A New Way To Compute The Pseudoscalar Screening Mass At Finite Chemical Potential

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Finite-Temperature Correlation Functions

- Correlation functions can provide useful information about the nature of the degrees of freedom in the thermal medium.

- Straightforward to calculate on the lattice for $\mu_B = 0$ e.g. if $\mathcal{M}_H$ is a mesonic operator, then

$$\left\langle \mathcal{M}_H^\dagger(\boldsymbol{x})\mathcal{M}_H(0)\right\rangle = \frac{1}{\mathcal{Z}(T)}\int \mathcal{D}U\ \det M(T)\, e^{-S_G(T)}\ \mathrm{tr}\left[P(\boldsymbol{x},0)\Gamma_H P^\dagger(\boldsymbol{x},0)\Gamma_H^\dagger\right]$$

- Here, $\boldsymbol{x} = (x, y, z, \tau)$ is a point in Euclidean spacetime. $P(\boldsymbol{x}, 0)$ is the fermion propagator while Γ_H is a Dirac matrix that depends upon the spin of the meson.

- Summing over x, y and τ projects the correlator on to $p_x = p_y = \omega = 0$ in Fourier space and gives us the screening correlator $C_H(z, T)$ at temperature T:

$$C_H(z,T) = \frac{1}{N_\sigma^2 N_\tau}\sum_{x,y,\tau}\left\langle \mathcal{M}_H^\dagger(\boldsymbol{x})\mathcal{M}_H(0)\right\rangle$$

Sign Problem and Taylor Series Expansion

- For μ_B real and non-zero, $\det M(T, \mu_B)$ becomes complex and importance sampling breaks down. This is the well-known sign problem of lattice QCD.

- No complete solution known in the case of QCD. All approaches involve extrapolation from either $\mu_B = 0$ (Taylor series expansions, various kinds of reweighting), or from imaginary μ_B (analytic continuation) where there is no sign problem.

- More recent approaches have also tried to work directly with the complex fermion determinant. (Lefschetz thimbles, complex Langevin, etc.)

- We will focus here on the **Taylor series** approach, in which the desired observable is expanded in a Taylor series in μ_B and the first few Taylor coefficients are calculated using lattice QCD. This yields an approximation to the exact observable in a neighborhood of $\mu_B = 0$.

The Method of Taylor Expansions

- The Taylor series method applies equally well to bulk observables as well as correlation functions.

- Originally applied to calculate the Equation of State at finite μ_B [R. Gavai and S. Gupta, Phys. Rev. D **64**, 074506 (2001); C. Allton *et al.* Phys. Rev. D **66**, 074507 (2002)]:

$$\frac{P(T,\mu_B)}{T^4} = \frac{1}{VT^3}\ln\mathcal{Z}(T,\mu_B) = \sum_{n=0}^{\infty}\frac{\chi_{2n}^{B}(T)}{(2n)!}\left(\frac{\mu_B}{T}\right)^{2n}$$

- Only even powers of μ_B appear due to invariance of the system under $\mu_B \to -\mu_B$ (particle-antiparticle symmetry).

- In this talk however, we will instead focus on the Taylor expansion of the finite-density screening correlator $C_H(z,T,\mu_B)$ [QCD-TARO, Phys. Rev. D **65**, 054501 (2002), Phys. Lett. B **609**, 265 (2005)]. Once again, only even powers of μ_B appear:

$$C_H(z,T,\mu_B) = \sum_{n=0}^{\infty}\frac{C_H^{(2n)}(z,T)}{(2n)!}\left(\frac{\mu_B}{T}\right)^{2n}$$

Setup of the Calculation

- Instead of μ_B, we will work with finite isoscalar chemical potential μ_ℓ. It is the two-flavor analog of μ_B:

	μ_u	μ_d	μ_s	μ_I
$\mu_B = \mu$	μ	μ	μ	0
$\mu_\ell = \mu$	μ	μ	0	0

- We will work with staggered fermions (Gamma matrices Γ_H replaced by phase factors η_H).
    - A generic staggered correlator couples simultaneously to **two** mesons of the same spin but opposite parities. This however is not true for the pion, hence we will only consider the pion correlator from here on.
    - Additionally, $\eta_H = 1$ everywhere for the pion case.
- We have computed the Taylor expansion of $C_H(z, T, \mu_B)$ to fourth order in μ_B. The derivatives act on both the quark propagator (correlator-like operators) as well as the fermion determinant (trace-like operators).

Screening Correlator: Free Theory

- The screening correlator for free massless quarks in the continuum is given by [M. Vepsalainen, JHEP **03**, 022 (2007)]

$$\frac{C_{\text{free}}(z,T,\mu_\ell)}{T^3} = \frac{3}{2}\frac{e^{-2\pi zT}}{zT}\left[\left(1+\frac{1}{2\pi zT}\right)\cos(2z\mu_\ell)+\frac{\mu_\ell}{\pi T}\sin(2z\mu_\ell)\right]+\mathcal{O}\left(e^{-4\pi zT}\right)$$

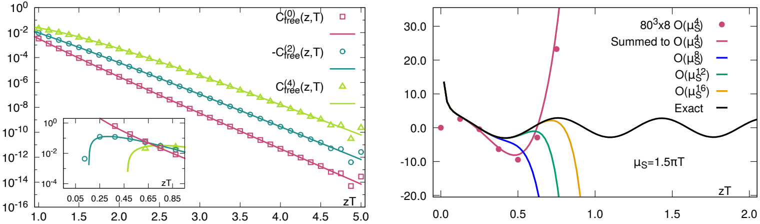
- For $\mu_\ell = 0$, we have:

$$\frac{C_{\text{free}}(z,T,0)}{T^3} = Ae^{-Mz} \quad \text{with} \quad M = 2\pi T \quad \text{and} \quad A = \frac{3}{2zT}\left(1+\frac{1}{2\pi zT}\right)$$

- We see that the effect of the chemical potential is to superpose an oscillatory component on the exponential decay of the correlator.

- This is reminiscent of Friedel oscillations in metals, where quantum effects superimpose oscillations on the exponentially decaying screening pattern predicted by the classical theory.

Screening Correlator: Free Theory



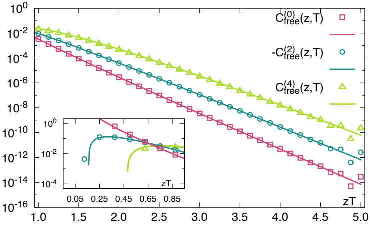
- By differentiating w.r.t. $\hat{\mu}_\ell \equiv \mu_\ell/T$, we obtain the first few Taylor coefficients as (with $\hat{z} \equiv zT$)

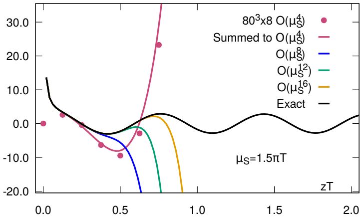
$$\frac{C^{(0)}_{\text{free}}(z,T)}{T^3} = \frac{3e^{-2\pi\hat{z}}}{2\hat{z}}\left(1+\frac{1}{2\pi\hat{z}}\right), \qquad \frac{C^{(2)}_{\text{free}}(z,T)}{T^3} = -6\hat{z}e^{-2\pi\hat{z}}\left(1-\frac{1}{2\pi\hat{z}}\right),$$

$$\frac{C^{(4)}_{\text{free}}(z,T)}{T^3} = 24\hat{z}^3e^{-2\pi\hat{z}}\left(1-\frac{3}{2\pi\hat{z}}\right), \qquad C^{(1)}_{\text{free}}(z,T) = C^{(3)}_{\text{free}}(z,T) = 0.$$

- The non-vanishing Taylor coefficients alternate in sign, which is a manifestation of the oscillatory nature of the correlator.

Screening Correlator: Free Theory



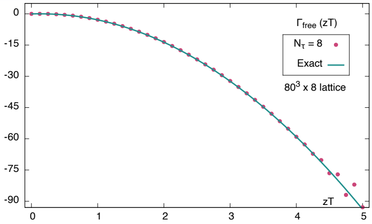


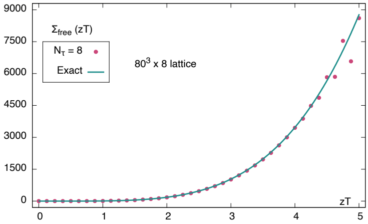
- Another way to see this is from the Taylor expansion of the amplitude of the screening correlator:

$$A_{\text{free}}(z,T,\mu_\ell) \equiv \left(\frac{C_{\text{free}}}{T^3}\right) \hat{z}\, e^{2\pi\hat{z}} = \sum_{k=0}^{\infty} \frac{A^{(k)}(z,T)}{k!} \left(\frac{\mu_\ell}{T}\right)^k .$$

- The first N terms of the sum reproduce the oscillation up to a certain value of $\hat{\mu}_\ell$, after which they diverge. The lattice data agree well with the $\mathcal{O}(\hat{\mu}_\ell^4)$ expression.

Screening Correlator: Free Theory





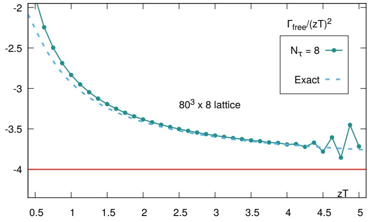
- Our free theory results are in very good agreement with the exact expressions, even at small $\hat{z}$.
- This agreement is also seen for the ratios:

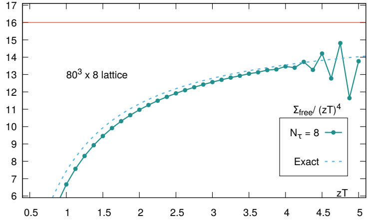
$$\Gamma_{\text{free}}(\hat{z}) \equiv \frac{C^{(2)}_{\text{free}}(z,T)}{C^{(0)}_{\text{free}}(z,T)} \qquad \text{and} \qquad \Sigma_{\text{free}}(\hat{z}) \equiv \frac{C^{(4)}_{\text{free}}(z,T)}{C^{(0)}_{\text{free}}(z,T)}$$

- The exponential factor cancels out in these ratios, resulting in a simple polynomial behavior at large $\hat{z}$:

$$\Gamma_{\text{free}}(\hat{z}) = -4\hat{z}^2 + \frac{4\hat{z}}{\pi} - \frac{2}{\pi^2} + \mathcal{O}\left(\hat{z}^{-1}\right), \quad \Sigma_{\text{free}}(\hat{z}) = 16\hat{z}^4 - \frac{32\hat{z}^3}{\pi} + \frac{16\hat{z}^2}{\pi^2} + \mathcal{O}\left(\hat{z}\right)$$

Free Theory Correlator: Approach to the Asymptotic Limit





- We see that Γ_{free} and Σ_{free} are respectively quadratic and quartic polynomials in $\hat{z}$.
- Therefore $\Gamma_{\text{free}}/\hat{z}^2$ and $\Sigma_{\text{free}}/\hat{z}^4$ should approach constant values as $\hat{z} \to \infty$. However, the approach to the asymptotic limit is seen to be quite slow.
- We will see later that the finite temperature screening mass Taylor coefficients can be determined from the coefficients of polynomial fits to $\Gamma(\hat{z})$ and $\Sigma(\hat{z})$. Hence it is necessary to fit these quantities correctly.

Fits to the Free Theory Correlator Derivatives

Fit range	$-\alpha_2$	α_1	$-\alpha_0$	β_4	$-\beta_3$	β_2
$1.0 \leq \hat{z} \leq 4.0$	3.985(3)	1.20(1)		15.97(5)	10.21(19)	
	4.018(6)	1.37(3)	0.20(4)	16.39(18)	12.9(1.1)	4.0(1.6)
$2.0 \leq \hat{z} \leq 4.0$	3.995(4)	1.24(1)		15.99(7)	10.29(24)	
	4.04(2)	1.53(11)	0.44(17)	16.63(33)	14.4(2.1)	6.6(3.4)
Exact	4	≈ 1.273	≈ 0.203	16	≈ 10.186	≈ 1.621

- We fit $\Gamma_{\text{free}}(\hat{z})$ and $\Sigma_{\text{free}}(\hat{z})$ to the following polynomials:

$$\Gamma_{\text{free}}(\hat{z}) = \alpha_2 \hat{z}^2 + \alpha_1 \hat{z} + \alpha_0, \qquad \Sigma_{\text{free}}(\hat{z}) = \beta_4 \hat{z}^4 + \beta_3 \hat{z}^3 + \beta_2 \hat{z}^2.$$

- Retaining more coefficients allowed us to fit over a wider range. Without the sub-leading coefficients, the fits yielded results that were very precise but $\sim 5\sigma$ away from the exact results in some cases!
- Overall however, we obtained better results by keeping fewer coefficients but fitting to larger zT.

Screening Mass Taylor Coefficients from Correlator Fits

- Remember that the free theory isoscalar correlator for massless quarks is given by

$$\frac{C_{\text{free}}(z,T,\mu_\ell)}{T^3} = \frac{3}{2}\frac{e^{-2\pi zT}}{zT}\left[\left(1+\frac{1}{2\pi zT}\right)\cos(2z\mu_\ell) + \frac{\mu_\ell}{\pi T}\sin(2z\mu_\ell)\right] + \mathcal{O}\left(e^{-4\pi zT}\right)$$

- For $\mu_\ell \neq 0$, we can still write the correlator as $C_{\text{free}}(z) = A(z)e^{-Mz}$ provided we allow A and M to take **complex values:**

$$\frac{C_{\text{free}}(z,T,\mu_\ell)}{T^3} = \text{Re}\left[A(\mu_\ell)e^{-zM(\mu_\ell)}\right] \quad \text{with}$$

$$A(\mu_\ell) = \frac{3}{2zT}\left(1+\frac{1}{2\pi zT}\right)\left(1 - i\frac{\mu_\ell}{\pi T}\right) \quad \text{and} \quad M(\mu_\ell) = 2\pi T + 2i\mu_\ell.$$

- We note that the real and imaginary parts of $C_{\text{free}}(z,T,\mu_\ell)$ are even and odd functions of μ_ℓ respectively. Since the QCD ground state is symmetric under $\mu_\ell \to -\mu_\ell$, $\text{Im}\, C_{\text{free}}(z,T,\mu_\ell)$ must vanish identically.

Screening Mass Taylor Coefficients from Correlator Fits

- The free theory is the $T = \infty$ limit of the interacting theory. For $T < \infty$, we make the following *ansatz*:

$$C(z, T, \mu_\ell) = \mathrm{Re}\Big[A(T, \mu_\ell) e^{-zM(T,\mu_\ell)}\Big] = e^{-zM_R}\Big[A_R \cos(zM_I) + A_I \sin(zM_I)\Big]$$

- A_R, A_I, M_R and M_I are functions of T and μ_ℓ. Taylor-expanding the correlator in μ_ℓ yields simple quadratic and quartic polynomials for $\Gamma(\hat{z})$ and $\Sigma(\hat{z})$:

$$\Gamma(\hat{z}) \equiv \frac{C^{(2)}(z, T, 0)}{C(z, T, 0)} = \alpha_2 \hat{z}^2 + \alpha_1 \hat{z} + \alpha_0$$

$$\Sigma(\hat{z}) \equiv \frac{C^{(4)}(z, T, 0)}{C(z, T, 0)} = \beta_4 \hat{z}^4 + \beta_3 \hat{z}^3 + \beta_2 \hat{z}^2 + \beta_1 \hat{z} + \beta_0$$

- The lowest-order screening mass corrections can be extracted from the polynomial coefficients ($\hat{M} \equiv M/T$) [R. Thakkar & PH, JHEP **07**, 171 (2023)]:

$$\hat{M}_I'(T, 0) \equiv \left.\frac{d\hat{M}_I}{d\hat{\mu}_\ell}\right|_{\mu_\ell = 0} = (-\alpha_2)^{1/2} = \beta_4^{1/4}$$

$$\hat{M}_R''(T, 0) \equiv \left.\frac{d^2 \hat{M}_R)}{d\hat{\mu}_\ell^2}\right|_{\mu_\ell = 0} = \frac{1}{4}\left(2\alpha_1 - \frac{\beta_3}{\alpha_2}\right)$$

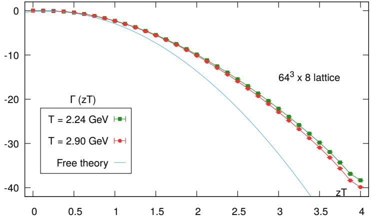
Setup of the Calculation

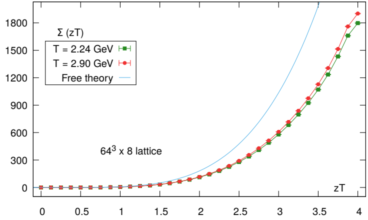
- Our calculations were done using $N_f = 2 + 1$ flavors of Highly Improved Staggered Quarks (HISQ) and a Symanzik-improved Wilson gauge action.
- The free theory calculation was done on an $80^3 \times 8$ lattice, while the finite temperature calculations were done using $64^3 \times 8$ lattices, with an additional ensemble of $32^3 \times 8$ at one of the temperatures to check for finite volume effects:

β	T [GeV]	N_σ	am_s	configurations
9.360	2.24	64	0.003691	6000
9.670	2.90	64	0.002798	6000
		32	0.002798	12700

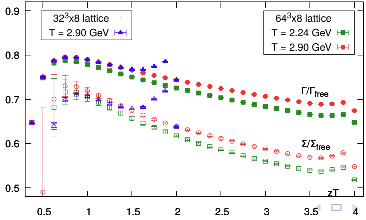
- The strange quark mass was chosen to lie on the LCP, and $m_l = m_s/20$ throughout.
- The correlator-like operators were calculated using 8 point sources per configuration placed at $n_i = 0$ or $N_\sigma/2$ for $i \in \{x, y, z\}$ keeping $n_t = 0$.
- The trace-like operators were estimated stochastically using 1000 Gaussian noise vectors per configuration.

$\Gamma(\hat{z})$ and $\Sigma(\hat{z})$ for finite temperature

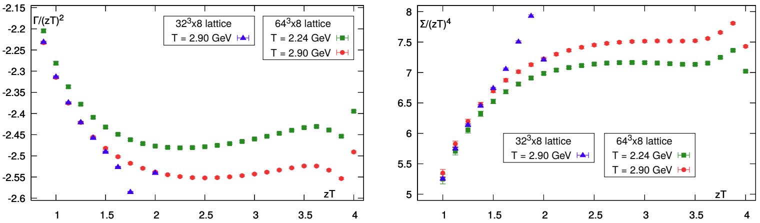




- We indeed observe polynomial-like behavior for $\Gamma(\hat{z})$ and $\Sigma(\hat{z})$. However, our results are very different from the free theory even at these high temperatures.
- The difference is around 30% for $\Gamma(\hat{z})$ and around 45% in the case of $\Sigma(\hat{z})$. Caveat: Results not continuum-extrapolated.



Fits to $\Gamma/\hat{z}^2$ and $\Sigma/\hat{z}^4$



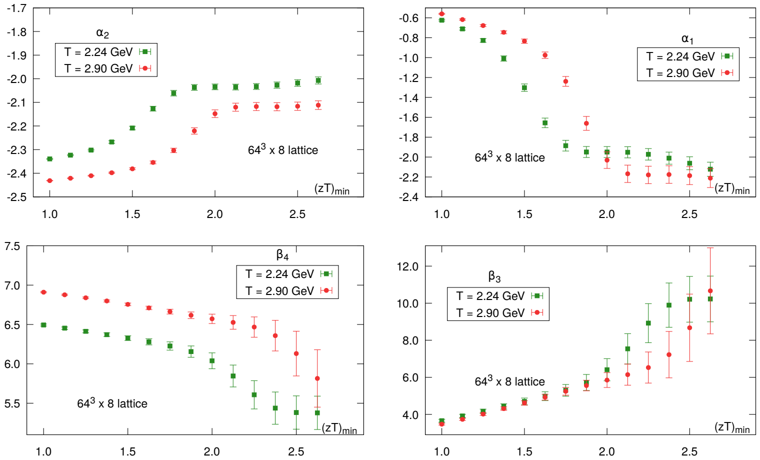
- Approach to the asymptotic limit non-monotonic unlike in the free theory. Hence, the coefficients α_1 and β_3 have opposite signs to the free theory.
- The extrema $\hat{z}_\Gamma$ and $\hat{z}_\Sigma$ can be identified from the fit ansatz as

$$\frac{\Gamma}{\hat{z}^2} = -|\alpha_2| - \frac{|\alpha_1|}{\hat{z}} + \frac{\alpha_0}{\hat{z}^2}, \qquad \frac{\Sigma}{\hat{z}^4} = \beta_4 - \frac{|\beta_3|}{\hat{z}} + \frac{\beta_2}{\hat{z}^2},$$

$$\hat{z}_\Gamma = -2\,\frac{\alpha_0}{\alpha_1}, \qquad \hat{z}_\Sigma = -2\,\frac{\beta_2}{\beta_3}$$

- $\hat{z}_\Gamma$ and $\hat{z}_\Sigma$ determined using spline fits and lowest-order coefficients α_0 and β_2 re-expressed in terms of $\hat{z}_\Gamma$ and $\hat{z}_\Sigma$ to reduce the number of fit coefficients.

Fits to $\Gamma/\hat{z}^2$ and $\Sigma/\hat{z}^4$



- Fit $\Gamma/\hat{z}^2$ and $\Sigma/\hat{z}^4$ in a window $[\hat{z}_{\min}, \hat{z}_{\max}]$ and look for a plateau while varying $\hat{z}_{\min}$ ($\hat{z}_{\max} = 3.25$ kept fixed).
- Good results for $\Gamma/\hat{z}^2$. Results for $\Sigma/\hat{z}^4$ need more work!

Final Results

Temperature	$\hat{z}_\Gamma$	α_1	$\hat{z}_\Sigma$	β_3
2.24 GeV	2.269(23)	-1.955(57)	2.860(50)	10.091(126)
2.90 GeV	2.500(16)	-2.175(87)	3.125(25)	10.667(232)
Free theory		$4/\pi \approx 1.273$		$-32/\pi \approx -10.186$

- As noted previously, α_1 and β_3 differ in sign from the free theory.
- Expand $\hat{M}_R$ and $\hat{M}_I$ in a Taylor series in μ_ℓ as:

$$\hat{M}_R(T,\mu_\ell) = \hat{M}_R(0) + \frac{1}{2}\hat{M}_R''(0)\mu_\ell^2 + \frac{1}{24}\hat{M}''''(0)\mu_\ell^4 \ldots,$$

$$\hat{M}_I(T,\mu_\ell) = \hat{M}_I'(0)\mu_\ell + \frac{1}{6}\hat{M}_I'''(0)\mu_\ell^3 + \ldots$$

- The biggest uncertainty in the determination of $M_I'(0)$ and $M_R''(0)$ is due to the uncertainties in β_2 and β_3.

Temperature	$\hat{M}_R(\hat{\mu}_\ell = 0)$	$\hat{M}_R''(\hat{\mu}_\ell = 0)$	$\hat{M}_I'(\hat{\mu}_\ell = 0)$
2.24 GeV	6.337(1)	0.263(169)	1.426(5)
2.90 GeV	6.352(1)	0.172(328)	1.455(6)
Free theory	$2\pi \approx 6.283$	0	2

Conclusions

- In this talk, we presented a new way of calculating the pion screening mass at finite density.

- Our approach is based on a Taylor expansion of the free theory expression for the pion correlator at finite μ_ℓ. Hence we expect our approach to be valid at high temperatures.

- As a first check of our formalism, we calculated up to the fourth derivative of the free theory pion screening correlator on an $80^3 \times 8$ lattice and compared our results with the known exact expressions.

- We then applied the same formalism to two temperatures viz. $T = 2.24$ GeV and $T = 2.90$ GeV.

- Although the correlator ratios showed the expected polynomial-like behavior, the non-monotonic nature of the ratios $\Gamma(\hat{z})$ and $\Sigma(\hat{z})$ and the resulting uncertainties in the fit coefficients led to significant errors for $\hat{M}''_R(0)$ and $\hat{M}'_I(0)$.

- However, our results seemed to indicate a positive value for $\hat{M}''_R(0)$ at these temperatures. Both $\hat{M}''_R(0)$ as well as $\hat{M}'_I(0)$ were also found to be very different from the free theory values.