rangle$$

- The extraction of the **parton physics**  $h(\lambda, \mu)$  ( $\lambda = z P_z$ )

$$\tilde{h}(z, P_z) = C(v, \alpha(\mu), z^2 \mu^2) \otimes h(v\lambda, \mu) + \mathcal{O}(z \Lambda_{\text{QCD}}) + \mathcal{O}(z^2 \Lambda_{\text{QCD}}^2) + \dots$$

Under current lattice technique,  $\mu \sim P_z \sim 2 \text{ GeV}$  or  $z \sim 0.1 \text{ fm}$

$$\alpha \text{ accuracy: } C(v, \alpha(\mu), z^2 \mu^2) = \mathcal{O}(1) + \mathcal{O}(\alpha(\mu)) + \mathcal{O}(\alpha^2(\mu)) + \dots$$

$\sim 0.30$                        $\sim 0.088$

$$\text{power accuracy: } \mathcal{O}(z \Lambda_{\text{QCD}}) + \mathcal{O}(z^2 \Lambda_{\text{QCD}}^2) + \dots$$

$\sim 0.15$                        $\sim 0.023$

- The leading power accuracy  $\mathcal{O}(z \Lambda_{\text{QCD}})$  is as important as  $\mathcal{O}(\alpha(\mu))$  accuracy

# A subtlety: the perturbation series is divergent

- Expand at the physical scale  $1/z$

$$C = 1 + \sum_{n=1} c_n \alpha^n (1/z)$$

where  $\alpha(\mu) = \frac{2\pi}{\beta_0 \ln(\frac{\mu}{\Lambda_{\text{QCD}}})}$

- At high orders,

$$c_n \sim \left(\frac{\beta_0}{2\pi}\right)^n n! \sim \left(\frac{\beta_0}{2\pi}\right)^n e^{n \ln n - n}$$

from bubble chain of Wilson link self energy

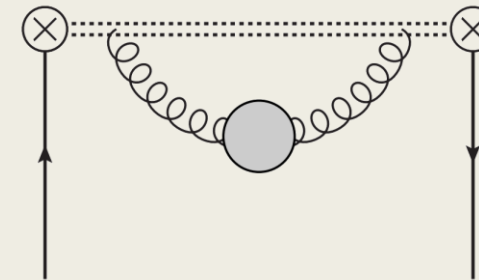
- Truncate near the minimum term

$$\frac{d \left(\frac{\beta_0}{2\pi}\right)^n e^{n \ln n - n} \alpha^n}{d n} = 0 \Rightarrow n = \frac{2\pi}{\alpha \beta_0}$$

- The uncertainty of the truncation

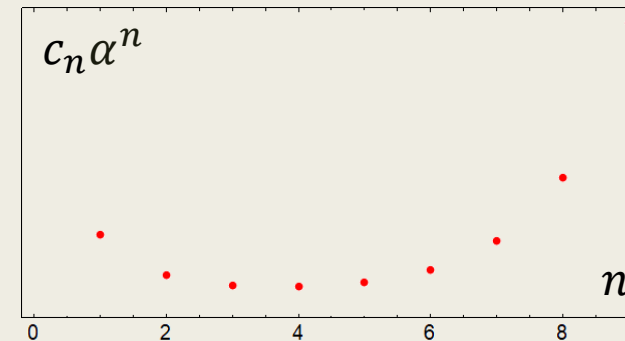
$$c_n \alpha^n \Big|_{n \rightarrow \frac{2\pi}{\alpha \beta_0}} = e^{-\frac{2\pi}{\alpha \beta_0}} = z \Lambda_{\text{QCD}}$$

Beneke, Phys.Rept. 317 (1999) 1-142



Braun, Vladimirov and Zhang, PRD 99 (2019) 1, 014013

$$\text{bubble chain} = \text{bubble} + \text{bubble}^2 + \text{bubble}^3 + \dots$$



Schematic diagram

The same order as the leading power correction

# Introduce a parameter $m_0(\tau)$

- No clear boundary between  $c_n(\nu, \ln z^2 \mu^2)$  and  $\mathcal{O}(z \Lambda_{\text{QCD}})$

$$\tilde{h}(z, P_z) = \sum_{n=0} \alpha^n(\mu) c_n(\nu, \ln z^2 \mu^2) \otimes h(\nu\lambda, \mu) + \mathcal{O}(z \Lambda_{\text{QCD}})$$

Uncertainty in regulating this divergent series  $\mathcal{O}(z \Lambda_{\text{QCD}})$

- A way to regulate the divergent series is called a summation scheme  $\tau$
- The leading power correction  $m_0(\tau)z$  depends on  $\tau$

$$\tilde{h}(z, P_z) = \sum_{n=0} \alpha^n(\mu) c_n(\nu, \ln z^2 \mu^2, \tau) \otimes h(\nu\lambda, \mu) + m_0(\tau)z$$

Ayala, Lobregat and Pineda, PRD 99 (2019) 7, 074019

- Knowing  $\tilde{h}(z, P_z)$  and  $c_n(\nu, \ln z^2 \mu^2, \tau)$ , one can extract  $m_0(\tau)$
- Make predictions with  $m_0(\tau)$ . Is  $m_0(\tau)$  a universal parameter?

$$\tilde{h}'(z, P_z) = \sum_{n=0} \alpha^n(\mu) c'_n(\nu, \ln z^2 \mu^2, \tau) \otimes h'(\nu\lambda, \mu) + m_0(\tau)z$$



# Physics of the divergent series

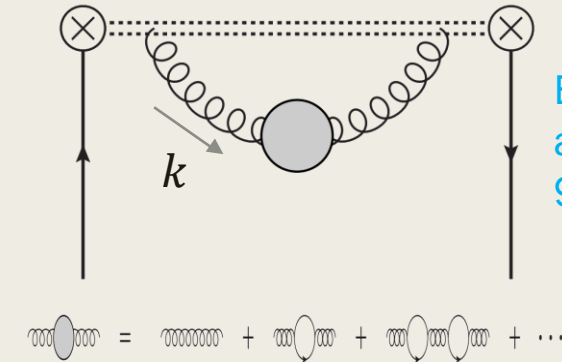
- Some low momentum modes in the perturbation theory lead to a divergent series,

e.g. Wilson link self energy with  $n$  bubbles

$$\sim \int_0 d^d k (-1)^n \ln^n(-k^2) \sim n!$$

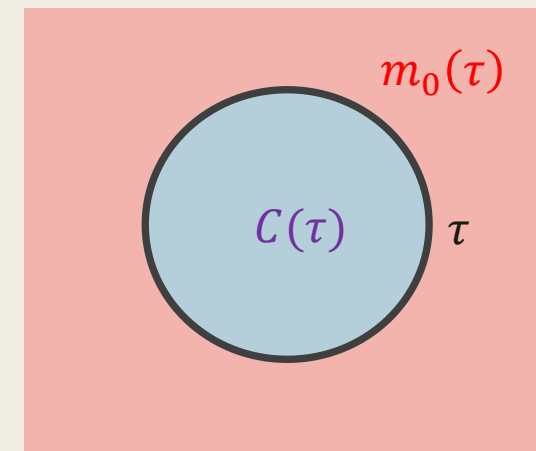
- Truncating the divergent series is to regulate the these low momentum modes

- Coordinate space picture: long distance physics leads to a divergent series
- Truncating the divergent series is to regulate the long distance physics
- Ambiguity in choosing  $\tau$ . No ambiguity in the full physics up to  $\mathcal{O}(z \Lambda_{\text{QCD}})$



Braun, Vladimirov  
and Zhang, PRD  
99 (2019) 1, 014013

Coordinate Space



$\tau$  is the summation scheme, intuitively understood as “the boundary” between UV and IR

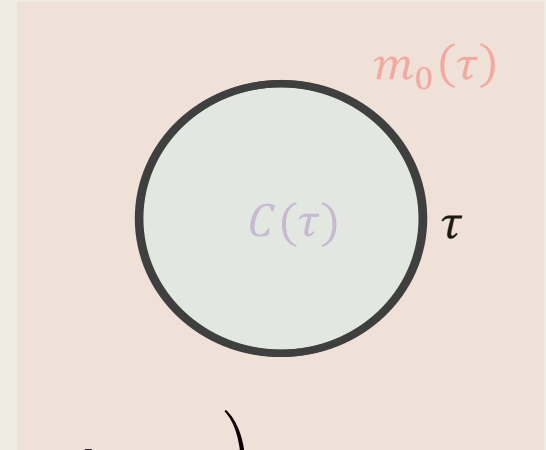
# Perturbative universality

- A universality class of the perturbative  $\tau$  dependence

Linear divergence series:  $\delta m(a) = \frac{1}{a} \sum_{n=0}^{\infty} c_n \alpha^{n+1} (1/a)$

Pole mass series:  $m_{OS} - m_{\overline{MS}} = \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}$

LaMET pert for 1D objects:  $\tilde{h}_{\text{pert}}^R = 1 + \alpha_s c_1 + \alpha_s^2 c_2 + \alpha_s^3 c_3 + \dots$



- Leading renormalon series [Beneke, PLB 344 \(1995\) 341-347](#)

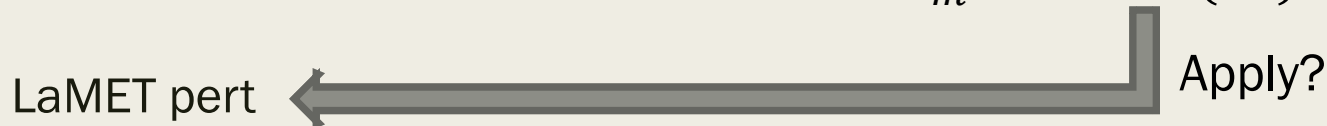
$$r_n = N_m \mu \left( \frac{\beta_0}{2\pi} \right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]} \left( 1 + \frac{b}{n+b} b_1 + \frac{b(b-1)}{(n+b)(n+b-1)} b_2 + \dots \right)$$

- $N_m$  determination [Bali, Bauer and Pineda, PRD 87 \(2013\) 094517](#)

Linear divergence  $\xrightarrow{\text{NSPT, } O[\alpha^{20}], n_f = 0}$   $N_m^{\text{latt}} = 19.0(16), N_m^{\overline{MS}} = \frac{N_m^{\text{latt}} \Lambda_{\text{latt}}}{\Lambda_{\overline{MS}}} = 0.660(56)$

Pole mass series  $\xrightarrow{\text{Ana, } O[\alpha^3], n_f = 0}$   $N_m^{\overline{MS}} = 0.622(23)$  Consistent

$n_f = 3 \rightarrow N_m^{\overline{MS}} = 0.575(13)$  [Pineda, JHEP 06 \(2001\) 022](#)



# The leading power correction

Ishikawa, Ma, Qiu and Yoshida, arXiv:1609.02018

Chen, Ji and Zhang, NPB 915, 1 (2017)

Constantinou and Panagopoulos, arXiv:1705.11193

Alexandrou et al., NPB 923 (2017)

Chen et al., PRD 97 (2018)

Musch, Hagler, Negele and Schafer, PRD 83 (2011)

Ji, Zhang and Zhao, PRL 120 (2018)

- Ambiguity during the Mass renormalization

$$\tilde{h}^R(\lambda, P_z)_\tau \sim \tilde{h}^B(\lambda, P_z) e^{(\delta m(a) + m_0(\tau))z}$$



Fourier transform

$$\tilde{f}(x, P_z)_\tau \sim \tilde{f}(x, P_z)_{\tau'} + \mathcal{O}\left(\frac{\Delta m_0}{xP_z}\right)$$

Ji et al., NPB 964 (2021)

- Ambiguity of perturbative matching kernel

$$c\left(\xi, \frac{\mu}{xP_z}\right)_\tau \sim c\left(\xi, \frac{\mu}{xP_z}\right)_{\tau'} + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right)$$

- Under the same scheme  $\tau$ , the leading power correction vanishes

$$\tilde{f}(x, P_z)_\tau = \int \frac{dy}{|y|} c\left(\frac{x}{y}, \frac{\mu}{xP_z}\right)_\tau f(y, \mu) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right) + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^2}{x^2 P_z^2}, \frac{\Lambda_{\text{QCD}}^2}{(1-x)^2 P_z^2}\right)$$

# Lattice data and renormalization

- Pion valence PDF matrix element from BNL/ANL collaboration:

$$h^{\text{lat}}(z, a, P_z) = \langle \pi^+(P_z) | \bar{u}(z) \gamma^t U(z, 0) u(0) - \bar{d}(z) \gamma^t U(z, 0) d(0) | \pi^+(P_z) \rangle$$

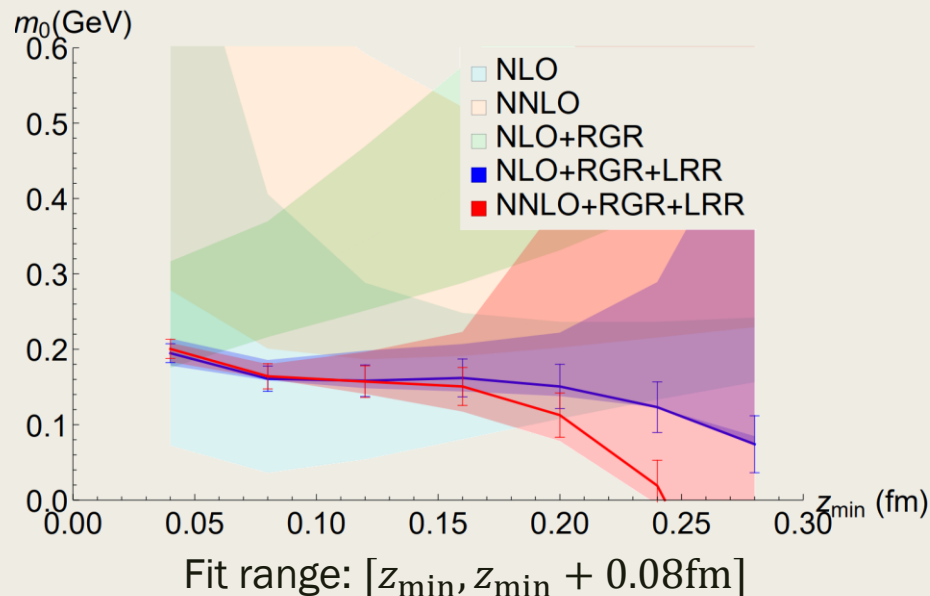
Gao et al., PRD (2020)  
 Gao et al., PRD (2021)  
 Gao et al., PRL (2022)

- Hybrid renormalized matrix element:

$$\tilde{h}^R(z, P_z) = \frac{\tilde{h}^{\text{lat}}(z, a, P_z)}{\tilde{h}^{\text{lat}}(z, a, 0)} \theta(z_s - |z|) + \frac{\tilde{h}^{\text{lat}}(z, a, P_z)}{Z^R(z, a, \mu) \tilde{h}^{\overline{\text{MS}}}(z_s, \mu, 0)} \theta(|z| - z_s)$$

where  $Z^R(z, a, \mu) \sim e^{-(\delta m(a) + m_0)z}$ .

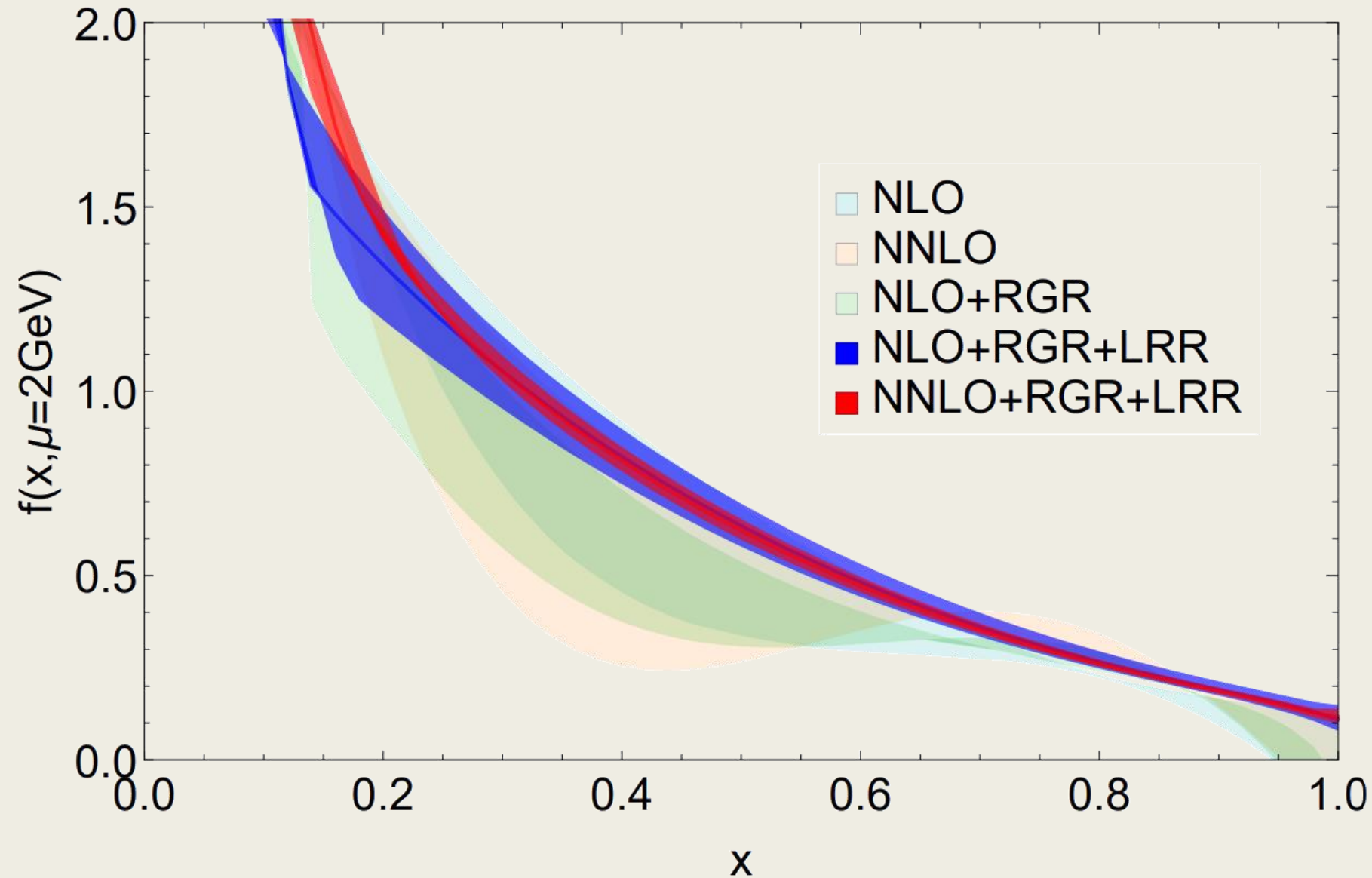
$m_0(\tau)$  is determined through requiring  $\frac{\tilde{h}^{\text{lat}}(z, a, P_z=0)}{Z^R(z, a, \mu)} = \tilde{h}^{\overline{\text{MS}}}(z, \mu, P_z = 0)$  for  $a < z \ll 1/\Lambda_{\text{QCD}}$



$$\tilde{h}^{\overline{\text{MS}}} = \left( \begin{array}{c} \text{NLO} \quad \text{NNLO} \\ 1 + \alpha_s c_1^c + \alpha_s^2 c_2^c + \alpha_s^3 c_3^c + \dots \\ \text{NLO} \quad \text{NNLO LRR:} \quad \text{RGR} \\ + \alpha_s c_1^{\text{LR}} + \alpha_s^2 c_2^{\text{LR}} + \alpha_s^3 c_3^{\text{LR}} + \dots \end{array} \right) \text{Exp} \left[ \int d\alpha_s \frac{\gamma}{\beta} \right]$$

Asymptotic Form to  $O[1/n^2]$   
 $\tau$  scheme: Borel sum with P.V.

# LRR: Matching with leading power correction



# Take home messages

- Under current lattice technique (scale  $\sim 2$  GeV), the leading power accuracy  $\mathcal{O}(z \Lambda_{\text{QCD}})$  is as important as  $\mathcal{O}(\alpha(\mu))$
- The definition of the leading power correction depends on the summation scheme  $\tau$  of the lead twist perturbation series
- After the leading power accuracy is controlled, the precisions of PDF are dramatically improved. The methods in our paper can be generalized to other LaMET 1D objects, such as GPD and DA.

# Appendix

# Renormalon divergence

- $R = \sum_{n=0} r_n \alpha_s^{n+1}$

- At high orders,

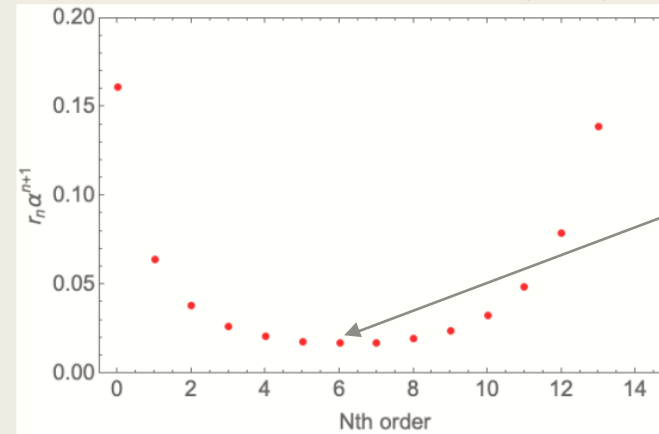
$$r_n \sim n!,$$

series is divergent for any  $\alpha_s$

- Borel transformation,  $B[t] = \sum_n \frac{r_n}{n!} t^n$

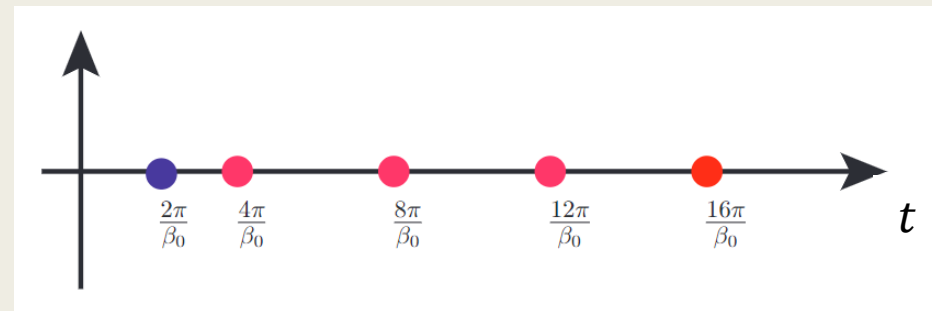
Renormalon divergences correspond to singularity poles on Borel Plane

M. Beneke, Phys.Rept. 317 (1999) 1-142



At minimum term:  
 $n \sim 1/\alpha_s$ , which  
 comes from  
 $\frac{d n! \alpha_s^{n+1}}{d n} = 0$

Data from Bali, et al. PRD(2013)





# Summation Schemes and Renormalon Ambiguity

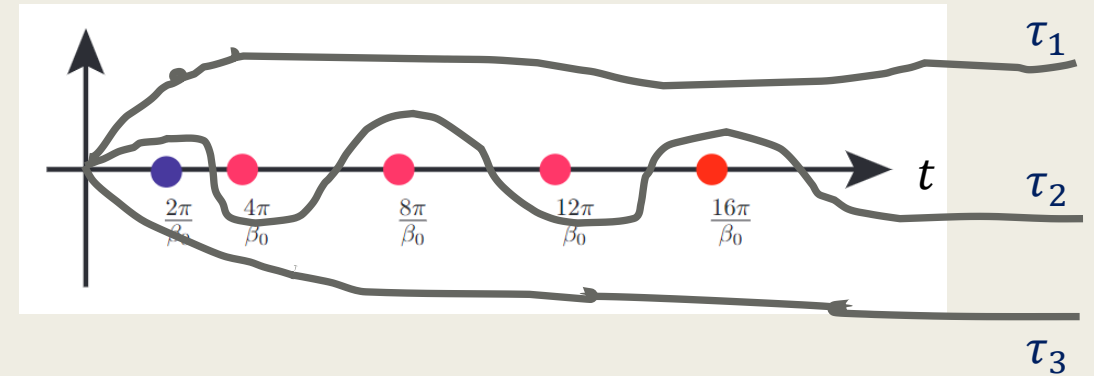
- Borel Sum

$$\tilde{R} = \int_0^{+\infty} dt e^{-\frac{t}{\alpha_s}} B[t]$$

which depends on the integral path

- To regularize the renormalon divergence, just choose an integral path that leads to a finite result
- Each integral path in Borel sum corresponds to a summation scheme  $\tau$

M. Beneke, Phys.Rept. 317 (1999) 1-142



- The resulting difference between summation schemes is called renormalon ambiguity, which can be estimated as the residues of the poles on Borel plane.
- e.g. the leading pole:

$$\begin{aligned} \Delta[\delta m(a)] &\sim \mathcal{O}(\Lambda_{\text{QCD}}) \\ \Delta[Z(v, z^2 \mu^2)] &\sim \mathcal{O}(z \Lambda_{\text{QCD}}) \\ \Delta\left[c\left(\xi, \frac{\mu}{xP_z}\right)\right] &\sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right) \end{aligned}$$

# Physical observables are renormalon free

Ayala et al., PRD  
99 (2019) 7, 074019

- The QCD physical observables (Suppose we measure them in a hadron state) should be independent of summation scheme  $\tau$

$$\langle O_1(Q^2) \rangle = \underbrace{C_1(Q^2, \tau)}_{\text{Pert}} + \frac{\langle O_G \rangle(\tau)}{Q^2} + \dots$$

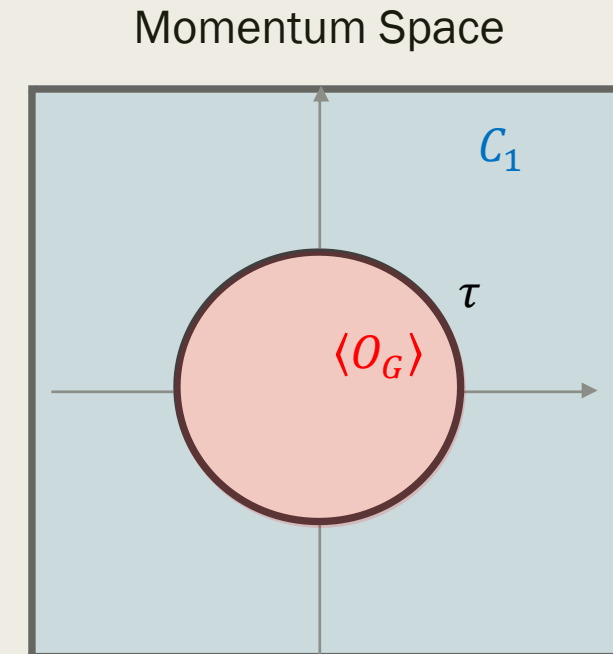
- The universality of summation scheme dependence

$$\langle O_1(Q^2) \rangle = C_1(Q^2, \tau) + \frac{\langle O_G \rangle(\tau)}{Q^2} + \dots$$

$$\langle O_2(Q^2) \rangle = C_2(Q^2, \tau) + \frac{\langle O_G \rangle(\tau)}{Q^2} + \dots$$

$$\langle O_3(Q^2) \rangle = C_3(Q^2, \tau) + \frac{\langle O_G \rangle(\tau)}{Q^2} + \dots$$

...

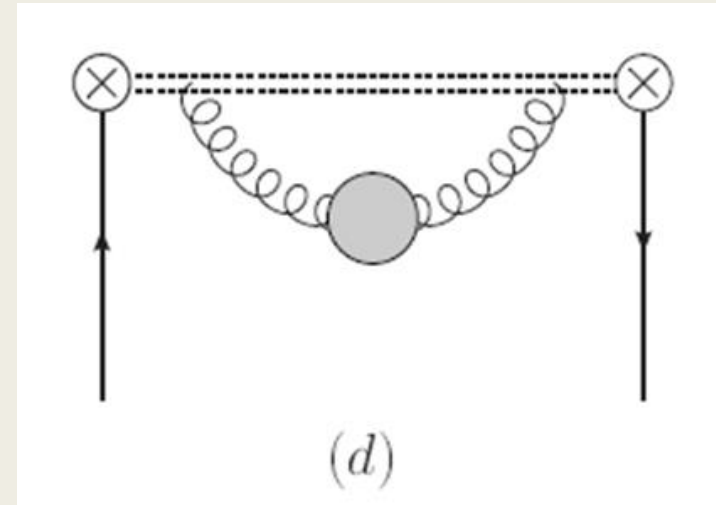


# The origin of the leading renormalon in LaMET

- The non-local quark bilinear operator with a Wilson link

$$\hat{O}(z) = \bar{\psi}_1(z)U(z, 0)\Gamma\psi_2(0)$$

- The leading renormalon comes from the self energy of the Wilson link (or as a subdiagram)



- During the renormalization

$$\tilde{H}^R(\lambda, P_z) \sim \tilde{H}^B(\lambda, P_z)e^{(\delta m - m_0)z}$$

where  $m_0$  is a nonperturbative mass which may be mixed with the leading renormalon ambiguity of  $\delta m$  and  $\tilde{H}_{\text{pert}}^R$

$$\Delta[\delta m(a)] \sim \mathcal{O}(\Lambda_{\text{QCD}})$$

$$\Delta[\tilde{H}_{\text{pert}}^R] \sim \mathcal{O}(z\Lambda_{\text{QCD}})$$

G. Bali et al., PRD 87 (2013) 094517

- In the matching kernel

$$c\left(\xi, \frac{\mu}{xP_z}\right) \sim c\left(\xi, \frac{\mu}{xP_z}\right)_\tau + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right)$$

where  $\mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right)$  denotes the leading renormalon ambiguity of the matching kernel

V. Braun et al., PRD 99 (2019) 1, 014013

# Twist-three accuracy

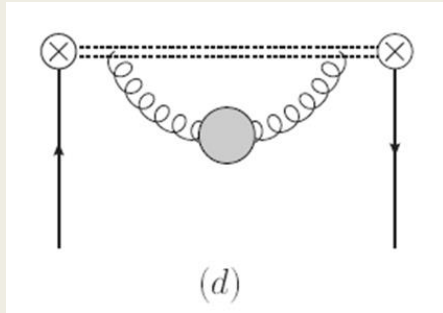
- The bare matrix element  $\tilde{H}^B(z, P_z, a) = \langle P_z | \bar{\psi}(z) U(z, 0) \gamma^t \psi(0) | P_z \rangle$

- The renormalized matrix element

$$\begin{aligned} \tilde{H}^R(z, P_z, \mu, \tau) &= e^{-m_0(\tau)z} \sum_{k=0} C_k(\alpha(\mu), z^2 \mu^2, \tau) \frac{\lambda^k}{k!} \langle x^k \rangle(\mu) + O(z^2 \Lambda_{\text{QCD}}^2) \\ &= \sum_{k=0} [C_k(\alpha(\mu), z^2 \mu^2, \tau) - m_0(\tau)z] \frac{\lambda^k}{k!} \langle x^k \rangle(\mu) + O(z \Lambda_{\text{QCD}} \alpha(\mu), z^2 \Lambda_{\text{QCD}}^2) \end{aligned}$$

- $m_0(\tau)$  is a twist-three non-perturbative parameter during the renormalization  
The leading renormalon ambiguity in the  $C_k(\alpha(\mu), z^2 \mu^2, \tau)$  is independent of  $k$

# The Leading Power Accuracy



Regularize the leading renormalon under the same summation scheme  $\tau$  for:

Perturbative Matrix Element:  $\tilde{H}_{\text{pert}}^R(\lambda, P_z)_\tau$

Perturbative Matching Kernel:  $C\left(\xi, \frac{\mu}{xP_z}\right)_\tau$

$$\frac{\text{F. T. } [\tilde{H}_{\text{pert}}^R(\lambda, P_z)_\tau]}{C\left(\xi, \frac{\mu}{xP_z}\right)_\tau} \sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right)$$



In our strategy,  $\tau$  dependence is perturbative

$$\frac{\text{F. T. } [\tilde{H}^R(\lambda, P_z)_\tau]}{C\left(\xi, \frac{\mu}{xP_z}\right)_\tau} \sim \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}}{xP_z}\right)$$

Renormalization:

$$\tilde{H}^R(\lambda, P_z)_\tau \sim \tilde{H}^B(\lambda, P_z) e^{(\delta m + m_0)z}$$

Determine  $m_0$  through requiring

$$\tilde{H}^R(\lambda, P_z)_\tau = \tilde{H}_{\text{pert}}^R(\lambda, P_z)_\tau \text{ for}$$

$$a < z \ll 1/\Lambda_{\text{QCD}}$$

Matching:

$$\begin{aligned} & \tilde{f}(x, P_z)_\tau \\ &= \int \frac{dy}{|y|} C\left(\frac{x}{y}, \frac{\mu}{xP_z}\right)_\tau f(y, \mu) \end{aligned}$$

Use the matching kernel under summation scheme  $\tau$

# Decomposition of LaMET perturbation series

- The LaMET perturbation series (e.g. Wilson coefficient and Matching kernel)

$$\tilde{H}_{\text{pert}}^R = 1 + \alpha_s c_1 + \alpha_s^2 c_2 + \alpha_s^3 c_3 + \dots$$

$$= 1 + \alpha_s c_1^c + \alpha_s^2 c_2^c + \alpha_s^3 c_3^c + \dots$$

$$+ \alpha_s c_1^{\text{LR}} + \alpha_s^2 c_2^{\text{LR}} + \alpha_s^3 c_3^{\text{LR}} + \dots$$

$$+ \alpha_s c_1^{\text{NLR}} + \alpha_s^2 c_2^{\text{NLR}} + \alpha_s^3 c_3^{\text{NLR}} + \dots$$

Convergent Series

keep

Leading Renormalon Series

Borel sum  $\tau$

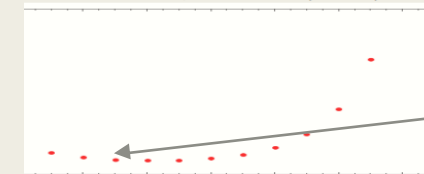
Higher Power Renormalon Series

ignore

- Conjecture I: the leading renormalon is important at the initial several orders in LaMET perturbation series. We expect good convergence after Borel summing the leading renormalon series.

Hint 1: the minimum term of the leading renormalon

$$\text{is at the order } n \sim \frac{1}{\beta_0 \alpha_s[\mu=2\text{GeV}]} \sim 2$$



NLO,  
NNLO

Hint 2: the pole mass series  $m_{OS} = m_{\overline{MS}} + \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}$

has the similar scales ( $m_c = 1.2\text{GeV}$ ,  $m_b = 4.2\text{GeV}$ ) as LaMET

$\tilde{r}_n = r_n/m_{\overline{MS}}$	$\tilde{r}_0$	$\tilde{r}_1$	$\tilde{r}_2$
exact ( $n_f = 3$ )	0.424413	1.04556	3.75086
Eq. (12) ( $n_f = 3$ )	0.617148	0.977493	3.76832

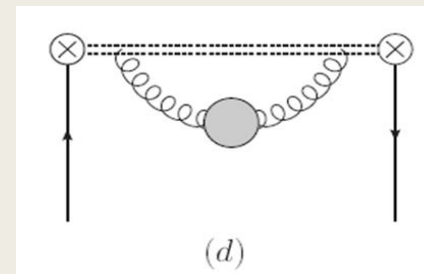
A. Pineda, JHEP 06 (2001) 022

Full Series

Leading Renormalon Series

# Large $\beta_0$ Approximation

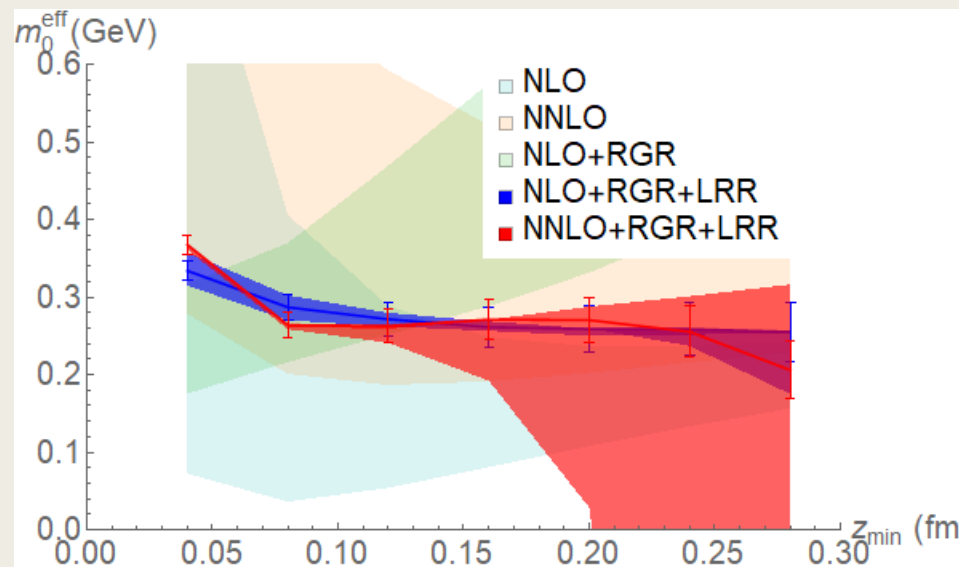
- To Borel sum the leading renormalon series, we need to know it up to all orders
- In large  $\beta_0$  approximation, only the bubble chain diagram of self energy contributes to the leading renormalon
- $m_0$  from fitting lattice data to perturbation series at short distance



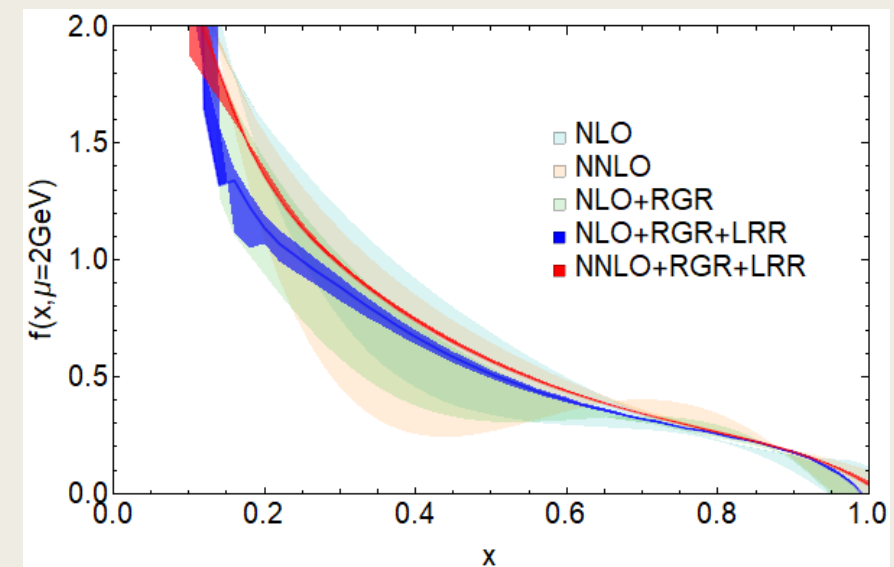
V. Braun et al., PRD 99 (2019) 1, 014013

$$\text{quark line with bubble chain} = \text{quark line with 1 loop} + \text{quark line with 2 loops} + \text{quark line with 3 loops} + \dots$$

- Pion PDF under LaMET with different perturbation series



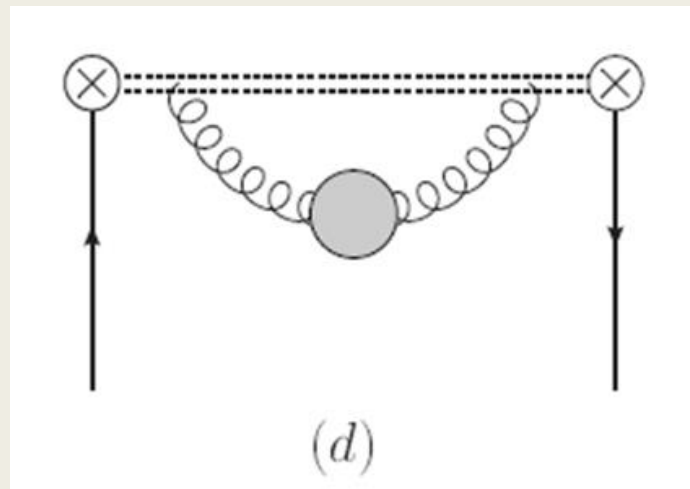
Data from BNL/ANL collaboration



# How to go beyond large $\beta_0$ ?

Large  $\beta_0$  approximation

Bubble Chain of one loop  $\beta$  function



$$\text{wavy line with bubble} = \text{wavy line} + \text{wavy line with bubble} + \text{wavy line with two bubbles} + \dots$$

How to improve the accuracy?



Unsolved in literature

Studied in literature.  
Applied to LaMET.

Higher Order corrections:  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$  ...

Method I:

Calculate high loop Feynman diagrams. Find a class of diagrams that contribute to the leading renormalon.

Method II:

Guess the renormalon series based on physics requirements. Verify the guess with data.



# Beyond Large $\beta_0$

- Conjecture II: the asymptotic form of the leading renormalon should satisfy the following conditions:
  - 1) It contains a pole at  $t = 2\pi/\beta_0$  on Borel plane;
  - 2) The leading renormalon ambiguity is renormalization scale independent;
  - 3) The leading renormalon ambiguity is renormalization scheme independent
- These conditions determine a unique asymptotic form except for an overall normalization factor in Borel plane  $B[t] = \sum_{n=0} \frac{r_n}{n!} t^n$

$$B[u] = N_m \mu \frac{1}{(1-2u)^{1+b}} (1 + c_1(1-2u) + c_2(1-2u)^2 + \dots)$$

M. Beneke, PLB  
344 (1995) 341-347

where  $u = \frac{\beta_0 t}{4\pi}$ ,  $b = \frac{\beta_1}{2\beta_0^2}$ ,  $c_1 = \frac{1}{4b\beta_0^3} \left( \frac{\beta_1^2}{\beta_0} - \beta_2 \right) \dots$

- The asymptotic form for the coefficients of the renormalon series  $R = \sum_{n=0} r_n \alpha_s^{n+1}$

$$r_n = N_m \mu \left( \frac{\beta_0}{2\pi} \right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]} \left( 1 + \frac{b}{n+b} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right)$$

Verified in the linear divergence perturbation series  $\delta m = \frac{1}{a} \sum_{n=0}^{\infty} c_n \alpha^{n+1} (1/a)$  to  $O[\alpha^{20}]$  in lattice scheme for  $n_f = 0$

G. Bali et al., PRD 87 (2013) 094517

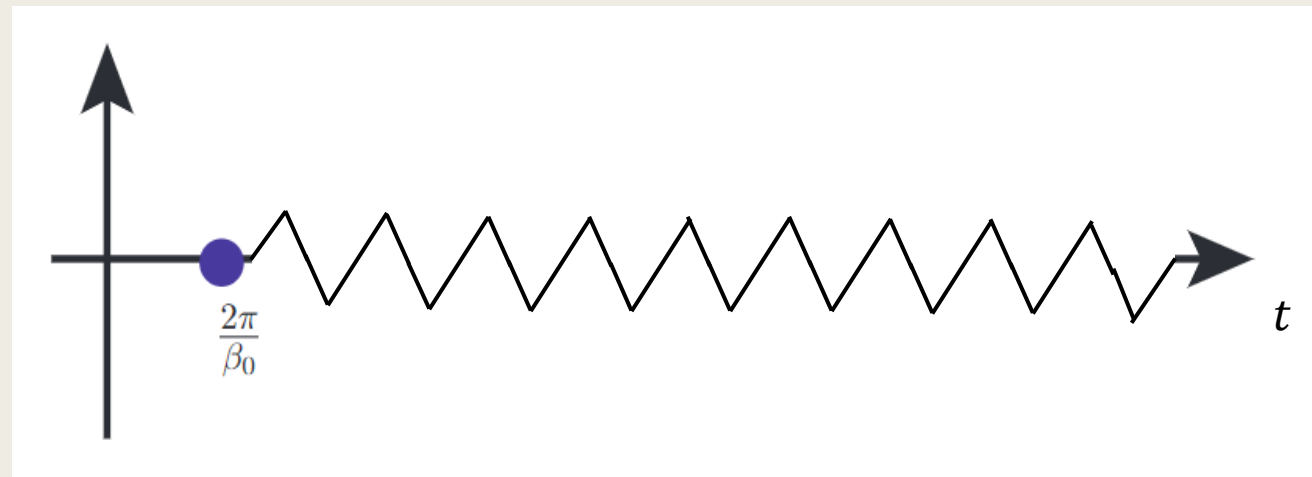
# The asymptotic form with the leading pole

- 1) It contains a pole at  $t = 2\pi/\beta_0$  on Borel plane ( $u = \frac{\beta_0 t}{4\pi}$ ):

$$B[u] = N_m \mu \frac{1}{(1-2u)^{1+b}} (1 + c_1(1-2u) + c_2(1-2u)^2 + \dots)$$

where  $N_m$ ,  $b$ ,  $c_1$ ,  $c_2$  ... are parameters to be determined

The renormalon ambiguity:  $\text{Im}[\tilde{R}] = \text{Im} \left[ \frac{4\pi}{\beta_0} \int_0^{+\infty} du e^{-\frac{4\pi u}{\beta_0 \alpha_s}} B[u + i\epsilon] \right]$



M. Beneke, PLB 344 (1995) 341-347

G. Cvetic, PRD 67 (2003) 074022

# The renormalization scale independence

$$B[u] = N_m \mu \frac{1}{(1-2u)^{1+b}} (1 + c_1(1-2u) + c_2(1-2u)^2 + \dots)$$

- 2) The leading renormalon ambiguity is renormalization scale independent.

The renormalon ambiguity (set  $M = \frac{2\pi}{\beta_0 \alpha_s}$ )

$$\begin{aligned} \text{Im}[\tilde{R}] &= \text{Im} \left[ \frac{4\pi}{\beta_0} \int_0^{+\infty} du e^{-\frac{4\pi u}{\beta_0 \alpha_s}} B[u + i\epsilon] \right] \\ &= N_m \frac{2\pi}{\beta_0} \frac{\Gamma[1-b] \text{Sin}[\pi b]}{b} \mu e^{-M} M^b \left( 1 + \frac{c_1 b}{M} + \frac{c_2 b(b-1)}{M^2} + \dots \right) \end{aligned}$$

The RG invariant scale ( $b', c'_1, c'_2$  are known)

[F. Karbstein, JHEP09\(2014\)114](#)

$$\Lambda_{\text{QCD}} = \mu \text{Exp} \left[ \int_{\alpha_s[\mu]}^{+\infty} \frac{d\alpha'}{\beta[\alpha']} \right] = \mu e^{-M} (2M)^{b'} \left( 1 + \frac{c'_1 b'}{M} + \frac{c'_2 b'(b'-1)}{M^2} + \dots \right)$$

If we choose  $b = b', c_1 = c'_1, c_2 = c'_2 \dots$ , the renormalon ambiguity is scale independent:

$$\text{Im}[\tilde{R}] = N_m \frac{2\pi}{\beta_0} \frac{\Gamma[1-b] \text{Sin}[\pi b]}{b 2^b} \Lambda_{\text{QCD}}$$

And we only have one choice for  $\{b, c_1, c_2 \dots\}$  since they are coefficients for different kinds of scale dependences

# The renormalization scheme independence

$$B[u] = N_m \mu \frac{1}{(1-2u)^{1+b}} (1 + c_1(1-2u) + c_2(1-2u)^2 + \dots)$$

- 3) The leading renormalon ambiguity is renormalization scheme independent.

The renormalon ambiguity

$$\text{Im}[\tilde{R}] = N_m \frac{2\pi \Gamma[1-b] \text{Sin}[\pi b]}{\beta_0 b 2^b} \Lambda_{\text{QCD}}$$

where  $b = \frac{\beta_1}{2\beta_0^2}$ .

Since  $\beta_0, \beta_1$  are renormalization scheme independent,  $N_m \Lambda_{\text{QCD}}$  is renormalization scheme independent.

$\Lambda_{\text{QCD}}$  is different for different schemes (e.g.  $\Lambda_{\text{lat}} \sim 10 \text{MeV}$ ,  $\Lambda_{\overline{\text{MS}}} \sim 300 \text{MeV}$ ). So the overall normalization factor is scheme dependent:

$$N_m^X \Lambda_X = N_m^R \Lambda_R$$

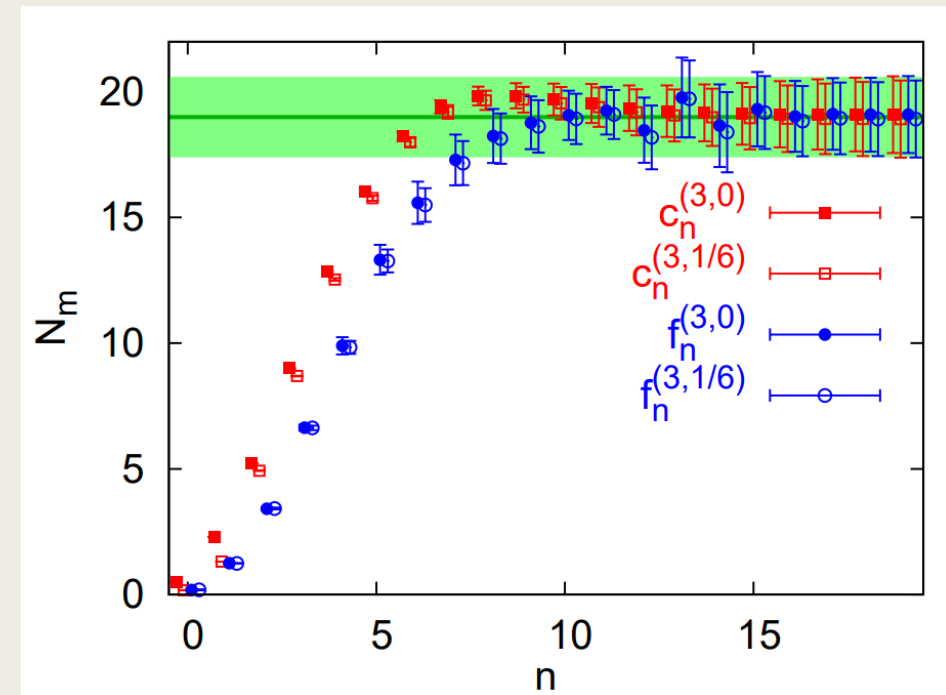
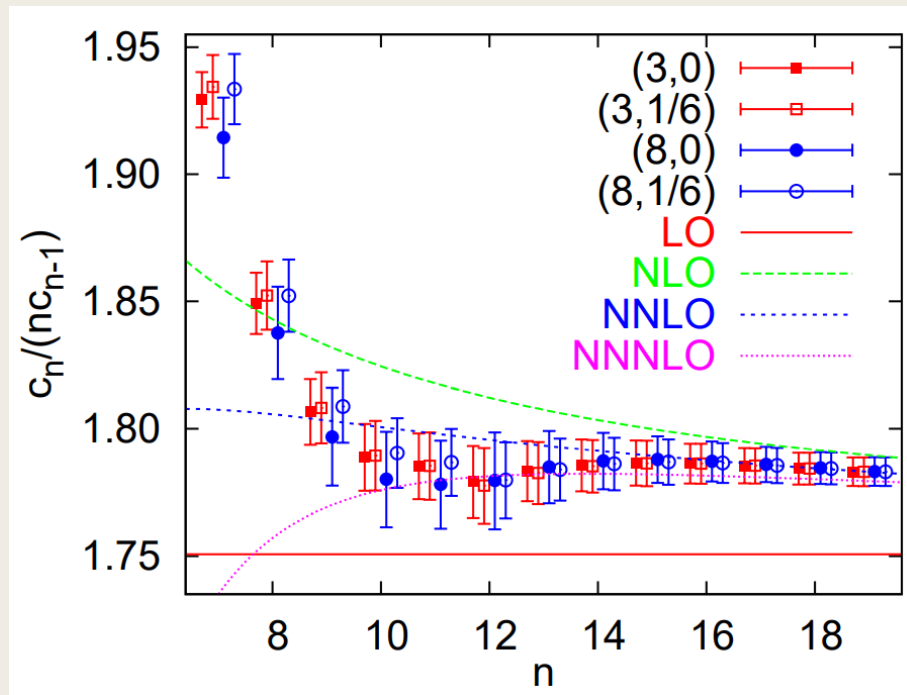
# Verify the asymptotic form

$$r_n = N_m \left( \frac{\beta_0}{2\pi} \right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]} \left( 1 + \frac{b}{n+b} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right)$$

- The linear divergence perturbation series  $\delta m = \frac{1}{a} \sum_{n=0}^{\infty} c_n \alpha^{n+1} (1/a)$  to  $O[\alpha^{20}]$  in lattice scheme for  $n_f = 0$

- Determine the  $N_m$

$$N_m = \lim_{n \rightarrow \infty} \frac{c_n}{r_n / N_m} \quad \begin{array}{l} \text{NSPT} \\ \text{Asymp} \end{array}$$



G. Bali et al., PRD 87 (2013) 094517

# Verify the asymptotic form

$$r_n = N_m \left( \frac{\beta_0}{2\pi} \right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]} \left( 1 + \frac{b}{n+b} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right)$$

- Calculate the linear divergence perturbation series up to  $O[\alpha^{20}]$  in lattice scheme through numerical stochastic perturbation theory (NSPT) for  $n_f = 0$ :

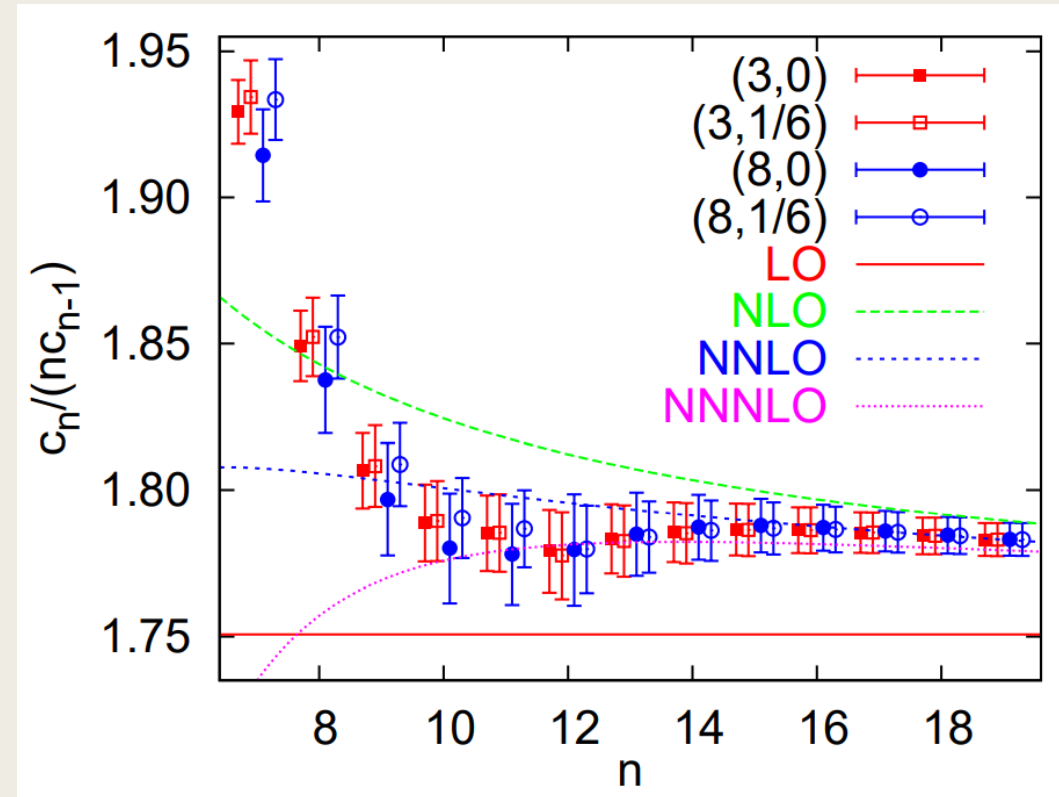
$$\delta m = \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(3,\rho)} \alpha^{n+1} (1/a) \text{ (fundamental)}$$

$$\delta m_{\tilde{g}} = \frac{1}{a} \sum_{n=0}^{\infty} c_n^{(8,\rho)} \alpha^{n+1} (1/a) \text{ (adjoint)}$$

- Verify the asymptotic form through the ratio

$$\frac{c_n}{n c_{n-1}} = \frac{r_n}{n r_{n-1}}$$

NSPT      Asymp



G. Bali et al., PRD 87 (2013) 094517

# Determine the overall factor $N_m$

$$r_n = N_m \left( \frac{\beta_0}{2\pi} \right)^n \frac{\Gamma[n+1+b]}{\Gamma[1+b]} \left( 1 + \frac{b}{n+b} c_1 + \frac{b(b-1)}{(n+b)(n+b-1)} c_2 + \dots \right)$$

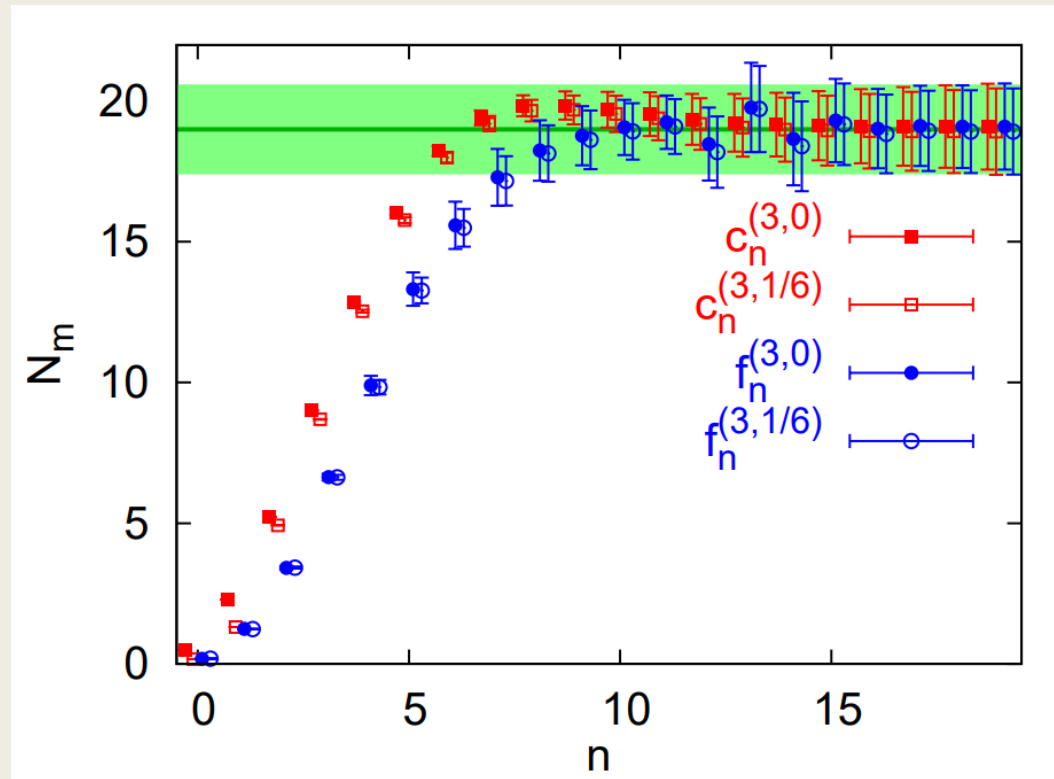
- Determine the  $N_m$

$$N_m = \lim_{n \rightarrow \infty} \frac{c_n}{r_n / N_m} \quad \begin{array}{l} \text{NSPT} \\ \text{Asymp} \end{array}$$

- $N_m$  is the same for HYP smeared and unsmeared cases for  $n_f = 0$ :

$$N_m^{\text{latt}}(\rho = 0) = 19.1(15)$$

$$N_m^{\text{latt}}(\rho = 1/6) = 18.9(15)$$



G. Bali et al., PRD 87 (2013) 094517

# A universality class for the leading renormalon

- Conjecture III: the following perturbation series share the same leading renormalon series (even the same overall factor  $N_m$ ):

Linear divergence series:  $\delta m = \frac{1}{a} \sum_{n=0}^{\infty} c_n \alpha^{n+1} (1/a)$

Pole mass series:  $m_{OS} - m_{\overline{MS}} = \sum_{n=0}^{\infty} r_n \alpha_s^{n+1}$

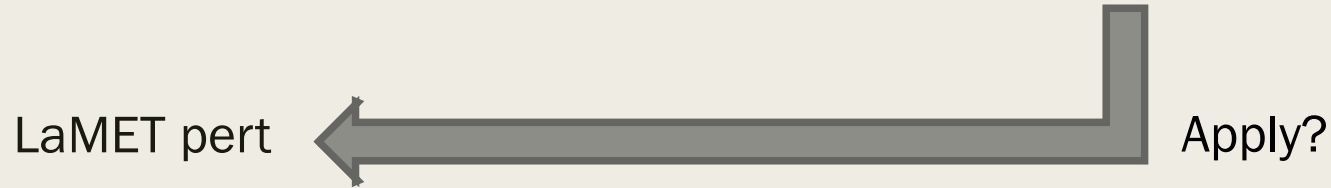
LaMET pert for 1D objects:  $\tilde{H}_{\text{pert}}^R = 1 + \alpha_s c_1 + \alpha_s^2 c_2 + \alpha_s^3 c_3 + \dots$

- $N_m$  determination

Linear divergence  $\xrightarrow{\text{NSPT, } O[\alpha^{20}], n_f = 0}$   $N_m^{\text{latt}} = 19.0(16), N_m^{\overline{MS}} = \frac{N_m^{\text{latt}} \Lambda_{\text{latt}}}{\Lambda_{\overline{MS}}} = 0.660(56)$  G. Bali et al., PRD 87 (2013) 094517

Pole mass series  $\xrightarrow{\text{Ana, } O[\alpha^3], n_f = 0}$   $N_m^{\overline{MS}} = 0.622(23)$  Consistent

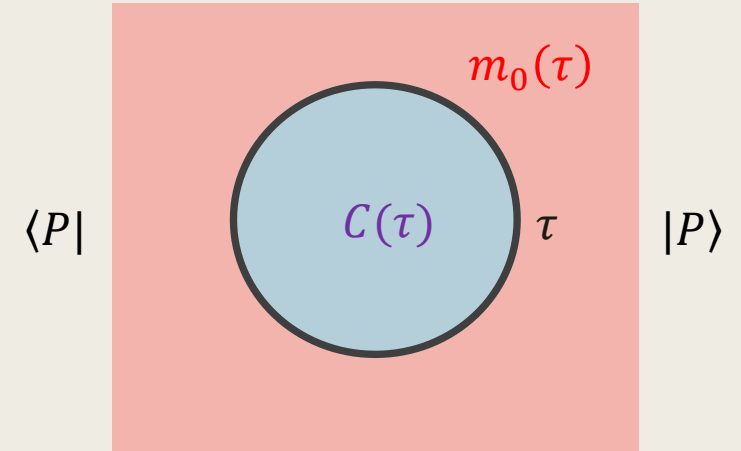
$\xrightarrow{n_f = 3}$   $N_m^{\overline{MS}} = 0.575(13)$  A. Pineda, JHEP 06 (2001) 022





# Non-perturbative universality

- $m_0(\tau)$  may depend on the hadron species  
e.g. pion and proton have different long distance physics  
Thus they may have different  $m_0(\tau)$



- $m_0(\tau)$  is momentum  $P$  independent  
 $n_z = (0,0,0,1)$  the direction of the gauge link

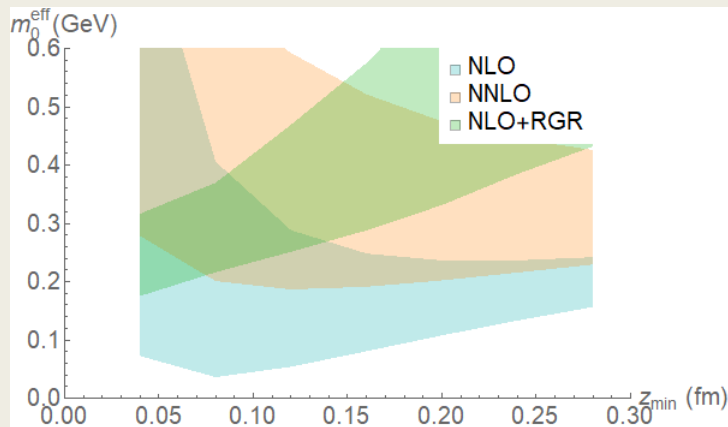
OPE with  $n_z \cdot z$

$$\boxed{\tilde{h}(n_z \cdot z, z \cdot P)} = \sum_{n=0} \alpha^n(\mu) c_n(v, \ln[(n_z \cdot z)^2 \mu^2], \tau) \otimes h(v\lambda, \mu) + \boxed{m_0(\tau)} \boxed{n_z \cdot z} + \dots$$

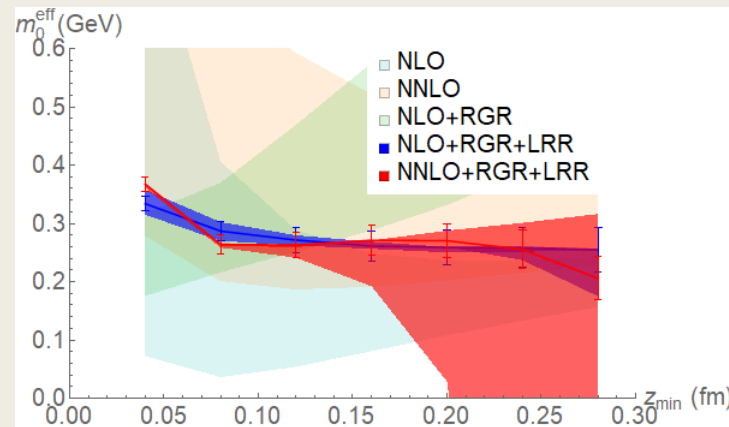
Lorentz invariant
Lorentz invariant
Lorentz invariant

# Review

No control on renormalon



Large  $\beta_0$  approximation  
Fine-Tuning



Asymptotic form  
beyond large  $\beta_0$

