Mass effects on the QCD β-function

M. Costa[1,*], D. Gavriel[1], H. Panagopoulos[1], G. Spanoudes[1]

[1]Department of Physics, University of Cyprus, 1 Panepistimiou Avenue, 2109 Aglantzia, Cyprus, *Speaker

Abstract

In this study we present lattice results on the QCD β-function in the presence of quark masses. The β-function is calculated to three loops in perturbation theory and for improved lattice actions; it is extracted from the renormalization of the coupling constant Z_g. The background field method is used to compute Z_g, where it is simply related to the background gluon field renormalization constant Z_A. We focus on the quark mass effects in the background gluon propagator; the dependence of the QCD β-function on the number of colors N_c, the number of fermionic flavors N_f and the quark masses, is shown explicitly. The perturbative results of the QCD β-function will be applied to the precise determination of the strong coupling constant, calculated by Monte Carlo simulations removing the mass effects from the nonperturbative Green's functions.

Introduction - Motivation

The renormalized β-function describes the dependence of the renormalized coupling constant g on the scale inherent in the renormalization scheme [1] (chosen in this work to be the $\overline{\rm MS}$ scheme). It encodes the underlying dynamics of QCD from low to high momentum regions. Nonperturbative estimations of the strong coupling α, in several renormalization schemes, through numerical simulations of the corresponding lattice theories, are being studied by a number of groups at present (See Refs. [2, 3] and references therein).

The three-loop bare QCD β-function [4] can be extracted from the two-loop calculation of the renormalization factor Z_g, which relates the bare running coupling α_0 to the $\overline{\rm MS}$-renormalized running coupling $\alpha_{\overline{\rm MS}}$ ($\bar\mu$ is the $\overline{\rm MS}$ renormalization scale and a is the lattice spacing) through:

$$\alpha_0 = Z_g^2(g_0, a\bar\mu) \times \alpha_{\overline{\rm MS}} \qquad (1)$$

The inclusion of the quark masses makes this calculation even more complicated [5]. Note that we are interested in the discretization errors proportional to the quark mass ($O(am)$ effects) on the β-function. For simplicity of notation, we denote all flavor masses by m; the case of different flavor masses can be trivially recovered from our results. This new direction is very important due to the fact that the $O(am)$ effects will be removed from the nonperturbative Green's functions entering the strong coupling, allowing for a more precise determination. Removing $O(am)$ effects will improve importantly any quantity that is calculated using Wilson-type fermions [6].

Computational setup and methods used to calculate Z_g

The renormalized β-function and the bare β-function on the lattice ($\beta_L(g_0)$) are defined:

$$\beta(g_{\overline{\rm MS}}) = \bar\mu \frac{dg_{\overline{\rm MS}}}{d\bar\mu}\Big|_{a,g_0}, \qquad \beta_L(g_0) = -a\frac{dg_0}{da}\Big|_{\bar\mu, g_{\overline{\rm MS}}} \qquad (2)$$

- In the asymptotic limit, one can write the expansion of Eq. (2) in powers of g_0:

$$\beta_L(g_0) = -b_0\, g_0^3 - b_1\, g_0^5 - b_2^L\, g_0^7 - ..., \quad \beta(g_{\overline{\rm MS}}) = -b_0\, g_{\overline{\rm MS}}^3 - b_1\, g_{\overline{\rm MS}}^5 - b_2\, g_{\overline{\rm MS}}^7 + ... (3)$$

  The coefficients b_0, b_1 are well-known universal constants (regularization independent) for the massless case; b_i^L ($i \geq 2$) (regularization dependent) must be calculated perturbatively. $\beta_L(g_0)$ and $\beta(g_{\overline{\rm MS}})$ can be related using the renormalization function Z_g, that is:

$$\beta_L(g_0) = \left(1 - g_0^2 \frac{\partial \ln Z_g^2}{\partial g_0^2}\right)^{-1} Z_g\; \beta(Z_g^{-1} g_0) \qquad (4)$$

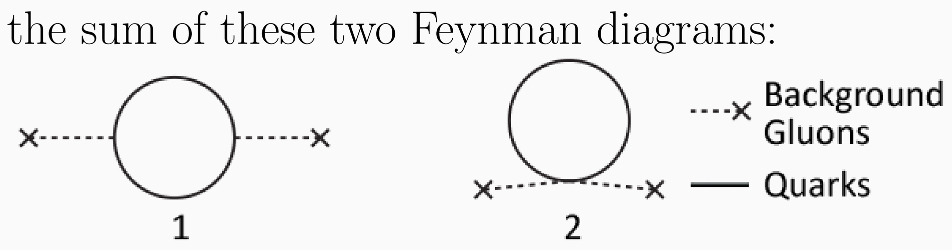
- We employ the clover action for fermions and the Symanzik improved action for gluons.
- The most convenient and economical way to proceed with the calculation of $Z_g(g_0, a\bar\mu)$ is to use the Background Field (BF) technique [7, 8], in which the following relation is valid.

$$Z_A(g_0, a\bar\mu) Z_g^2(g_0, a\bar\mu) = 1 \qquad (5)$$

  where Z_A is the BF renormalization function. In the lattice version of the BF technique, the link variable takes the form: $U_\mu(x) = e^{iag_0 Q_\mu(x)} \cdot e^{iaA_\mu(x)}$ (Q_μ: quantum field, A_μ: background field). In this framework, instead of calculating Z_g, it suffices to compute Z_A. For the above lattice calculation, we consider the 2-point BF 1PI Green's function $\langle A_\mu(x) A_\nu(y)\rangle$; we focus on the quark mass effects $O(am)$.

One-loop Results

- The mass effects, which contribute to the 2-point Green's function $\langle A_\mu(x) A_\nu(y)\rangle$, are associated with the Feynman diagrams with at least one fermion line. At one-loop order $\langle A_\mu(x) A_\nu(y)\rangle^{1-loop}$ is the sum of these two Feynman diagrams:



- The one-loop result of the 2-pt lattice Green's function is:

$$\langle A^\alpha_\mu A^\beta_\nu\rangle^{1-loop} = \delta^{\alpha\beta} N_f \left(\delta_{\mu\nu} q^2 - q_\nu q_\mu\right)\left\{F_1(aq) + F_2\left(\frac{m^2}{q^2}\right) + am\left[F_3(aq) + F_4\left(\frac{m^2}{q^2}\right)\right]\right\} (6)$$

$$F_1(aq) = -0.0137322 + 0.0050467\, c_{sw} - 0.0298435\, c_{sw}^2 + \frac{2}{3}\frac{1}{16\pi^2}\log(a^2q^2)$$

$$F_2\left(\frac{m^2}{q^2}\right) = \frac{8}{3}\frac{1}{16\pi^2}\frac{m^2}{q^2} - \frac{8}{3}\frac{1}{16\pi^2}\left(-\frac{1}{2} + \frac{m^2}{q^2}\right)\sqrt{1+4\frac{m^2}{q^2}}\,\mathrm{arccoth}\left(\sqrt{1+4\frac{\mathrm{m}^2}{\mathrm{q}^2}}\right) + \frac{2}{3}\frac{1}{16\pi^2}\log\left(\frac{\mathrm{m}^2}{\mathrm{q}^2}\right)$$

$$F_3(aq) = 0.0272837 - 0.0223503 c_{sw} + 0.0070667 c_{sw}^2 - (1-c_{sw})\frac{2}{16\pi^2}\log\left(a^2q^2\right)$$

$$F_4\left(\frac{m^2}{q^2}\right) = -\frac{4}{16\pi^2}\frac{m^2}{q^2} + \frac{4}{16\pi^2}\left[(-1+c_{sw})\left(1+4\frac{m^2}{q^2}\right) + 4\left(\frac{m^2}{q^2}\right)^2\right]\frac{\mathrm{arccoth}\left(\sqrt{1+4\frac{\mathrm{m}^2}{\mathrm{q}^2}}\right)}{\sqrt{1+4\frac{m^2}{q^2}}} - (1-c_{sw})\frac{2}{16\pi^2}\log\left(\frac{m^2}{q^2}\right)$$

- We define the BF coupling to one-loop order as (for $c_{sw} = 1 + O(g_0^2)$):

$$g^2_{BF}(q,m) = g_0^2 + g_0^4\left\{F_1(aq) + F_2\left(\frac{m^2}{q^2}\right) + am\left[F_3(aq) + F_4\left(\frac{m^2}{q^2}\right)\right]\right\}\Big|_{c_{sw}=1} + O(g_0^6) \quad (7)$$

  $g^2_{BF}(q,m)$ can be expressed in terms of the renormalized coupling $g_{\overline{\rm MS}}$ through $Z_g^{L,\overline{\rm MS}}$ where $g^2_{\overline{\rm MS}} = Z_g^{L,\overline{\rm MS}} g_0^2$, $\quad Z_g^{L,\overline{\rm MS}} = 1 - g_0^2\left(b \log\left(a^2\bar\mu^2\right) - a\, m\, b_g\right) + O(g_0^4)$. It easy to show that in order to remove the unwanted lattice contributions ($\log(a)$ and (am)) the coefficients b and b_g must be:

$$b = -\frac{1}{24\pi^2}, \qquad b_g = 0.01200 \qquad (8)$$

Large-mass expansion of the background field coupling

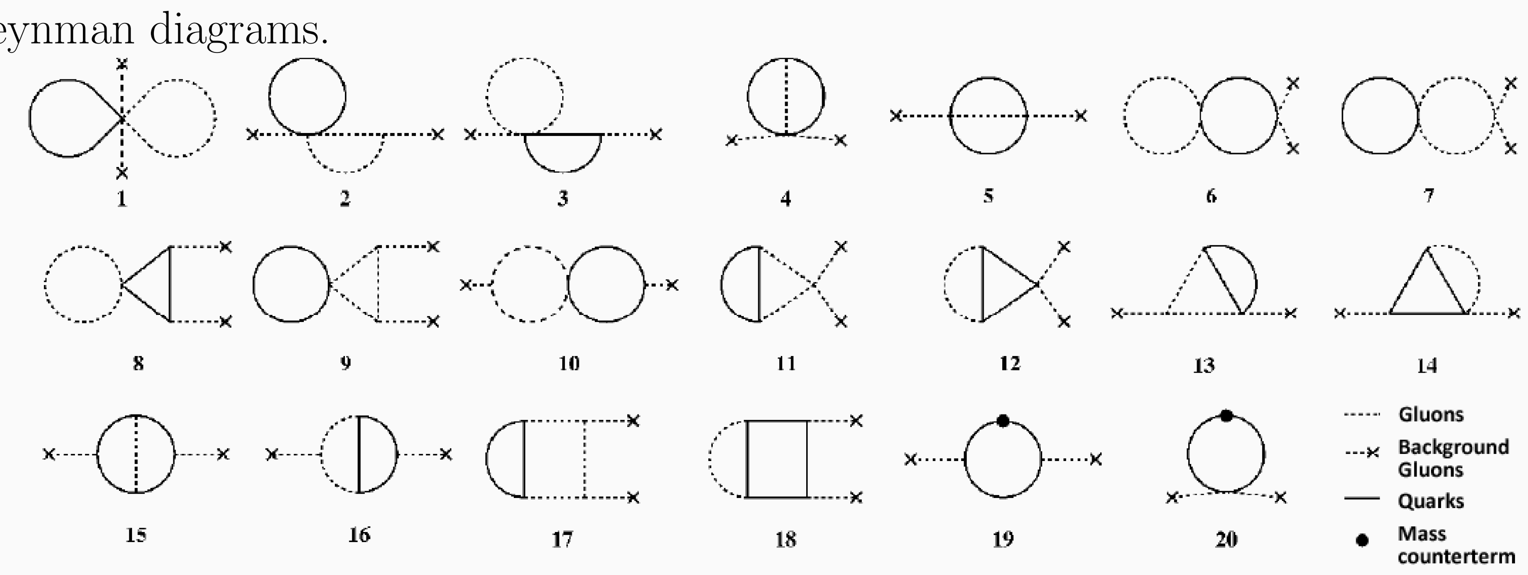
At one loop order, expressing $g^2_{BF}(q,m)$ in terms of renormalized quantities ($g_{\overline{\rm MS}}$, $m_{\overline{\rm MS}} = m\left(1 - \frac{1}{2}am\right)$ and taking the limit $z \to \infty$ ($z = m^2_{\overline{\rm MS}}/q^2$) we get:

$$\lim_{z\to\infty} g^2_{BF}(q, m_{\overline{\rm MS}}) = g^2_{\overline{\rm MS}} + g^4_{\overline{\rm MS}}\left(-0.0314928 + \frac{1}{24\pi^2}\log\left(m^2_{\overline{\rm MS}}/\bar\mu^2\right)\right) + O(a^2, g^6_{\overline{\rm MS}}) \quad (9)$$

Eq. (9) shows the logarithmic mass dependence of the heavy quarks in the continuum limit.

Two-loop Calculations

- The calculation of the two-loop Feynman diagrams is currently underway.
- The two-loop result of the 2-pt lattice Green's function is the sum of the following twenty Feynman diagrams.



- Since we are interested in the $O(am)$ corrections, we use the relation for the tree-level fermion propagator in momentum space: $\langle\psi\bar\psi\rangle = \frac{-i\mathring{\not{q}} + M(q,m)}{\mathring{q}^2 + M(q,m)^2}$, where: $\mathring{\not{q}} = \sum_\mu \gamma_\mu \frac{1}{a}\sin(aq_\mu)$ and $M(q,m) = m + \frac{2}{a}\sum_\mu \sin^2(aq_\mu/2)$.

$$\frac{1}{\mathring{q}^2 + M(q,m)^2} = \frac{1}{\mathring{q}^2 + M(q,0)^2}\left(1 - \frac{4m\frac{1}{a}\sum_\mu \sin^2(aq_\mu/2)}{\mathring{q}^2 + M(q,0)^2} + O(a^2m^2)\right) \qquad (10)$$

- One main difficulty in this computation, as compared to the $O((am)^0)$ calculation, stems from the fact that the fermion propagator now contains contributions of $O(q^{-2})$; this amplifies the presence of potential IR divergences, which must be carefully addressed. Also, the sheer number of terms which must be integrated over the two loop momenta is of the order of $\sim 10^9$; this has necessitated the creation of special-purpose integration routines, in order to overcome the severe contraints on CPU and memory.

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