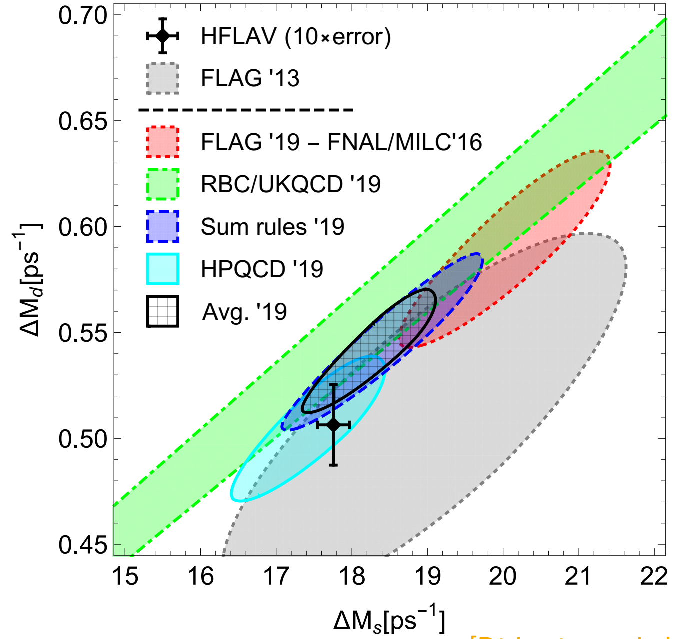
# Using Gradient Flow to Renormalise Matrix Elements for Meson Mixing and Lifetimes

Matthew Black

In collaboration with:
R. Harlander, F. Lange, A. Rago, A. Shindler, O. Witzel

August 2, 2023

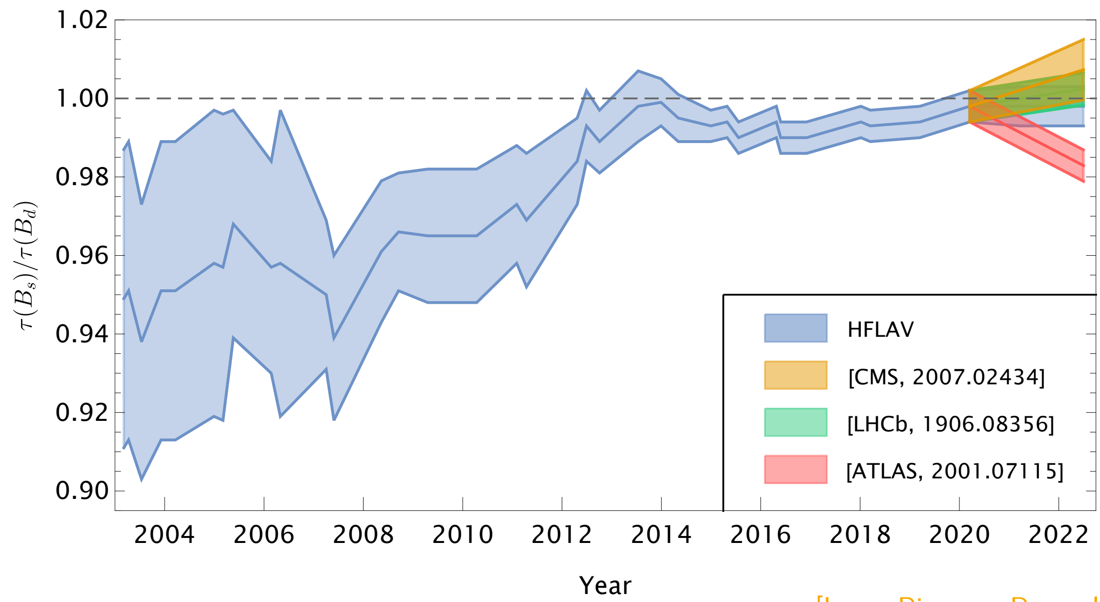
- $B$-meson mixing and lifetimes are measured experimentally to high precision
  - ➥ Key observables for probing New Physics ➡ **high precision in theory needed!**




[Di Luzio et al. '19]

➤ $B$-meson mixing and lifetimes are measured experimentally to high precision

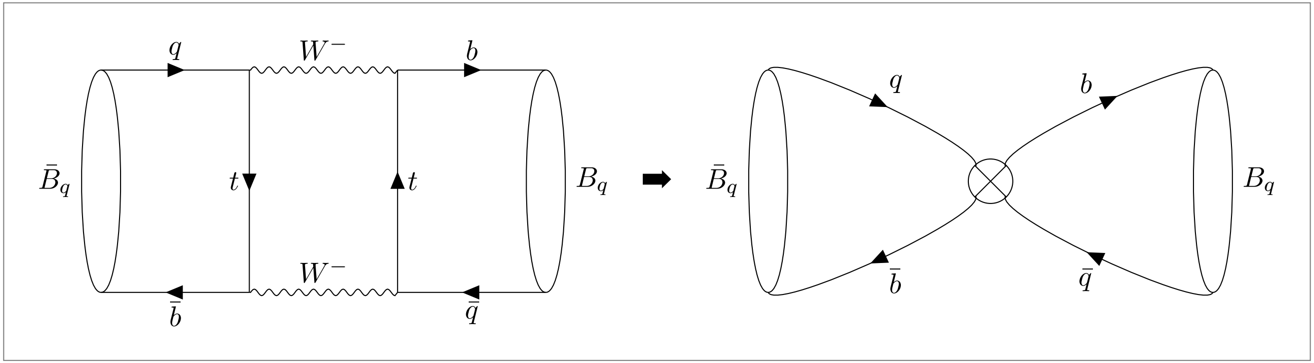
➥ Key observables for probing New Physics ➡ **high precision in theory needed!**




[Lenz, Piscopo, Rusov '22]

- $B$-meson mixing and lifetimes are measured experimentally to high precision
  - ➥ Key observables for probing New Physics ➡ **high precision in theory needed!**
- For lifetimes and decay rates, we use the **Heavy Quark Expansion**

$$\Gamma_{B_q} = \Gamma_3 \langle \mathcal{O}_{D=3} \rangle + \Gamma_5 \frac{\langle \mathcal{O}_{D=5} \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_{D=6} \rangle}{m_b^3} + \ldots + 16\pi^2 \left[ \tilde{\Gamma}_6 \frac{\langle \tilde{\mathcal{O}}_{D=6} \rangle}{m_b^3} + \tilde{\Gamma}_7 \frac{\langle \tilde{\mathcal{O}}_{D=7} \rangle}{m_b^4} + \ldots \right]$$




- Factorise observables into ➡ perturbative QCD contributions
  ➡ **Non-Perturbative Matrix Elements**

- Four-quark $\Delta B = 0$ and $\Delta B = 2$ matrix elements can be determined from lattice QCD simulations
- $\Delta B = 2$ well-studied by several groups ➡ precision increasing [Tsang, Wed 09:20]
  - ➥ preliminary $\Delta K = 2$ for Kaon mixing study with gradient flow [Suzuki et al. '20], [Taniguchi, Lattice '19]
- $\Delta B = 0$ ➡ exploratory studies from ∼20 years ago + new developments for lifetime ratios [Lin, Detmold, Meinel '22]
  - ➥ contributions from disconnected diagrams
  - ➥ mixing with lower dimension operators in renormalisation [Lin, Wed 09:00]

- Four-quark $\Delta B = 0$ and $\Delta B = 2$ matrix elements can be determined from lattice QCD simulations
- $\Delta B = 2$ well-studied by several groups ➡ precision increasing [Tsang, Wed 09:20]
  - ➥ preliminary $\Delta K = 2$ for Kaon mixing study with gradient flow [Suzuki et al. '20], [Taniguchi, Lattice '19]
- $\Delta B = 0$ ➡ exploratory studies from ∼20 years ago + new developments for lifetime ratios [Lin, Detmold, Meinel '22]
  - ➥ contributions from disconnected diagrams
  - ➥ mixing with lower dimension operators in renormalisation [Lin, Wed 09:00]

---

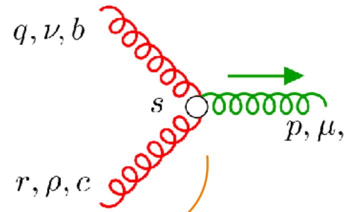
1. Establish gradient flow renormalisation procedure with $\Delta B = 2$ matrix elements
2. Pioneer **connected** $\Delta B = 0$ matrix element calculation
3. Tackle disconnected contributions

- Formulated by [Lüscher '10], [Lüscher '13] ➡ scale setting, RG $\beta$-function, **renormalisation**...
- Introduce auxiliary dimension, **flow time** $\tau$ as a way to regularise the UV
- Extend gauge and fermion fields in flow time and express dependence with first-order differential equations:

$$\partial_t B_\mu(\tau, x) = \mathcal{D}_\nu(\tau) G_{\nu\mu}(\tau, x), \quad B_\mu(0, x) = A_\mu(x),$$
$$\partial_t \chi(\tau, x) = \mathcal{D}^2(\tau)\chi(\tau, x), \quad \chi(0, x) = q(x).$$

- Re-express effective Hamiltonian in terms of 'flowed' operators:

$$\mathcal{H}_{\text{eff}} = \sum_n C_n \mathcal{O}_n = \sum_n \widetilde{C}_n(\tau)\widetilde{\mathcal{O}}_n(\tau).$$



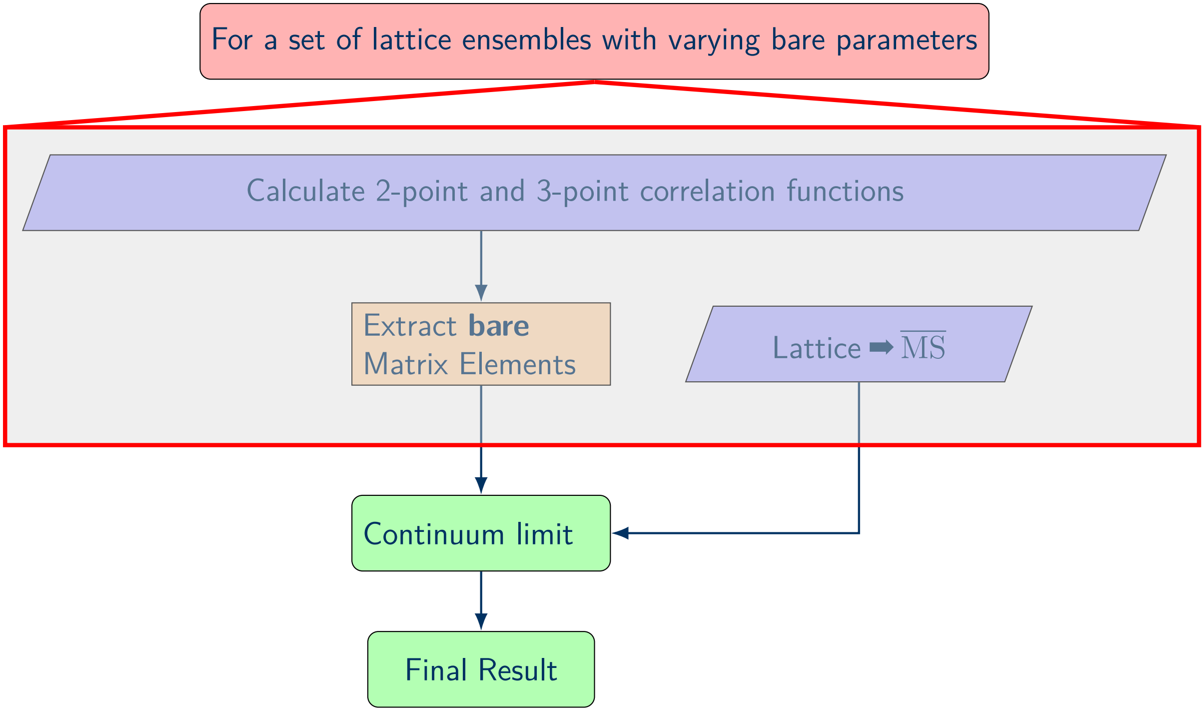

new Feynman diagrams

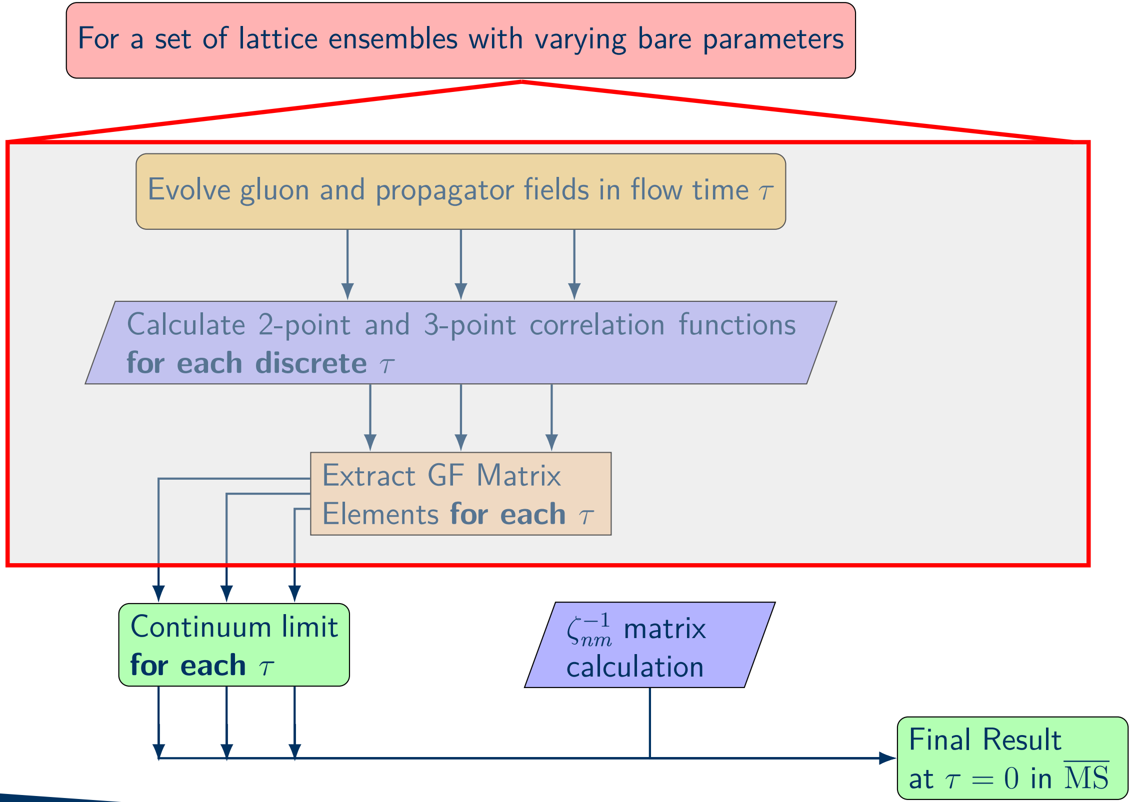
- Relate to regular operators in 'short-flow-time expansion':

$$\widetilde{\mathcal{O}}_n(\tau) = \sum_m \zeta_{nm}(\tau)\mathcal{O}_m + O(\tau)$$

'flowed' MEs calculated on lattice replacing $A_\mu, q \to B_\mu, \chi$

matching matrix calculated perturbatively


For a set of lattice ensembles with varying bare parameters
Calculate 2-point and 3-point correlation functions
Extract **bare** Matrix Elements
Lattice ➡ $\overline{\mathrm{MS}}$
Continuum limit
Final Result


For a set of lattice ensembles with varying bare parameters
Evolve gluon and propagator fields in flow time $\tau$
Calculate 2-point and 3-point correlation functions **for each discrete** $\tau$
Extract GF Matrix Elements **for each** $\tau$
Continuum limit **for each** $\tau$
$\zeta_{nm}^{-1}$ matrix calculation
Final Result at $\tau = 0$ in $\overline{\mathrm{MS}}$

➤ We will consider RBC/UKQCD's 2+1 flavour Shamir DWF + Iwasaki gauge action ensembles
[Shamir '93] [Iwasaki, Yoshie '84] [Iwasaki '85]

| | $L$ | $T$ | $a^{-1}$/GeV | $am_l^{\text{sea}}$ | $am_s^{\text{sea}}$ | $M_\pi$/MeV | srcs $\times$ N$_{\text{conf}}$ |
|---|---|---|---|---|---|---|---|
| C1 | 24 | 64 | 1.7848 | 0.005 | 0.040 | 340 | 32 $\times$ 101 |
| C2 | 24 | 64 | 1.7848 | 0.010 | 0.040 | 433 | 32 $\times$ 101 |
| M1 | 32 | 64 | 2.3833 | 0.004 | 0.030 | 302 | 32 $\times$ 79 |
| M2 | 32 | 64 | 2.3833 | 0.006 | 0.030 | 362 | |
| M3 | 32 | 64 | 2.3833 | 0.008 | 0.030 | 411 | |
| F1S | 48 | 96 | 2.785 | 0.002144 | 0.02144 | 267 | |

[Allton et al. '08]
[Aoki et al. '10]
[Blum et al. '14]
[Boyle et al. '17]

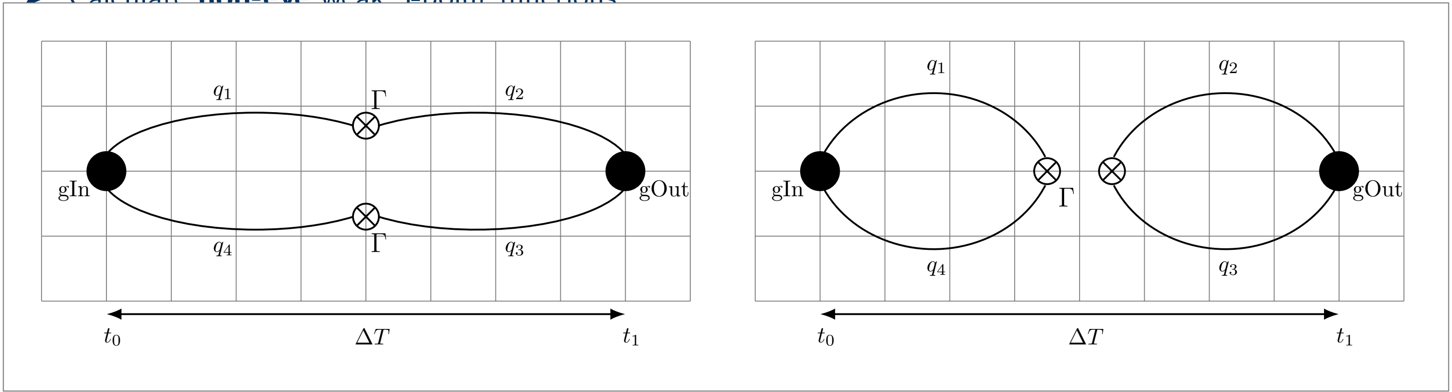
➤ Exploratory simulations on C1, C2, M1 so far

➤ To remove additional extrapolations in valence sector, we simulate at physical charm and strange

➥ "neutral $D_s$" meson mixing

- Fully-relativistic DWF for all valence quarks
- Strange quarks tuned to physical value (Shamir DWF)
- Heavy $c$ quarks tuned for physical $D_s$ mass (Möbius DWF)
  - $am_c \lesssim 0.7$ with stout smearing of gauge fields [Morningstar, Peardon '03]
- Z2 wall sources for all quark propagators [Boyle et al. '08]
  - Sources for strange propagators are also Gaussian smeared [Allton et al. '93]
- Calculate **non-eye** weak 3-point functions

- Fully-relativistic DWF for all valence quarks
- Strange quarks tuned to physical value (Shamir DWF)
- Heavy $c$ quarks tuned for physical $D_s$ mass (Möbius DWF)
  - $am_c \lesssim 0.7$ with stout smearing of gauge fields [Morningstar, Peardon '03]
- Z2 wall sources for all quark propagators [Boyle et al. '08]
  - Sources for strange propagators are also Gaussian smeared [Allton et al. '93]
- Calculate **non-eye** weak 3-point functions



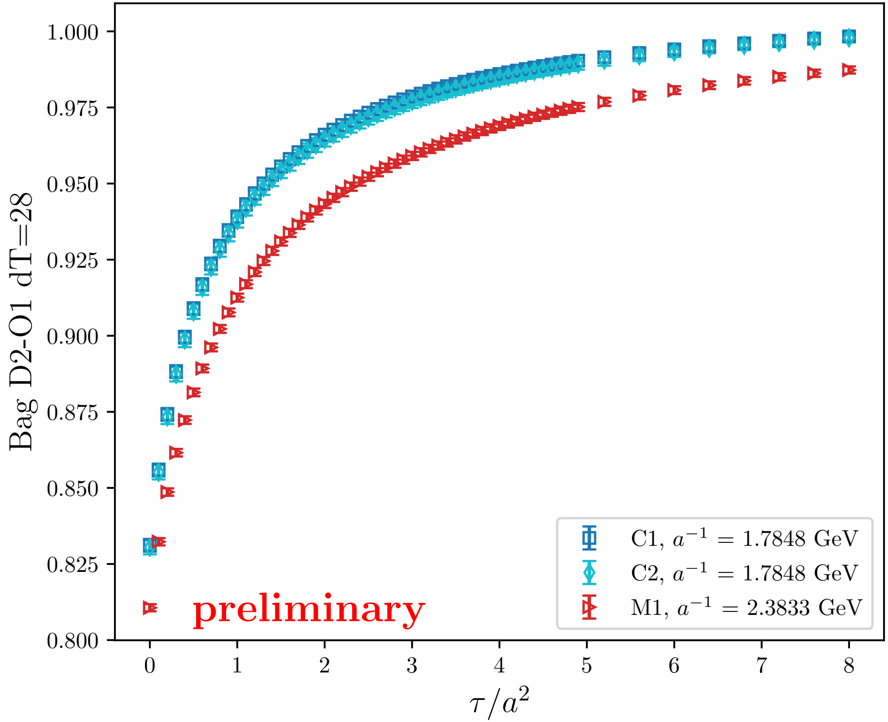
- Fully-relativistic DWF for all valence quarks
- Strange quarks tuned to physical value (Shamir DWF)
- Heavy $c$ quarks tuned for physical $D_s$ mass (Möbius DWF)
  - $am_c \lesssim 0.7$ with stout smearing of gauge fields [Morningstar, Peardon '03]
- Z2 wall sources for all quark propagators [Boyle et al. '08]
  - Sources for strange propagators are also Gaussian smeared [Allton et al. '93]
- Calculate **non-eye** weak 3-point functions
- Valence simulations carried out using Hadrons [Portelli et al. '22]
- Implemented gradient flow in Hadrons with 4D Laplacian for fermion evolution
- Gauge and fermion fields evolved with $\epsilon = 0.01$
- Measurements taken every 10 steps for $\tau/a^2 < 5$

- Look at Bag parameters and their behaviour with the flow time $\tau$

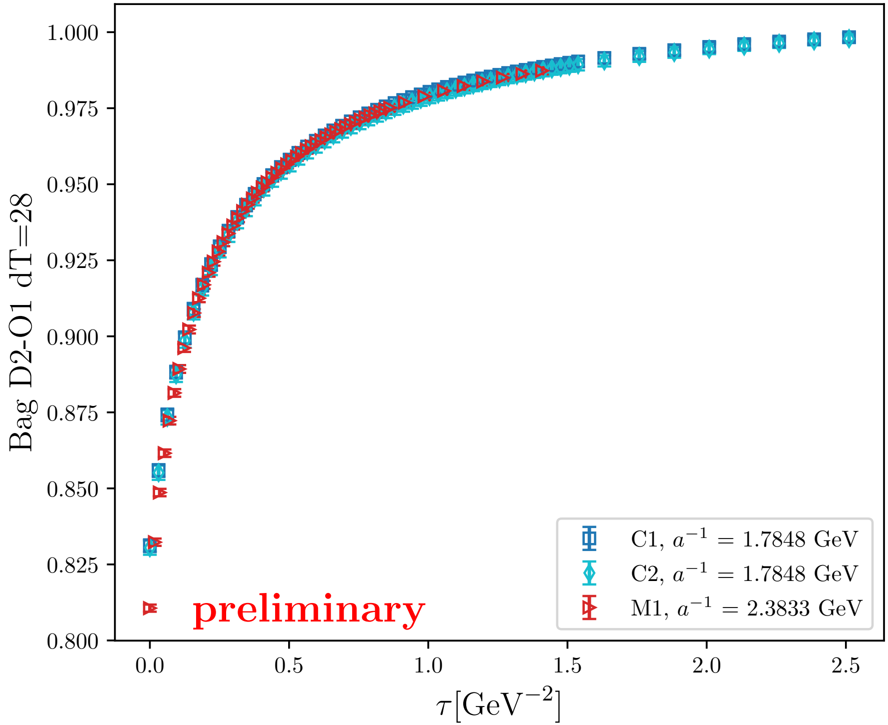
$$R_1(\tau) = \frac{C^{\rm 3pt}_{\mathcal{O}_1}(t,\Delta t,\tau)}{C^{\rm 2pt}_{AP}(t,\tau)\,C^{\rm 2pt}_{PA}(\Delta T - t,\tau)} \to B_1(\tau),$$

$$R_i(\tau) = \frac{C^{\rm 3pt}_{\mathcal{O}_i}(t,\Delta t,\tau)}{C^{\rm 2pt}_{PP}(t,\tau)\,C^{\rm 2pt}_{PP}(\Delta T - t,\tau)} \to B_i(\tau),\, i = 2 \to 5$$

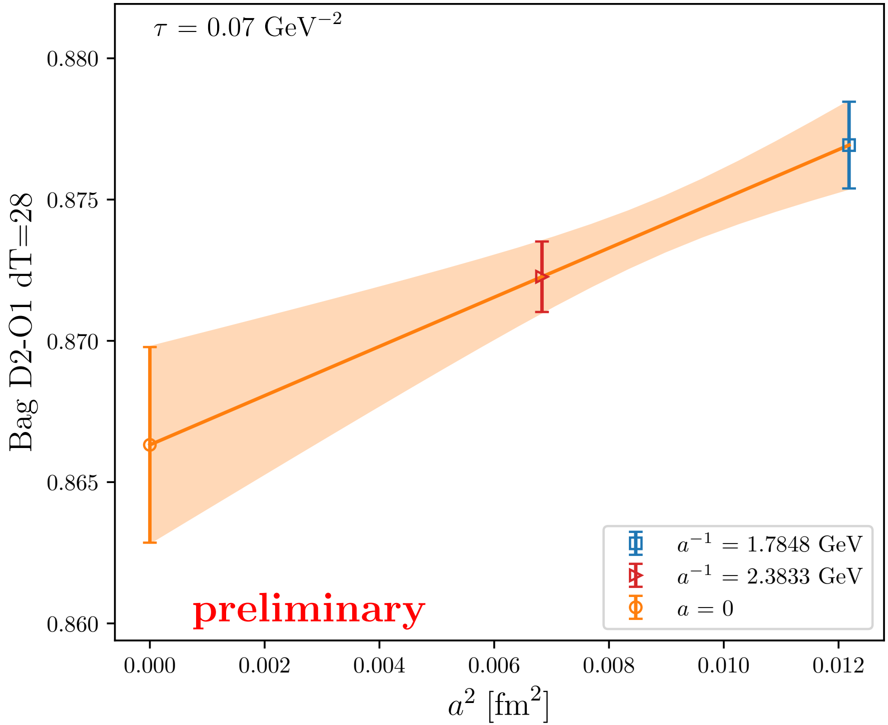
- We seek a **suitable window** in flow time where the $\tau$-dependences of flowed matrix elements and perturbative coefficients cancel

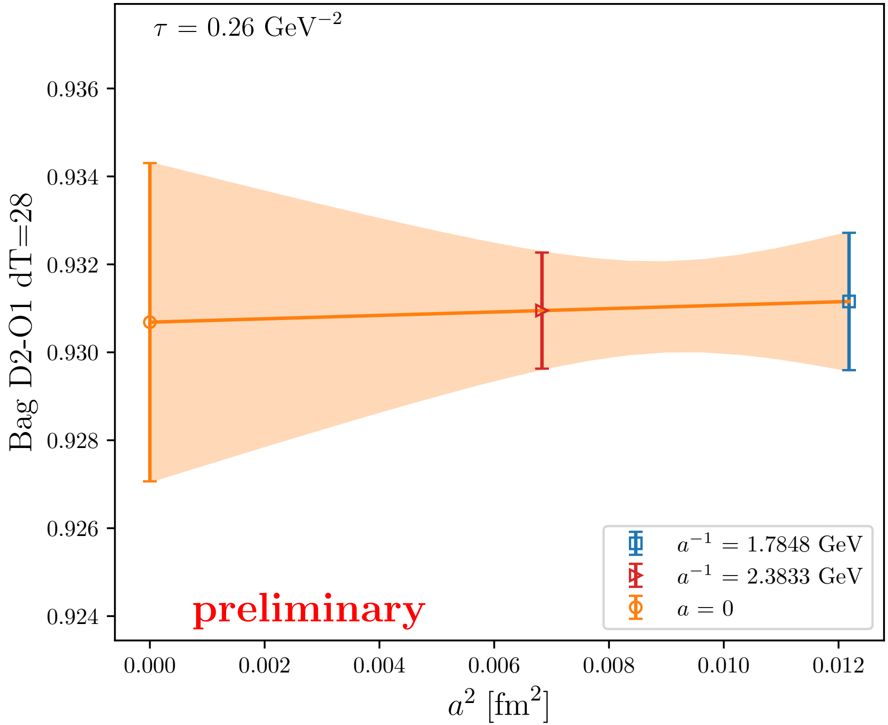


Bag D2-O1 dT=28
1.000
0.975
0.950
0.925
0.900
0.875
0.850
0.825
0.800
0
1
2
3
4
5
6
7
8
$\tau/a^2$
C1, $a^{-1}$ = 1.7848 GeV
C2, $a^{-1}$ = 1.7848 GeV
M1, $a^{-1}$ = 2.3833 GeV
preliminary




➤ different lattice spacings overlap in physical flow time ➡ mild continuum limit


$\tau = 0.07$ GeV$^{-2}$
0.880
0.875
0.870
0.865
0.860
Bag D2-O1 dT=28
preliminary
$a^{-1} = 1.7848$ GeV
$a^{-1} = 2.3833$ GeV
$a = 0$
0.000
0.002
0.004
0.006
0.008
0.010
0.012
$a^2$ [fm$^2$]




➤ continuum limit very flat at positive flow time ✔

➤ Relate to regular operators in 'short-flow-time expansion':

$$\widetilde{\mathcal{O}}_n(\tau) = \sum_m \zeta_{nm}(\tau)\mathcal{O}_m + O(\tau)$$

'flowed' MEs calculated on lattice

matching matrix
calculated perturbatively

- Relate to regular operators in 'short-flow-time expansion':

$$\widetilde{\mathcal{O}}_n(\tau) = \sum_m \zeta_{nm}(\tau)\mathcal{O}_m + O(\tau)$$

'flowed' MEs calculated on lattice

matching matrix calculated perturbatively

$$\sum_n \zeta^{-1}_{nm}(\mu,\tau)\langle\widetilde{\mathcal{O}}_n\rangle(\tau) = \langle\mathcal{O}_m\rangle(\mu)$$

- Relate to regular operators in 'short-flow-time expansion':

$$\widetilde{\mathcal{O}}_n(\tau) = \sum_m \zeta_{nm}(\tau)\mathcal{O}_m + O(\tau)$$

'flowed' MEs calculated on lattice

matching matrix
calculated perturbatively

$$\sum_n \zeta^{-1}_{nm}(\mu,\tau)\langle\widetilde{\mathcal{O}}_n\rangle(\tau) = \langle\mathcal{O}_m\rangle(\mu)$$

- Calculated at two-loop for $\mathcal{B}_1$ based on [Harlander, Lange '22]:

$$\zeta^{-1}_{\mathcal{B}_1}(\mu,\tau) = 1 + \frac{a_s}{4}\left(-\frac{11}{3} - 2L_{\mu\tau}\right) + \frac{a_s^2}{43200}\Big[-2376 - 79650L_{\mu\tau} - 24300L_{\mu\tau}^2 + 8250n_f + 6000\,n_f L_{\mu\tau}$$
$$+ 1800\,n_f L_{\mu\tau}^2 - 2775\pi^2 + 300\,n_f\pi^2 - 241800\log 2$$
$$+ 202500\log 3 - 110700\,\mathrm{Li}_2\left(\frac{1}{4}\right)\Big]$$

$$L_{\mu\tau} = \log(2\mu^2\tau) + \gamma_E, \quad \mu = \frac{C}{\sqrt{\tau}}$$

- Choose $C$ such that logs remain small

➤ Relate to regul

'flowed' MEs c

rbatively

➤ Calculated at t

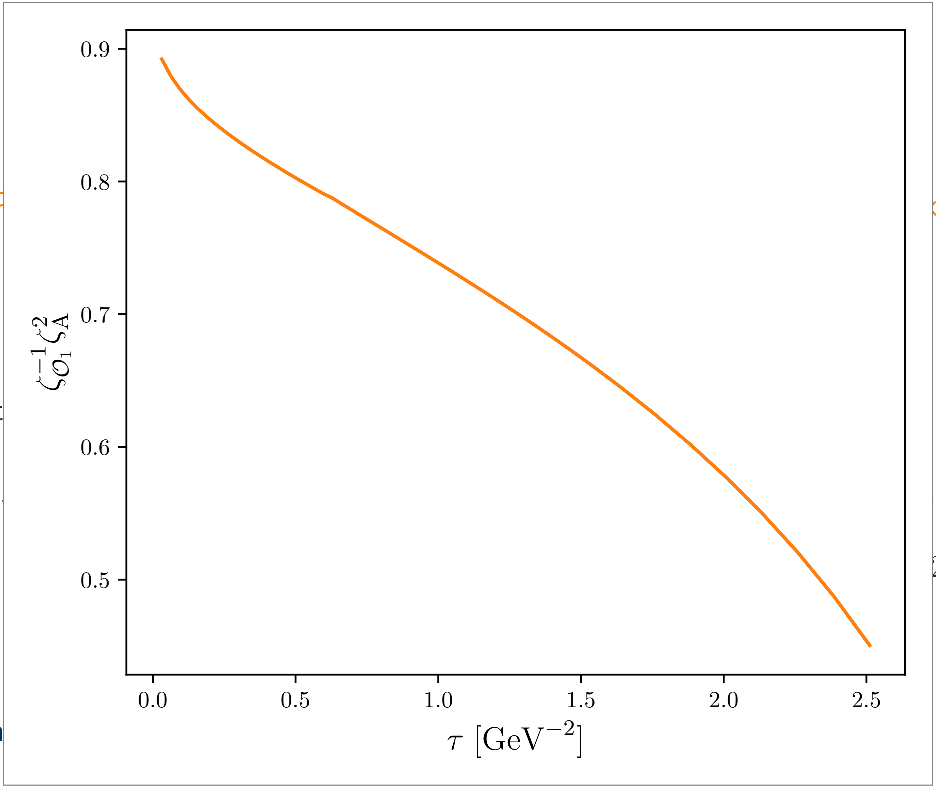
$\zeta_{\mathcal{B}_1}^{-1}(\mu,\tau) = 1$

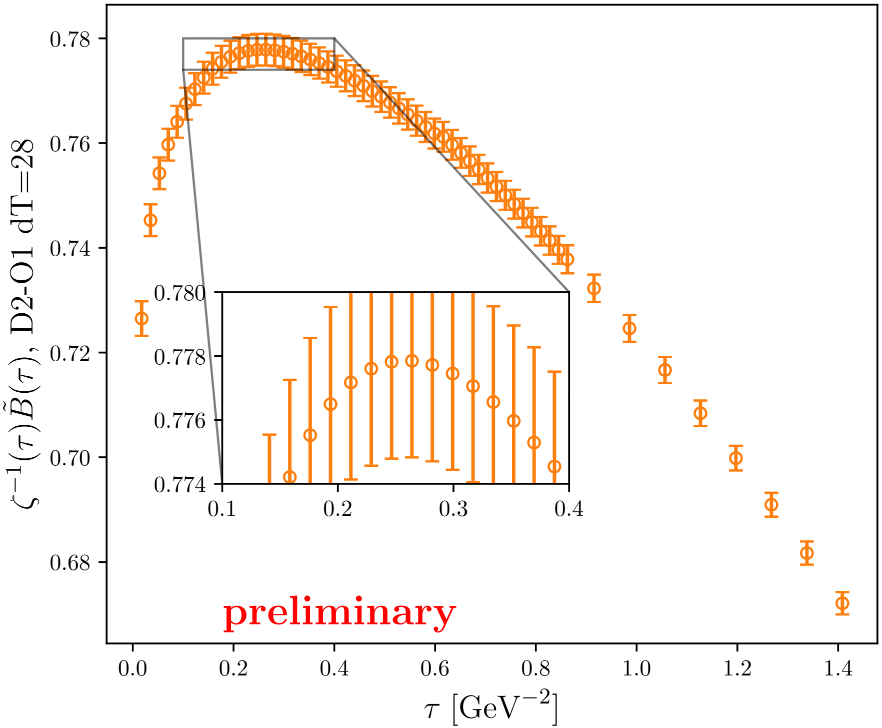
$0 n_f + 6000\, n_f L_{\mu\tau}$

$241800 \log 2$

$L_{\mu\tau} = \log(2\mu^2$

➤ Choose $C$ such






➤ Better choices for $\mu$ may exist to extend 'short-flow-time region'

- $\Delta B = 0$ four-quark matrix elements are strongly-desired quantities
  - Standard renormalisation introduces mixing with operators of lower mass dimension
  - We aim to use the fermionic gradient flow as a non-perturbative renormalisation procedure
- Testing method first with $\Delta Q = 2$ matrix elements
- Shown first simulations for $\Delta C = 2$

---

**Next Steps:**

- Simulate on all ensembles with multiple valence quark masses
- Extrapolate to physical $B_{(s)}$ meson mixing
- Repeat analysis for quark-line connected $\Delta B = 0$ matrix elements
- Consider disconnected contributions