Using Gradient Flow to Renormalise Matrix Elements for Meson Mixing and Lifetimes

Matthew Black

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**Introduction**

- $B$-meson mixing and lifetimes are measured experimentally to high precision
- Key observables for probing New Physics ➤ **high precision in theory needed!**

![Graph showing key observables for probing New Physics](image)

**Using GF to Renormalise**

**Matrix Elements for Mixing and Lifetimes**

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B-meson mixing and lifetimes are measured experimentally to high precision

Key observables for probing New Physics ➔ high precision in theory needed!

Using GF to Renormalise Matrix Elements for Mixing and Lifetimes

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**Introduction**

- $B$-meson mixing and lifetimes are measured experimentally to high precision
- Key observables for probing New Physics ⇒ **high precision in theory needed!**
- For lifetimes and decay rates, we use the **Heavy Quark Expansion**

$$
\Gamma_{B_q} = \Gamma_3 \langle O_{D=3} \rangle + \Gamma_5 \frac{\langle O_{D=5} \rangle}{m_b^2} + \Gamma_6 \frac{\langle O_{D=6} \rangle}{m_b^3} + \ldots + 16\pi^2 \left[ \tilde{\Gamma}_6 \frac{\langle \tilde{O}_{D=6} \rangle}{m_b^3} + \tilde{\Gamma}_7 \frac{\langle \tilde{O}_{D=7} \rangle}{m_b^4} + \ldots \right]
$$

- Factorise observables into ⇒ perturbative QCD contributions
  ⇒ **Non-Perturbative Matrix Elements**

**Using GF to Renormalise**

**Matrix Elements for Mixing and Lifetimes**

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Four-quark $\Delta B = 0$ and $\Delta B = 2$ matrix elements can be determined from lattice QCD simulations

$\Delta B = 2$ well-studied by several groups ➔ precision increasing [Tsang, Wed 09:20]
  ➔ preliminary $\Delta K = 2$ for Kaon mixing study with gradient flow [Suzuki et al. '20], [Taniguchi, Lattice '19]

$\Delta B = 0$ ➔ exploratory studies from $\sim$20 years ago + new developments for lifetime ratios [Lin, Detmold, Meinel '22]
  ➔ contributions from disconnected diagrams
  ➔ mixing with lower dimension operators in renormalisation [Lin, Wed 09:00]
Four-quark $\Delta B = 0$ and $\Delta B = 2$ matrix elements can be determined from lattice QCD simulations.

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Preliminary $\Delta K = 2$ for Kaon mixing study with gradient flow [Suzuki et al. '20], [Taniguchi, Lattice '19].

$\Delta B = 0$ are exploratory studies from ~20 years ago plus new developments for lifetime ratios [Lin, Detmold, Meinel '22].

Contributions from disconnected diagrams.

Mixing with lower dimension operators in renormalisation [Lin, Wed 09:00].

1. Establish gradient flow renormalisation procedure with $\Delta B = 2$ matrix elements.

2. Pioneer **connected** $\Delta B = 0$ matrix element calculation.

3. Tackle disconnected contributions.
Formulated by [Lüscher '10], [Lüscher '13] ➔ scale setting, RG β-function, renormalisation...

Introduce auxiliary dimension, flow time $\tau$ as a way to regularise the UV

Extend gauge and fermion fields in flow time and express dependence with first-order differential equations:

$$
\partial_t B_\mu(\tau, x) = D_\nu(\tau) G_{\nu\mu}(\tau, x), \quad B_\mu(0, x) = A_\mu(x),
\partial_t \chi(\tau, x) = D^2(\tau) \chi(\tau, x), \quad \chi(0, x) = q(x).
$$

Re-express effective Hamiltonian in terms of 'flowed' operators:

$$
\mathcal{H}_{\text{eff}} = \sum_n C_n O_n = \sum_n \tilde{C}_n(\tau) \tilde{O}_n(\tau).
$$

Relate to regular operators in 'short-flow-time expansion':

$$
\tilde{O}_n(\tau) = \sum_m \zeta_{nm}(\tau) O_m + O(\tau)
$$

'flowed' MEs calculated on lattice replacing $A_\mu, q \rightarrow B_\mu, \chi$
For a set of lattice ensembles with varying bare parameters

Calculate 2-point and 3-point correlation functions

Extract \textit{bare} Matrix Elements

Lattice \Rightarrow \overline{\text{MS}}

Continuum limit

Final Result
For a set of lattice ensembles with varying bare parameters

Evolve gluon and propagator fields in flow time $\tau$

Calculate 2-point and 3-point correlation functions for each discrete $\tau$

Extract GF Matrix Elements for each $\tau$

Continuum limit for each $\tau$

$\zeta_{nm}^{-1}$ matrix calculation

Final Result at $\tau = 0$ in $\overline{\text{MS}}$
We will consider RBC/UKQCD's 2+1 flavour Shamir DWF + Iwasaki gauge action ensembles

<table>
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<th></th>
<th>L</th>
<th>T</th>
<th>(a^{-1}/\text{GeV})</th>
<th>(am_l^{\text{sea}})</th>
<th>(am_s^{\text{sea}})</th>
<th>(M_\pi/\text{MeV})</th>
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<td>267</td>
<td></td>
</tr>
</tbody>
</table>

Exploratory simulations on C1, C2, M1 so far

To remove additional extrapolations in valence sector, we simulate at physical charm and strange

"neutral \(D_s\)" meson mixing
Lattice Simulation

- Fully-relativistic DWF for all valence quarks
- Strange quarks tuned to physical value (Shamir DWF)
- Heavy $c$ quarks tuned for physical $D_s$ mass (Möbius DWF)
  - $am_c \lesssim 0.7$ with stout smearing of gauge fields [Morningstar, Peardon '03]
- $Z_2$ wall sources for all quark propagators [Boyle et al. '08]
  - Sources for strange propagators are also Gaussian smeared [Allton et al. '93]
- Calculate non-eye weak 3-point functions
- Fully-relativistic DWF for all valence quarks
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- Sources for strange propagators are also Gaussian smeared [Allton et al. '93]

Calculate non-eye weak 3-point functions

Valence simulations carried out using Hadrons [Portelli et al. '22]

Implemented gradient flow in Hadrons with 4D Laplacian for fermion evolution

Gauge and fermion fields evolved with $\epsilon = 0.01$

Measurements taken every 10 steps for $\tau/a^2 < 5$
Look at Bag parameters and their behaviour with the flow time $\tau$

\[
R_1(\tau) = \frac{C^{3\text{pt}}_{O_1}(t, \Delta t, \tau)}{C^{2\text{pt}}_{AF}(t, \tau) C^{2\text{pt}}_{PA} (\Delta T - t, \tau)} \rightarrow B_1(\tau),
\]

\[
R_i(\tau) = \frac{C^{3\text{pt}}_{O_i}(t, \Delta t, \tau)}{C^{2\text{pt}}_{PP}(t, \tau) C^{2\text{pt}}_{PP} (\Delta T - t, \tau)} \rightarrow B_i(\tau), \quad i = 2 \rightarrow 5
\]

We seek a suitable window in flow time where the $\tau$-dependences of flowed matrix elements and perturbative coefficients cancel.
Using GF to Renormalise Matrix Elements for Mixing and Lifetimes

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Continuum Limit

\[ \tau = 0.26 \text{ GeV}^{-2} \]

\[ a^2 [\text{fm}^2] \]

-\[ a^{-1} = 1.7848 \text{ GeV} \]
-\[ a^{-1} = 2.3833 \text{ GeV} \]
-\[ a = 0 \]

-\[ \text{continuum limit very flat at positive flow time} \]
Combine with perturbative matching $\rightarrow \overline{\text{MS}}$

- Relate to regular operators in 'short-flow-time expansion':

$$\tilde{O}_n(\tau) = \sum_m \zeta_{nm}(\tau)O_m + O(\tau)$$

'flowed' MEs calculated on lattice

matching matrix calculated perturbatively

Using GF to Renormalise
Matrix Elements for Mixing and Lifetimes
Combine with perturbative matching $\rightarrow \overline{\text{MS}}$

- Relate to regular operators in 'short-flow-time expansion':

\[
\tilde{O}_n(\tau) = \sum_m \zeta_{nm}(\tau) O_m + O(\tau)
\]

'flowed' MEs calculated on lattice

\[
\sum_n \zeta_{nm}^{-1}(\mu, \tau) \langle \tilde{O}_n(\tau) \rangle = \langle O_m(\mu) \rangle
\]

mathcing matrix calculated perturbatively

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Combine with perturbative matching \( \rightarrow \overline{\text{MS}} \)

- Relate to regular operators in 'short-flow-time expansion':

\[
\tilde{O}_n(\tau) = \sum_m \zeta_{nm}(\tau) O_m + O(\tau)
\]

'flowed' MEs calculated on lattice

\[
\sum_n \zeta_{nm}^{-1}(\mu, \tau) \langle \tilde{O}_n \rangle(\tau) = \langle O_m \rangle(\mu)
\]

- Calculated at two-loop for \( B_1 \) based on [Harlander, Lange '22]:

\[
\zeta_{B_1}^{-1}(\mu, \tau) = 1 + \frac{a_s}{4} \left( -\frac{11}{3} - 2L_{\mu\tau} \right) + \frac{a_s^2}{43200} \left[ -2376 - 79650L_{\mu\tau} - 24300L_{\mu\tau}^2 + 8250n_f + 6000n_fL_{\mu\tau} \\
+ 1800n_fL_{\mu\tau}^2 - 2775\pi^2 + 300n_f\pi^2 - 241800 \log 2 \\
+ 202500 \log 3 - 110700 \text{Li}_2 \left( \frac{1}{4} \right) \right]
\]

\[
L_{\mu\tau} = \log(2\mu^2\tau) + \gamma_E, \quad \mu = \frac{C}{\sqrt{\tau}}
\]

- Choose \( C \) such that logs remain small
Using GF to Renormalise Matrix Elements for Mixing and Lifetimes

![Graph showing the relationship between \( \tau \) and \( \zeta^{-1} \)]

- Combine with perturbative matching \( \rightarrow \overline{\text{MS}} \)
- Relate to regular operators in 'short-flow-time expansion':
  \[
  \sim O_n(\tau) = \sum m \zeta^{-1}_{nm}(\tau) O_m(\tau) + O(\tau)
  \]

- Calculated at two-loop for \( B_1 \) based on [Harlander, Lange '22]:
  \[
  \zeta^{-1}_{B_1}(\mu, \tau) = 1 + a_s^4 \left( -\frac{11}{3} - 2L_{\mu\tau} \right) + 2a_s^2 \left( 0n_f + 6000 n_f L_{\mu\tau} \right) + 241800 \log 2
  \]

- Choose \( C \) such that logs remain small

\( L_{\mu\tau} = \log(2\mu^2/\tau) \)
Better choices for $\mu$ may exist to extend 'short-flow-time region'
Summary

- $\Delta B = 0$ four-quark matrix elements are strongly-desired quantities
  - Standard renormalisation introduces mixing with operators of lower mass dimension
  - We aim to use the fermionic gradient flow as a non-perturbative renormalisation procedure

- Testing method first with $\Delta Q = 2$ matrix elements

- Shown first simulations for $\Delta C = 2$

Next Steps:
- Simulate on all ensembles with multiple valence quark masses
- Extrapolate to physical $B_{(s)}$ meson mixing
- Repeat analysis for quark-line connected $\Delta B = 0$ matrix elements
- Consider disconnected contributions