A NEW WAY TO COMPUTE THE PSEUDOSCALAR SCREENING MASS AT FINITE CHEMICAL POTENTIAL

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Finite-Temperature Correlation Functions

- Correlation functions can provide useful information about the nature of the degrees of freedom in the thermal medium.

- Straightforward to calculate on the lattice for $\mu_B = 0$ e.g. if $\mathcal{M}_H$ is a mesonic operator, then

$$\left\langle \mathcal{M}_H^\dagger(x) \mathcal{M}_H(0) \right\rangle = \frac{1}{\mathcal{Z}(T)} \int \mathcal{D}U \ det M(T) e^{-S_G(T)} \ tr \left[ P(x,0) \Gamma_H P^\dagger(x,0) \Gamma_H^\dagger \right]$$

- Here, $x = (x, y, z, \tau)$ is a point in Euclidean spacetime. $P(x,0)$ is the fermion propagator while $\Gamma_H$ is a Dirac matrix that depends upon the spin of the meson.

- Summing over $x$, $y$ and $\tau$ projects the correlator on to $p_x = p_y = \omega = 0$ in Fourier space and gives us the screening correlator $C_H(z,T)$ at temperature $T$:

$$C_H(z,T) = \frac{1}{N_\sigma^2 N_\tau} \sum_{x,y,\tau} \left\langle \mathcal{M}_H^\dagger(x) \mathcal{M}_H(0) \right\rangle$$
• For $\mu_B$ real and non-zero, $\det M(T, \mu_B)$ becomes complex and importance sampling breaks down. This is the well-known sign problem of lattice QCD.

• No complete solution known in the case of QCD. All approaches involve extrapolation from either $\mu_B = 0$ (Taylor series expansions, various kinds of reweighting), or from imaginary $\mu_B$ (analytic continuation) where there is no sign problem.

• More recent approaches have also tried to work directly with the complex fermion determinant. (Lefschetz thimbles, complex Langevin, etc.)

• We will focus here on the Taylor series approach, in which the desired observable is expanded in a Taylor series in $\mu_B$ and the first few Taylor coefficients are calculated using lattice QCD. This yields an approximation to the exact observable in a neighborhood of $\mu_B = 0$. 
The Method of Taylor Expansions

- The Taylor series method applies equally well to bulk observables as well as correlation functions.


\[
\frac{P(T, \mu_B)}{T^4} = \frac{1}{VT^3} \ln Z(T, \mu_B) = \sum_{n=0}^{\infty} \frac{\chi_{2n}^B(T)}{(2n)!} \left( \frac{\mu_B}{T} \right)^{2n}
\]

- Only even powers of $\mu_B$ appear due to invariance of the system under $\mu_B \rightarrow -\mu_B$ (particle-antiparticle symmetry).

- In this talk however, we will instead focus on the Taylor expansion of the finite-density screening correlator $C_H(z, T, \mu_B)$ [QCD-TARO, Phys. Rev. D 65, 054501 (2002), Phys. Lett. B 609, 265 (2005)]. Once again, only even powers of $\mu_B$ appear:

\[
C_H(z, T, \mu_B) = \sum_{n=0}^{\infty} \frac{C_H^{(2n)}(z, T)}{(2n)!} \left( \frac{\mu_B}{T} \right)^{2n}
\]
Setup of the Calculation

- Instead of $\mu_B$, we will work with finite isoscalar chemical potential $\mu_\ell$. It is the two-flavor analog of $\mu_B$:

<table>
<thead>
<tr>
<th></th>
<th>$\mu_u$</th>
<th>$\mu_d$</th>
<th>$\mu_s$</th>
<th>$\mu_I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_B = \mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>0</td>
</tr>
<tr>
<td>$\mu_\ell = \mu$</td>
<td>$\mu$</td>
<td>$\mu$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- We will work with staggered fermions (Gamma matrices $\Gamma_H$ replaced by phase factors $\eta_H$).
  - A generic staggered correlator couples simultaneously to two mesons of the same spin but opposite parities. This however is not true for the pion, hence we will only consider the pion correlator from here on.
  - Additionally, $\eta_H = 1$ everywhere for the pion case.

- We have computed the Taylor expansion of $C_H(z, T, \mu_B)$ to fourth order in $\mu_B$. The derivatives act on both the quark propagator (correlator-like operators) as well as the fermion determinant (trace-like operators).
Screening Correlator: Free Theory

- The screening correlator for free massless quarks in the continuum is given by [M. Vepsalainen, JHEP 03, 022 (2007)]

\[
\frac{C_{\text{free}}(z,T,\mu_\ell)}{T^3} = \frac{3}{2} \frac{e^{-2\pi zT}}{zT} \left[ \left(1 + \frac{1}{2\pi zT}\right) \cos(2z\mu_\ell) + \frac{\mu_\ell}{\pi T} \sin(2z\mu_\ell) \right] + \mathcal{O}\left(e^{-4\pi zT}\right)
\]

- For \(\mu_\ell = 0\), we have:

\[
\frac{C_{\text{free}}(z,T,0)}{T^3} = Ae^{-Mz} \quad \text{with} \quad M = 2\pi T \quad \text{and} \quad A = \frac{3}{2zT} \left(1 + \frac{1}{2\pi zT}\right)
\]

- We see that the effect of the chemical potential is to superpose an oscillatory component on the exponential decay of the correlator.

- This is reminiscent of Friedel oscillations in metals, where quantum effects superimpose oscillations on the exponentially decaying screening pattern predicted by the classical theory.
• By differentiating w.r.t. $\hat{\mu}_\ell \equiv \mu_\ell / T$, we obtain the first few Taylor coefficients as (with $\hat{z} \equiv zT$)

$$
\frac{C_{\text{free}}^{(0)}(z, T)}{T^3} = \frac{3e^{-2\pi \hat{z}}}{2\hat{z}} \left(1 + \frac{1}{2\pi \hat{z}}\right), \\
\frac{C_{\text{free}}^{(2)}(z, T)}{T^3} = -6\hat{z}e^{-2\pi \hat{z}} \left(1 - \frac{1}{2\pi \hat{z}}\right), \\
\frac{C_{\text{free}}^{(4)}(z, T)}{T^3} = 24\hat{z}^3e^{-2\pi \hat{z}} \left(1 - \frac{3}{2\pi \hat{z}}\right), \\
C_{\text{free}}^{(1)}(z, T) = C_{\text{free}}^{(3)}(z, T) = 0.
$$

• The non-vanishing Taylor coefficients alternate in sign, which is a manifestation of the oscillatory nature of the correlator.
Another way to see this is from the Taylor expansion of the amplitude of the screening correlator:

$$A_{\text{free}}(z, T, \mu \ell) \equiv \left( \frac{C_{\text{free}}}{T^3} \right) \hat{z} e^{2\pi \hat{z}} = \sum_{k=0}^{\infty} \frac{A^{(k)}(z, T)}{k!} \left( \frac{\mu \ell}{T} \right)^k.$$  

The first $N$ terms of the sum reproduce the oscillation up to a certain value of $\hat{\mu} \ell$, after which they diverge. The lattice data agree well with the $O(\hat{\mu}^4 \ell)$ expression.
• Our free theory results are in very good agreement with the exact expressions, even at small $\hat{z}$.
• This agreement is also seen for the ratios:

$$\Gamma_{\text{free}}(\hat{z}) \equiv \frac{C_{\text{free}}^{(2)}(z, T)}{C_{\text{free}}^{(0)}(z, T)} \quad \text{and} \quad \Sigma_{\text{free}}(\hat{z}) \equiv \frac{C_{\text{free}}^{(4)}(z, T)}{C_{\text{free}}^{(0)}(z, T)}$$

• The exponential factor cancels out in these ratios, resulting in a simple polynomial behavior at large $\hat{z}$:

$$\Gamma_{\text{free}}(\hat{z}) = -4\hat{z}^2 + \frac{4\hat{z}}{\pi} - \frac{2}{\pi^2} + \mathcal{O}(\hat{z}^{-1}), \quad \Sigma_{\text{free}}(\hat{z}) = 16\hat{z}^4 - \frac{32\hat{z}^3}{\pi} + \frac{16\hat{z}^2}{\pi^2} + \mathcal{O}(\hat{z})$$
We see that $\Gamma_{\text{free}}$ and $\Sigma_{\text{free}}$ are respectively quadratic and quartic polynomials in $\hat{z}$.

Therefore $\Gamma_{\text{free}}/\hat{z}^2$ and $\Sigma_{\text{free}}/\hat{z}^4$ should approach constant values as $\hat{z} \to \infty$. However, the approach to the asymptotic limit is seen to be quite slow.

We will see later that the finite temperature screening mass Taylor coefficients can be determined from the coefficients of polynomial fits to $\Gamma(\hat{z})$ and $\Sigma(\hat{z})$. Hence it is necessary to fit these quantities correctly.
Fits to the Free Theory Correlator Derivatives

<table>
<thead>
<tr>
<th>Fit range</th>
<th>$-\alpha_2$</th>
<th>$\alpha_1$</th>
<th>$-\alpha_0$</th>
<th>$\beta_4$</th>
<th>$-\beta_3$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0 \leq \hat{z} \leq 4.0$</td>
<td>3.985(3)</td>
<td>1.20(1)</td>
<td>15.97(5)</td>
<td>10.21(19)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.018(6)</td>
<td>1.37(3)</td>
<td>16.39(18)</td>
<td>12.9(1.1)</td>
<td>4.0(1.6)</td>
<td></td>
</tr>
<tr>
<td>$2.0 \leq \hat{z} \leq 4.0$</td>
<td>3.995(4)</td>
<td>1.24(1)</td>
<td>15.99(7)</td>
<td>10.29(24)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.04(2)</td>
<td>1.53(11)</td>
<td>16.63(33)</td>
<td>14.4(2.1)</td>
<td>6.6(3.4)</td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td>4</td>
<td>$\approx 1.273$</td>
<td>$\approx 0.203$</td>
<td>16</td>
<td>$\approx 10.186$</td>
<td>$\approx 1.621$</td>
</tr>
</tbody>
</table>

- We fit $\Gamma_{\text{free}}(\hat{z})$ and $\Sigma_{\text{free}}(\hat{z})$ to the following polynomials:

  \[
  \Gamma_{\text{free}}(\hat{z}) = \alpha_2 \hat{z}^2 + \alpha_1 \hat{z} + \alpha_0, \quad \Sigma_{\text{free}}(\hat{z}) = \beta_4 \hat{z}^4 + \beta_3 \hat{z}^3 + \beta_2 \hat{z}^2.
  \]

- Retaining more coefficients allowed us to fit over a wider range. Without the sub-leading coefficients, the fits yielded results that were very precise but $\sim 5\sigma$ away from the exact results in some cases!

- Overall however, we obtained better results by keeping fewer coefficients but fitting to larger $zT$. 

Screening Mass Taylor Coefficients from Correlator Fits

- Remember that the free theory isoscalar correlator for massless quarks is given by

\[
\frac{C_{\text{free}}(z, T, \mu_\ell)}{T^3} = \frac{3}{2} e^{-\frac{2\pi z T}{z T}} \left[ \left( 1 + \frac{1}{2\pi z T} \right) \cos(2z\mu_\ell) + \frac{\mu_\ell}{\pi T} \sin(2z\mu_\ell) \right] + \mathcal{O} \left( e^{-4\pi z T} \right)
\]

- For \( \mu_\ell \neq 0 \), we can still write the correlator as \( C_{\text{free}}(z) = A(z)e^{-Mz} \) provided we allow \( A \) and \( M \) to take complex values:

\[
\frac{C_{\text{free}}(z, T, \mu_\ell)}{T^3} = \text{Re} \left[ A(\mu_\ell)e^{-zM(\mu_\ell)} \right] \quad \text{with}
\]

\[
A(\mu_\ell) = \frac{3}{2zT} \left( 1 + \frac{1}{2\pi z T} \right) \left( 1 - i \frac{\mu_\ell}{\pi T} \right) \quad \text{and} \quad M(\mu_\ell) = 2\pi T + 2i\mu_\ell.
\]

- We note that the real and imaginary parts of \( C_{\text{free}}(z, T, \mu_\ell) \) are even and odd functions of \( \mu_\ell \) respectively. Since the QCD ground state is symmetric under \( \mu_\ell \rightarrow -\mu_\ell \), \( \text{Im} C_{\text{free}}(z, T, \mu_\ell) \) must vanish identically.
Screening Mass Taylor Coefficients from Correlator Fits

• The free theory is the $T = \infty$ limit of the interacting theory. For $T < \infty$, we make the following ansatz:

$$C(z, T, \mu_\ell) = \text{Re} \left[ A(T, \mu_\ell) e^{-zM(T, \mu_\ell)} \right] = e^{-zM_R} \left[ A_R \cos(zM_I) + A_I \sin(zM_I) \right]$$

• $A_R$, $A_I$, $M_R$ and $M_I$ are functions of $T$ and $\mu_\ell$. Taylor-expanding the correlator in $\mu_\ell$ yields simple quadratic and quartic polynomials for $\Gamma(\hat{z})$ and $\Sigma(\hat{z})$:

$$\Gamma(\hat{z}) \equiv \frac{C^{(2)}(z, T, 0)}{C(z, T, 0)} = \alpha_2 \hat{z}^2 + \alpha_1 \hat{z} + \alpha_0$$

$$\Sigma(\hat{z}) \equiv \frac{C^{(4)}(z, T, 0)}{C(z, T, 0)} = \beta_4 \hat{z}^4 + \beta_3 \hat{z}^3 + \beta_2 \hat{z}^2 + \beta_1 \hat{z} + \beta_0$$

• The lowest-order screening mass corrections can be extracted from the polynomial coefficients ($\hat{M} \equiv M/T$) [R. Thakkar & PH, JHEP 07, 171 (2023)]:

$$\hat{M}'_I(T, 0) \equiv \frac{d\hat{M}_I}{d\hat{\mu}_\ell} \bigg|_{\mu_\ell=0} = (-\alpha_2)^{1/2} = \beta_4^{1/4}$$

$$\hat{M}''_R(T, 0) \equiv \frac{d^2\hat{M}_R}{d\hat{\mu}_\ell^2} \bigg|_{\mu_\ell=0} = \frac{1}{4} \left( 2\alpha_1 - \frac{\beta_3}{\alpha_2} \right)$$
Setup of the Calculation

- Our calculations were done using $N_f = 2 + 1$ flavors of Highly Improved Staggered Quarks (HISQ) and a Symanzik-improved Wilson gauge action.

- The free theory calculation was done on an $80^3 \times 8$ lattice, while the finite temperature calculations were done using $64^3 \times 8$ lattices, with an additional ensemble of $32^3 \times 8$ at one of the temperatures to check for finite volume effects:

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$T$ [GeV]</th>
<th>$N_\sigma$</th>
<th>$am_s$</th>
<th>configurations</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.360</td>
<td>2.24</td>
<td>64</td>
<td>0.003691</td>
<td>6000</td>
</tr>
<tr>
<td>9.670</td>
<td>2.90</td>
<td>64</td>
<td>0.002798</td>
<td>6000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>32</td>
<td>0.002798</td>
<td>12700</td>
</tr>
</tbody>
</table>

- The strange quark mass was chosen to lie on the LCP, and $m_l = m_s/20$ throughout.

- The correlator-like operators were calculated using 8 point sources per configuration placed at $n_i = 0$ or $N_\sigma/2$ for $i \in \{x, y, z\}$ keeping $n_t = 0$.

- The trace-like operators were estimated stochastically using 1000 Gaussian noise vectors per configuration.
We indeed observe polynomial-like behavior for $\Gamma(\hat{z})$ and $\Sigma(\hat{z})$. However, our results are very different from the free theory even at these high temperatures.

The difference is around 30% for $\Gamma(\hat{z})$ and around 45% in the case of $\Sigma(\hat{z})$.

*Caveat:* Results not continuum-extrapolated.
Fits to $\Gamma/\hat{z}^2$ and $\Sigma/\hat{z}^4$

- Approach to the asymptotic limit non-monotonic unlike in the free theory. Hence, the coefficients $\alpha_1$ and $\beta_3$ have opposite signs to the free theory.

- The extrema $\hat{z}_\Gamma$ and $\hat{z}_\Sigma$ can be identified from the fit ansatz as

$$\frac{\Gamma}{\hat{z}^2} = -|\alpha_2| - \frac{|\alpha_1|}{\hat{z}} + \frac{\alpha_0}{\hat{z}^2}, \quad \frac{\Sigma}{\hat{z}^4} = \beta_4 - \frac{|\beta_3|}{\hat{z}} + \frac{\beta_2}{\hat{z}^2},$$

$$\hat{z}_\Gamma = -2 \frac{\alpha_0}{\alpha_1}, \quad \hat{z}_\Sigma = -2 \frac{\beta_2}{\beta_3}$$

- $\hat{z}_\Gamma$ and $\hat{z}_\Sigma$ determined using spline fits and lowest-order coefficients $\alpha_0$ and $\beta_2$ re-expressed in terms of $\hat{z}_\Gamma$ and $\hat{z}_\Sigma$ to reduce the number of fit coefficients.
Fits to $\Gamma/\hat{z}^2$ and $\Sigma/\hat{z}^4$

- Fit $\Gamma/\hat{z}^2$ and $\Sigma/\hat{z}^4$ in a window $[\hat{z}_{\text{min}}, \hat{z}_{\text{max}}]$ and look for a plateau while varying $\hat{z}_{\text{min}}$ ($\hat{z}_{\text{max}} = 3.25$ kept fixed).
- Good results for $\Gamma/\hat{z}^2$. Results for $\Sigma/\hat{z}^4$ need more work!
Final Results

<table>
<thead>
<tr>
<th>Temperature</th>
<th>( \hat{\Gamma} )</th>
<th>( \alpha_1 )</th>
<th>( \hat{\Sigma} )</th>
<th>( \beta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.24 GeV</td>
<td>2.269(23)</td>
<td>-1.955(57)</td>
<td>2.860(50)</td>
<td>10.091(126)</td>
</tr>
<tr>
<td>2.90 GeV</td>
<td>2.500(16)</td>
<td>-2.175(87)</td>
<td>3.125(25)</td>
<td>10.667(232)</td>
</tr>
<tr>
<td>Free theory</td>
<td></td>
<td>4/\pi \approx 1.273</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- As noted previously, \( \alpha_1 \) and \( \beta_3 \) differ in sign from the free theory.

- Expand \( \hat{M}_R \) and \( \hat{M}_I \) in a Taylor series in \( \mu_{\ell} \) as:

\[
\hat{M}_R(T, \mu_{\ell}) = \hat{M}_R(0) + \frac{1}{2} \hat{M}_R''(0) \mu_{\ell}^2 + \frac{1}{24} \hat{M}_R'''(0) \mu_{\ell}^4 \ldots,
\]

\[
\hat{M}_I(T, \mu_{\ell}) = \hat{M}_I'(0) \mu_{\ell} + \frac{1}{6} \hat{M}_I'''(0) \mu_{\ell}^3 + \ldots
\]

- The biggest uncertainty in the determination of \( M_I'(0) \) and \( M_R''(0) \) is due to the uncertainties in \( \beta_2 \) and \( \beta_3 \).
Conclusions

• In this talk, we presented a new way of calculating the pion screening mass at finite density.

• Our approach is based on a Taylor expansion of the free theory expression for the pion correlator at finite $\mu_\ell$. Hence we expect our approach to be valid at high temperatures.

• As a first check of our formalism, we calculated up to the fourth derivative of the free theory pion screening correlator on an $80^3 \times 8$ lattice and compared our results with the known exact expressions.

• We then applied the same formalism to two temperatures viz. $T = 2.24$ GeV and $T = 2.90$ GeV.

• Although the correlator ratios showed the expected polynomial-like behavior, the non-monotonic nature of the ratios $\Gamma(\hat{z})$ and $\Sigma(\hat{z})$ and the resulting uncertainties in the fit coefficients led to significant errors for $\hat{M}_R''(0)$ and $\hat{M}_I'(0)$.

• However, our results seemed to indicate a positive value for $\hat{M}_R''(0)$ at these temperatures. Both $\hat{M}_R''(0)$ as well as $\hat{M}_I'(0)$ were also found to be very different from the free theory values.