Simulations of Hyperbolic Ising Model

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Motivations

- Our main goal in this talk is to develop classical and quantum simulations to probe AdS/CFT correspondence.
- A great candidate for this is the Hyperbolic Ising chain since it can be easily simulated using DMRG and TEBD algorihtms as well as usual quantum simulation techniques.
- Ising Hamiltonian matches very closely to Rydberg Hamiltonian which opens posibilities to use Rydberg Arrays for quantum simulating this model.
- Information spread in hyperbolic spaces has many applications both in physics and quantum information sciences.

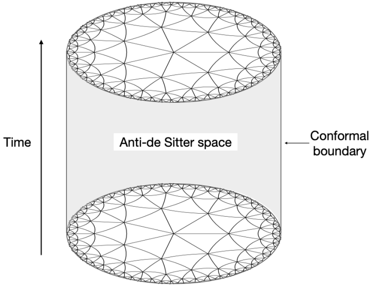
Hyperbolic Space & AdS/CFT

- The AdS/CFT correspondence is a very powerful tool that provides a duality between strongly coupled d-dimensional critical systems and weakly coupled $d+1$ dimensional gravitational theories on a negatively curved background
- The the d-dimensional non-gravitational conformal theory resides on the boundary of AdS_{d+1} which makes this duality holographic in it's nature.

AdS Space

Euclidean AdS_{d+1} has the following metric which has the topology of a cylinder $\mathbb{R} \times \mathbb{H}^d$. [1]

$$ds^2 = g_{00}dt^2 + ds^2_{\mathbb{H}^d} \tag{1}$$



$$ds^2 = \pm\ell^2 \cosh^2 \rho dt^2 + \ell^2(d\rho^2 + \sinh^2 \rho d\Omega^2_{d-1}) \tag{2}$$

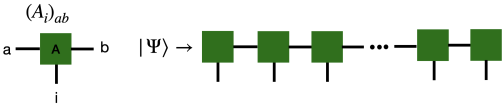
[1] Image taken from R. Brower et al. arXiv:2202.03464

AdS_2 Ising Hamiltonian

- For AdS_2 where $d = 1$ this cylindrical form reduces to a strip with $1D$ conformal quantum mechanics at the each end and leads to the following Ising Hamiltonian

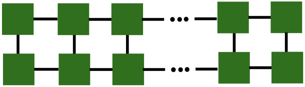
$$H_{AdS} = -J \sum_{<ij>} (\frac{\cosh(\rho_i) + \cosh(\rho_j)}{2}) \sigma_i^z \sigma_j^z + h \sum_i \cosh(\rho_i) \sigma_i^x + m \sum_i \cosh(\rho_i) \sigma_i^z$$

A Matrix Product State (MPS) $|\psi\rangle$ can be represented as,



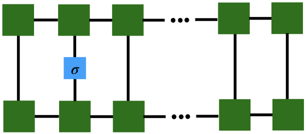
Inner products are given as,

$\langle\Psi|\Psi\rangle \rightarrow$



And expectation values,

$\langle\Psi|\sigma|\Psi\rangle \rightarrow$



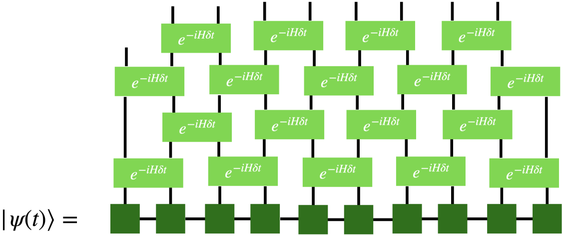
- Density Matrix Renormalization Group (DMRG) is a variational optimization technique that allows us to find the ground state of a system that's represented by a MPS.
- The idea is to minimize,

$$\langle\psi_A|\hat{H}|\psi_A\rangle - \lambda(\langle\psi_A|\psi_A\rangle - 1) \tag{3}$$

- This minimization corresponds to solving a generalized eigenvalue problem which can be performed one site at a time and sweeping across the chain.
- When we obtain the desired accuracy for the ground state energy E the corresponding MPS state gives an approximation of the ground state.

MPS,DMRG & TEBD

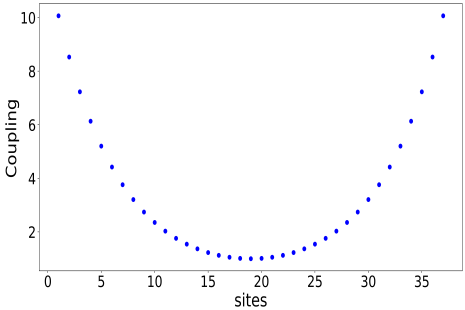
- Time-evolving block decimation (TEBD) algorithm is used to simulate the time evolution of one dimensional quantum many body systems.
- The basic idea is to represent an initial state $|\psi\rangle$ as an MPS and applying the trotterized time evolution operators that are expressed in an MPO form.



DMRG simulation of the AdS_2 Hamiltonian

- For simulations of this model using DMRG, we need to find a way to control the hyperbolic deformation $\cosh(\rho_i)$ for any given chain size N

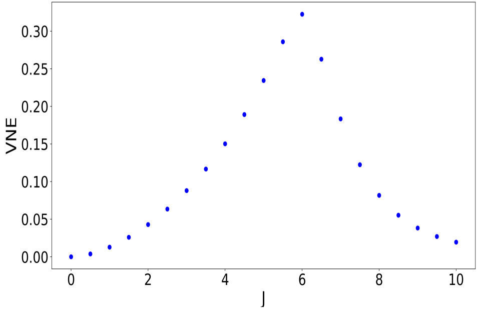
1. Replace $\cosh(\rho_i)$ with $\cosh(l_i)$ where l_i ranges from $-l_{max}$ to l_{max} from the first site to the last one.
2. We start at the first site with $\cosh(-l_{max})$ Then we increase l_i in increments of $\delta_l = 2l/(N-1)$ until we reach $\cosh(l_{max})$ at the last site.



Ground State Properties

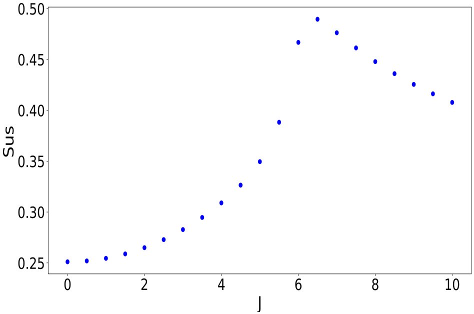
Using the DMRG algorithm we can investigate the ground state properties of the Hyperbolic Ising Model,

- First we calculate the half-chain Von-Neumann entropy for $N = 37, l_{max} = 3.0, h = 3.0, m = 0.25$



Magnetic Susceptibility

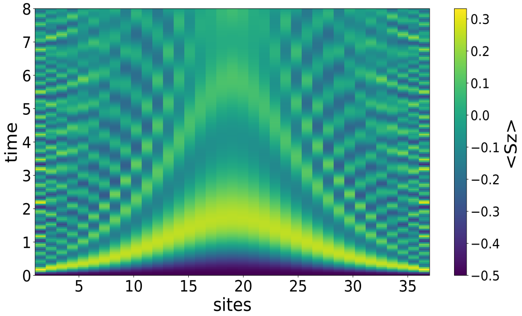
Next, we calculate the Magnetic Susceptibility for the same parameters,



Both the entropy and susceptibility peaks around $J/h = 2.0$ signaling a possible phase transition in the model

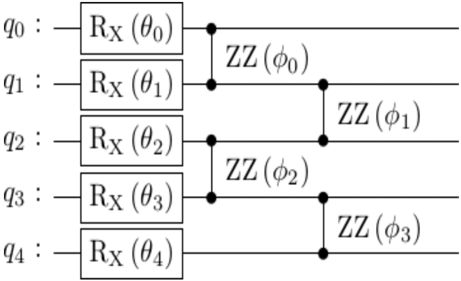
Time-Evolution of the Model

- To obtain the time evolution we use the TEBD algorithm.
- Below we plot the time evolution of $\langle S_z \rangle$ for $N = 37, l_{max} = 3.0, h = 2.0$, and $J = 2.0, m = 0.25$



- This interesting warping effect in the bulk can be related to the time-dilation of the coefficient $g_{00}(\rho)$

- This Hamiltonian can be easily generalized to obtain the Suzuki-Trotter evolution on a Universal Quantum Computer



For the results of our quantum simulations see the next talk by Muhammad Asaduzzaman

Rydberg Simulation

We have the following Hamiltonian for the Rydberg atoms which can be related to the Ising Hamiltonian as followsl,

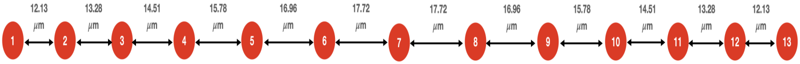
$$\hat{H}_R(t) = \sum_j \frac{\Omega_j(t)}{2} \underbrace{\left(e^{i\phi_j(t)} |g_j\rangle \langle r_j| + e^{-i\phi_j(t)} |r_j\rangle \langle g_j|\right)}_{\sigma_x} - \sum_j \Delta_j(t) \underbrace{\hat{n}_j}_{\sigma_z} + \sum_{j<k} V_{jk} \underbrace{\hat{n}_j \hat{n}_k}_{\sigma_z \sigma_z}$$

Where $V_{jk} = C_6/|r_j - r_k|^6$ and $C_6 = 2\pi \times 862690 \text{MHz}\mu\text{m}^6$

Rydberg Simulation

- To get a hyperbolic Ising like model with this Hamiltonian we need to adjust the separation between the atoms such that V_{jk} matches the form of the hyperbolic deformation by iteratively solving

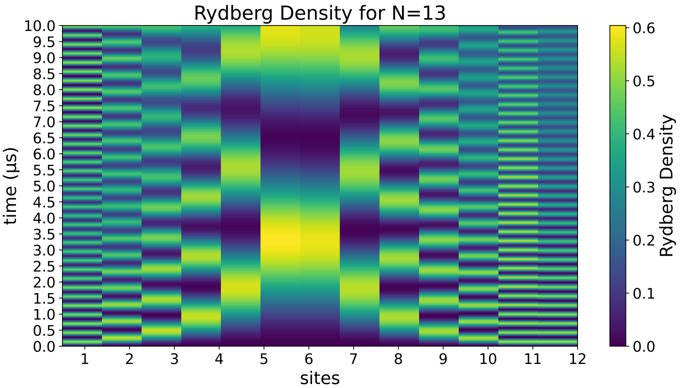
$$\delta_{i+1} = (A/\eta_i)^{1/6} + r_i \quad (4)$$



Doing this procedure for $l_{max} = 3$ results with $N = 13$ results in distances that range in between 12.13μm to 17.72μm, and the furthest atom being located at 180.77μm.

Simulation of the Rydberg Hamiltonian

- With all the ingredients of the Hamiltonian set we can calculate the corresponding Rydberg density $\langle n_i \rangle$ for the system.
- For Rydberg Simulations we used the Bloqade Software package developed by QuEra.



Out of Time Ordered Correlators & Information Spread

- Now we focus on the question of how information spreads in the Hyperbolic Ising chain, for that we calculate Out of Time Ordered Correlators (OTOCs)

In general OTOCs have the following form

$$F_r(t) = Tr(W(t)^\dagger V_r^\dagger W(t) V_r) \tag{5}$$

For the Ising case W and V can be taken as local Pauli operators.

OTOCs & Information Spread

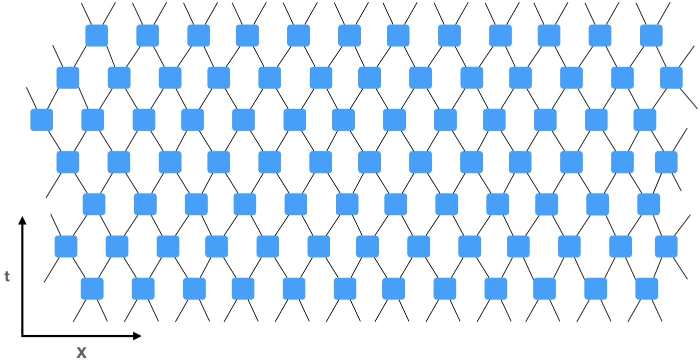
The connection between OTOC and operator growth can be made explicit by introducing the squared commutator.

$$C(r,t) = \frac{1}{Tr\mathbb{I}} Tr([W(t), V_r]^\dagger [W(t), V_r]) = 2 - 2F(r,t) \tag{6}$$

- The squared commutator depends on the number of the degrees of freedom $W(t)$ acts on.
- At $t = 0$ $W(t)$ acts only on one site and commutes with V_r that is located away from W so $C(r,t) = 0$
- As the system evolves under time, $W(t)$ becomes more and more non-local and starts to overlap with V_r which results in an increase in $C(r,t)$

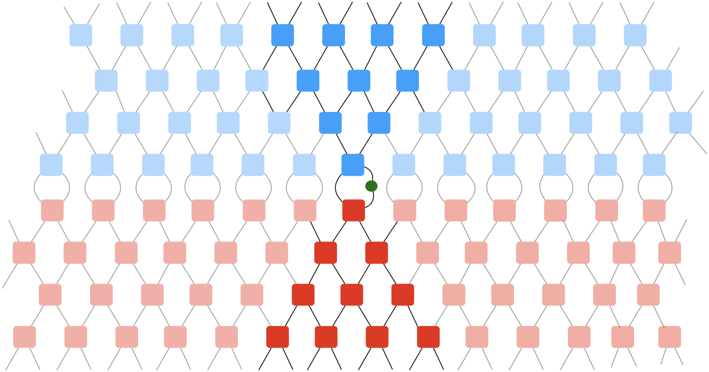
Lightcone for Operator Growth

To make this relation more explicit consider the unitary time evolution operator constructed out of local two qubit unitaries.



Lightcone for Operator Growth

Then the Heisenberg time evolution for a local operator $A(-t) = UAU^{\dagger}$ is given by,



OTOCs & Information Spread

- In our calculations for the OTOC using TEBD we used this specific form. [2]

$$O(t) = Tr(\rho W(t) V^{\dagger} W(t) V) / Tr(\rho W(t)^2 V^{\dagger} V) \tag{7}$$

This definition ensures that $O(t) = 1$ when $W(t)$ and V commute

- Taking $\rho \sim \mathbb{I}/D$ one can easily take the infinite temperature limit
- Instead of calculating the trace we can also calculate the expectation value of this operator on eigenstates of the Hamiltonian

[2] B. Vermersch et al. arXiv:1807.09087

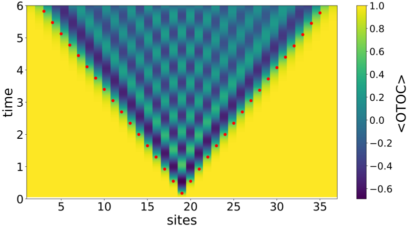
OTOCs & Information Spread

- How $O(t)$ spreads through the chain distinguishes between different kinds of scrambling

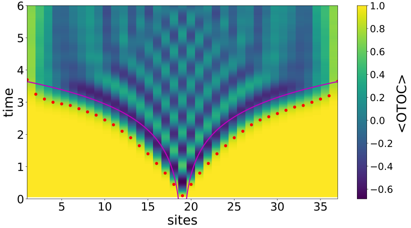
1. $O(t) \sim log(\lambda d) \rightarrow$ fast scramblers like the SYK model and Black Holes.
2. $O(t) \sim \lambda d^n \rightarrow$ systems with infinite/long range interactions
3. $O(t) \sim \lambda d \rightarrow$ systems that saturate the Lieb-Robinson bound

OTOCs in Hyperbolic Ising Model

Let's start our discussion with OTOC calculations at the infinite temperature limit.

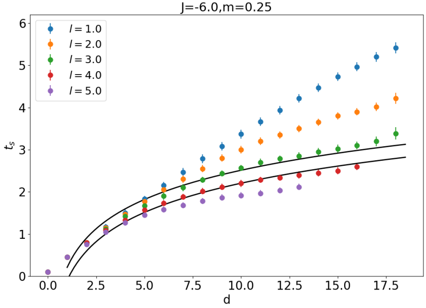


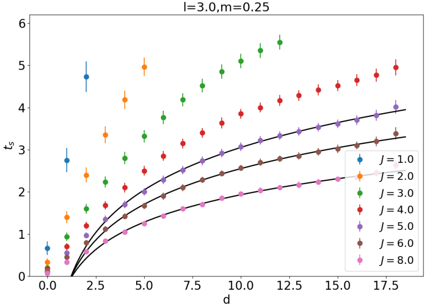
$N = 37, J = 6.0, h = 3.05,$
$m = 0.25, l_{max} = 0.0$



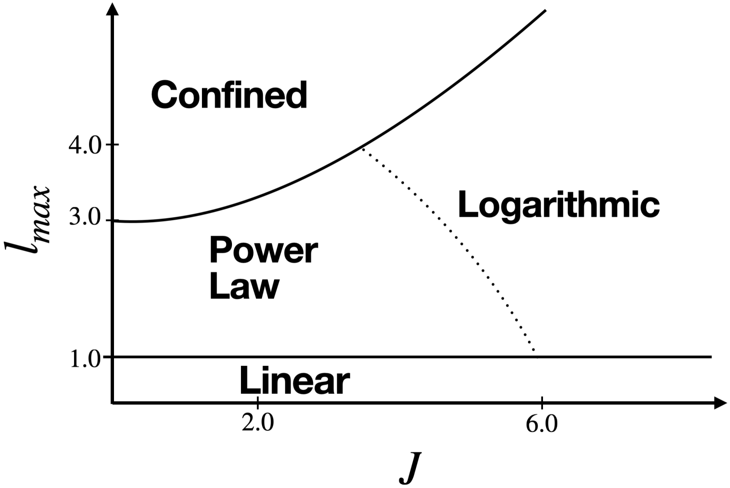
$N = 37, J = 6.0, h = 3.05,$
$m = 0.25, l_{max} = 3.0$

OTOCs in Hyperbolic Ising Model





Phase Diagram for the OTOCs



Conclusions

- We developed MPS/MPO simulations for the Hyperbolic Ising Model.

We see that for suitable parameters Hyperbolic Ising model propagates information in a logarithmic fashion.

This is important for a few reasons

- Models that exhibits this kind of behavior are dependent on long range interactions, time dependent Hamiltonians or random interactions.
- Our simple model which only has nearest neighbor and on-site interactions managed to achive that wihtout the above complications.
- This makes the hyperbolic Ising model very unique and a worthwhile testbed for information propagation in quantum systems.

Thanks for listening.

- We obtain $W(t)$ by expressing W as an MPO state and applying Heisenberg time evolution using TEBD.