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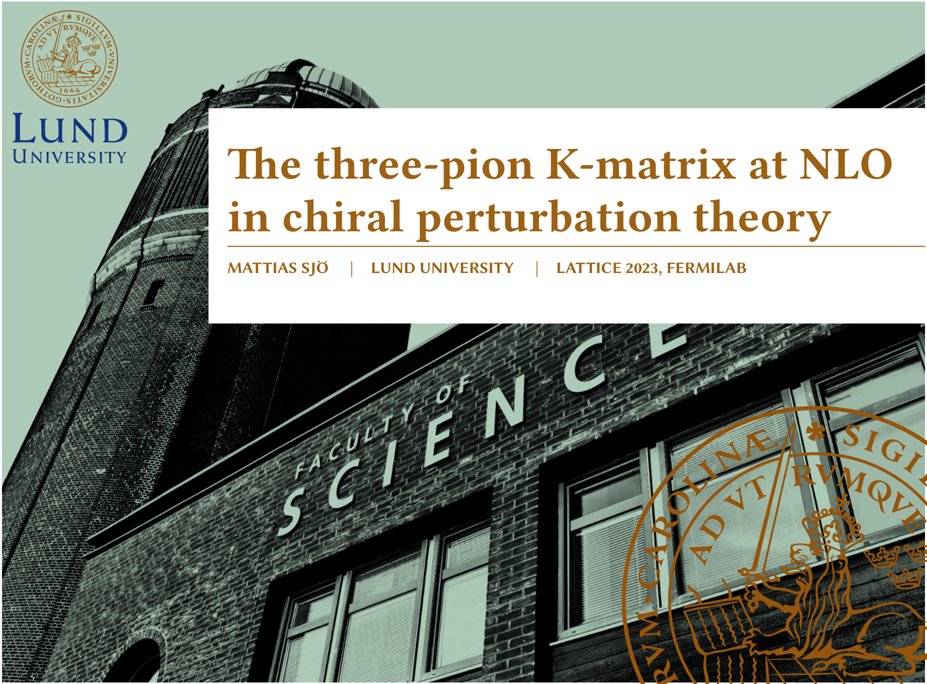
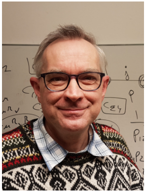
The three-pion K-matrix at NLO in chiral perturbation theory

MATTIAS SJÖ | LUND UNIVERSITY | LATTICE 2023, FERMILAB



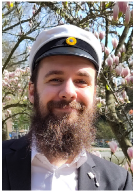
Some background

The People



Hans Bijnens,
Lund U.

Tomáš Husek,
Lund U./Charles U.



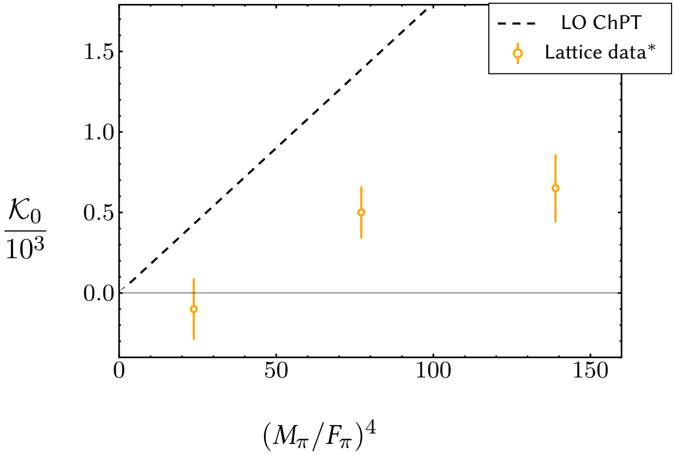
Mattias Sjö,
Lund U.

Stephen Sharpe,
U. of Washington

Fernando
Romero-López,
MIT

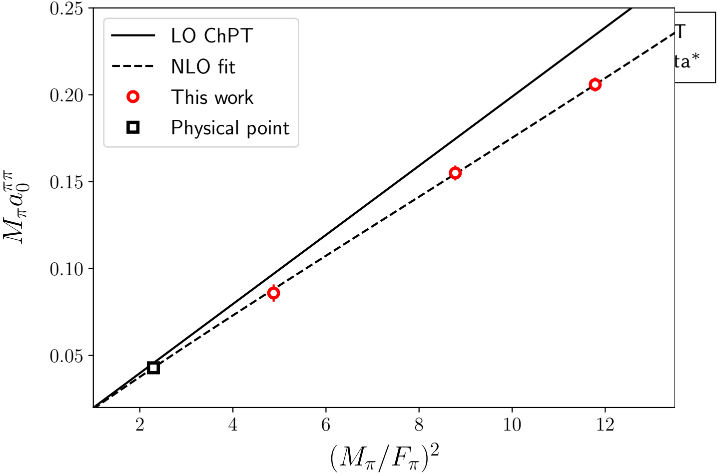
Jorge
Baeza-Ballesteros,
U. de València

The status quo



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, `2021.06144[hep-lat]`

The status quo



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe,
"*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, `2021.06144[hep-lat]`

The $\pi\pi\pi \to \pi\pi\pi$ amplitude

Bijnens & Husek, "*Six-pion amplitude*"

Phys.Rev.D, `2107.06291[hep-ph]`

Bijnens, Husek & **Sjö**, "*Six-meson amplitude in QCD-like theories*"

Phys.Rev.D, `2206.14212[hep-ph]`

Dusting off the LO

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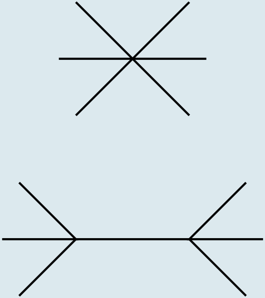
Leading order (pre-ChPT)



Osborn (1969)

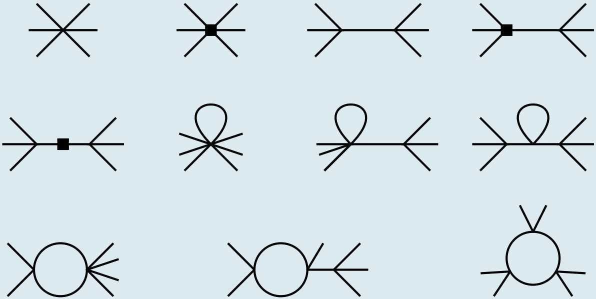
Susskind & Frye (1970)

Six-Meson Diagrams

Vertices

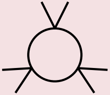
 = LO vertex  = NLO vertex

All the LO and NLO diagrams



One-Loop Integrals

Three-propagator integral

 $\sim$

$$\int \frac{d^d k}{(2\pi)^d} \frac{\{1, k^\mu, k^\mu k^\nu, k^\mu k^\nu k^\rho\}}{(k^2 - M^2)\left[(k-q_1)^2 - M^2\right]\left[(k+q_2)^2 - M^2\right]}$$

In principle: one master integral $\bar{J}$ — impractical

Instead: elegant but redundant basis

$$\{\bar{J}, C, C_{11}, C_{21}, C_3\}(p_1, \ldots, p_6)$$

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The $\pi\pi\pi$ K-matrix

B.-B., Bijnens, Husek, R.-L., Sharpe & **Sjö**, *"The isospin-3 three-particle K-matrix at NLO in ChPT"* *JHEP*, `2303.13206[hep-ph]`

Quantization condition

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Energy spectrum: n particles, box of size L

$$\det\left[F_n^{-1}(E, \boldsymbol{P}, L) + \mathcal{K}_n(E^*)\right] = 0$$

$n = 2$ particles

Lüscher, "*Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories*" *Commun. Math. Phys.* (1986)

$n = 3$ particles

Hansen & Sharpe, "*Relativistic, model-independent, three-particle quantization condition*" *Phys. Rev. D*, `1408.5933[hep-lat]`

(and several other approaches)

Properties of $\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}}$

Lorentz invariant,
Scheme dependent

NLO in ChPT
→ reduces to
algebraic relation

Related to $\mathcal{M}_3^{\mathrm{NLO}}$
through integral
equation

$\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}}$

Divergence-free
(no poles or cuts)

Anatomy of $\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}}$

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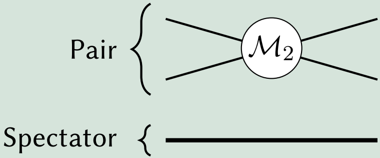
3 particles, 2 scattering



Form of $\mathcal{M}_2$

$\to \mathcal{M}_2(\boldsymbol{p})_{lm,l'm'}$

$\boldsymbol{p}$ — spectator momentum

$Y_m^l(\theta, \phi)$ — pair angular momentum

Anatomy of $\mathcal{K}_{\text{df},3}^{\text{NLO}}$

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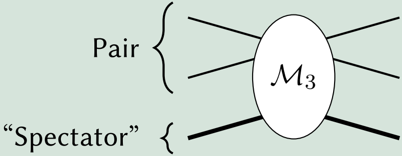
3 particles, 3 scattering



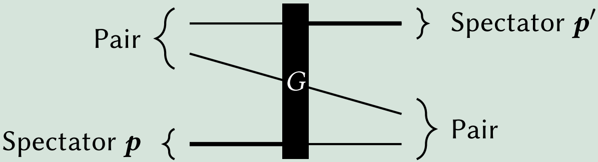
Form of $\mathcal{M}_3$

$\to \mathcal{M}_3(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'}$

Particle exchange $\Leftrightarrow$ Spectator choice

Anatomy of $\mathcal{K}^{\text{NLO}}_{\text{df},3}$

On-shell propagator substitute



Properties of G

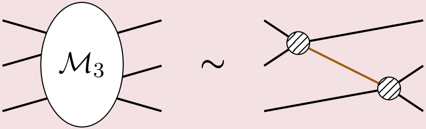
On-shell only — propagator-like near pole:

$$G(\boldsymbol{p}, \boldsymbol{p}')_{lm,l'm'} \sim \frac{1}{(P - p - p')^2 - M_\pi + i\epsilon}$$

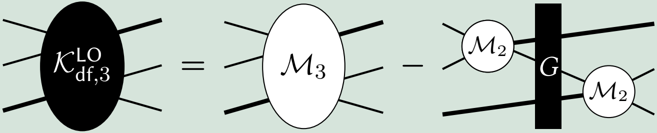
Smooth cutoff far from pole — non-analytic

$\mathcal{K}_{\text{df},3}$ at leading order

One-particle exchange pole

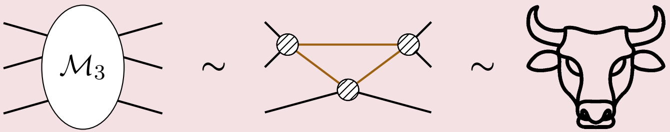


One-particle exchange subtraction

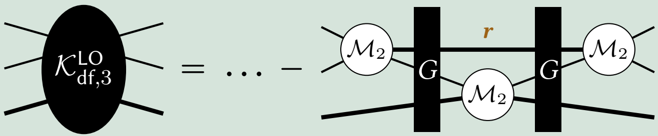


$\mathcal{K}_{\mathrm{df},3}$ at next-to-leading order

Bull's head cut



Bull's head subtraction



Threshold expansion

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Expansion parameters

$$\Delta \propto P^2 - (3M_\pi)^2 \qquad \text{(system above-thr'ness)}$$

$$\Delta_i^{(\prime)} \propto (P - p_i^{(\prime)})^2 - (2M_\pi)^2 \qquad \text{(pair above-thr'ness)}$$

$$\tilde{t}_{ij} \propto (p_i - p_j')^2 \qquad \text{(spectator above-thr'ness)}$$

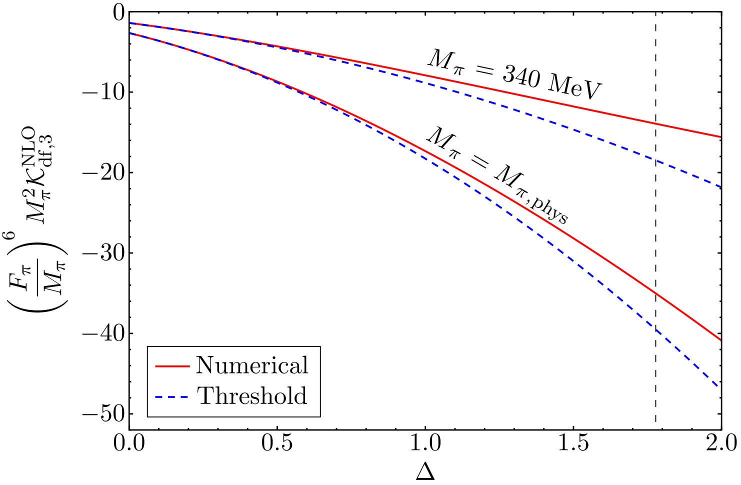
Compound parameters

$$\Delta_A = \sum_i (\Delta_i^2 + \Delta_i'^2) - \Delta^2 \qquad \Delta_B = \sum_{ij} \tilde{t}_{ij}^2 - \Delta^2$$

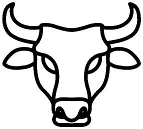
Threshold expansion

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_0 + \mathcal{K}_1\Delta + \mathcal{K}_2\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B + \mathcal{O}(\Delta^3)$$

Accuracy of threshold expansion



Expanding the bull's head



Expand

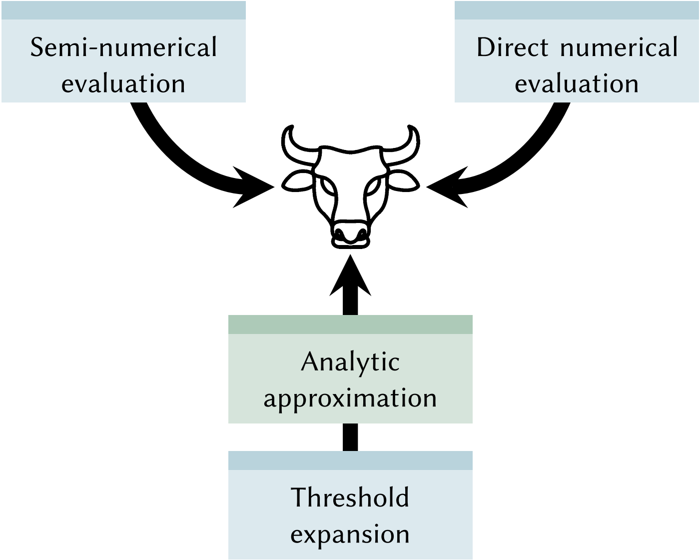
$$\sum \int \mathrm{d}x\ [\text{End-point divergent}](x)$$

Hadamard finite part

$$\sum \int \mathrm{d}x\ [\text{Finite}](x)$$

$$\underbrace{\sum \int \mathrm{d}x\ [\text{Simple}](x)}_{\text{analytic}} + \underbrace{\int \mathrm{d}x\ [\text{Small}](x)}_{\text{numeric}}$$

Different approaches



Results!

$$\mathcal{K}_0 = \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-3\kappa(35 + 12\log 3) - \mathcal{D}_0 + 111L + \ell^{\mathrm{r}}_{(0)}\right],$$

$$\mathcal{K}_1 = \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-\frac{\kappa}{20}(1999 + 1920\log 3) - \mathcal{D}_1 + 384L + \ell^{\mathrm{r}}_{(1)}\right],$$

$$\mathcal{K}_2 = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{207\kappa}{1400}(2923 - 420\log 3) - \mathcal{D}_2 + 360L + \ell^{\mathrm{r}}_{(2)}\right],$$

$$\mathcal{K}_{\mathrm{A}} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{9\kappa}{560}(21809 - 1050\log 3) - \mathcal{D}_{\mathrm{A}} - 9L + \ell^{\mathrm{r}}_{(\mathrm{A})}\right],$$

$$\mathcal{K}_{\mathrm{B}} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{27\kappa}{1400}(6698 - 245\log 3) - \mathcal{D}_{\mathrm{B}} + 54L + \ell^{\mathrm{r}}_{(\mathrm{B})}\right].$$

$$\mathcal{D}_0 \approx -0.0563476589\,, \qquad \mathcal{D}_1 \approx 0.129589681\,, \qquad \mathcal{D}_2 \approx 0.432202370\,,$$
$$\mathcal{D}_{\mathrm{A}} \approx 9.07273890 \cdot 10^{-4}\,, \qquad \mathcal{D}_{\mathrm{B}} \approx 1.62394747 \cdot 10^{-4}\,,$$

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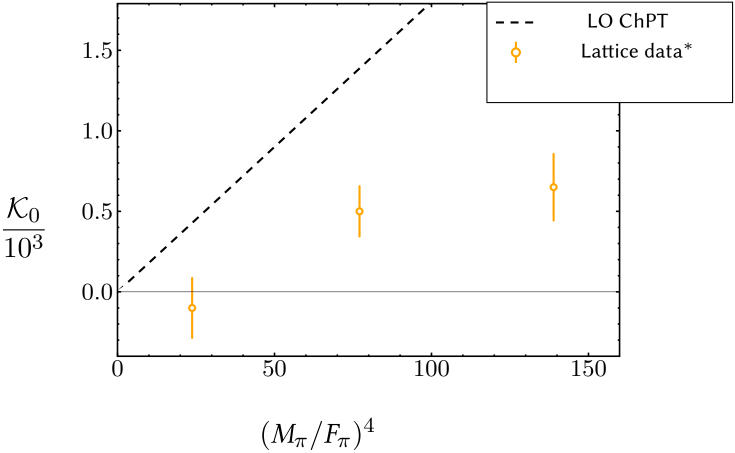
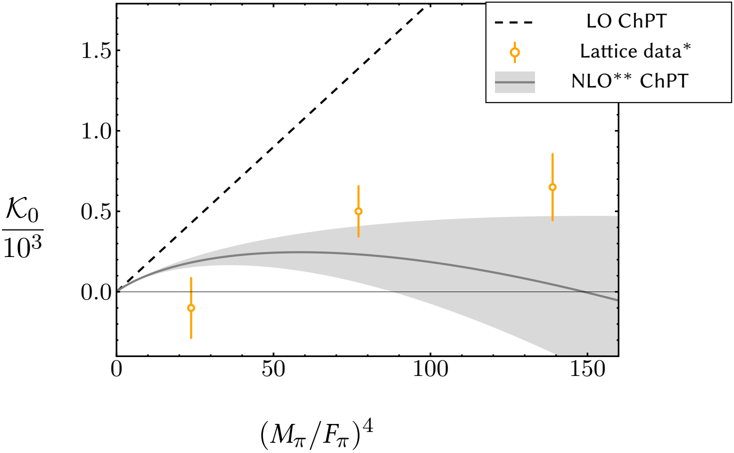
Reconciliation with the lattice



* Blanton, Hanlon, Hörz, Morningstar, Romero-López & Sharpe, "*Three-body interactions from the finite-volume QCD spectrum*" *Phys.Rev.D*, 2021.06144[hep-lat]

Reconciliation with the lattice



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** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" *Nucl.Phys.B*, hep-ph/0103088

Reconciliation with the lattice

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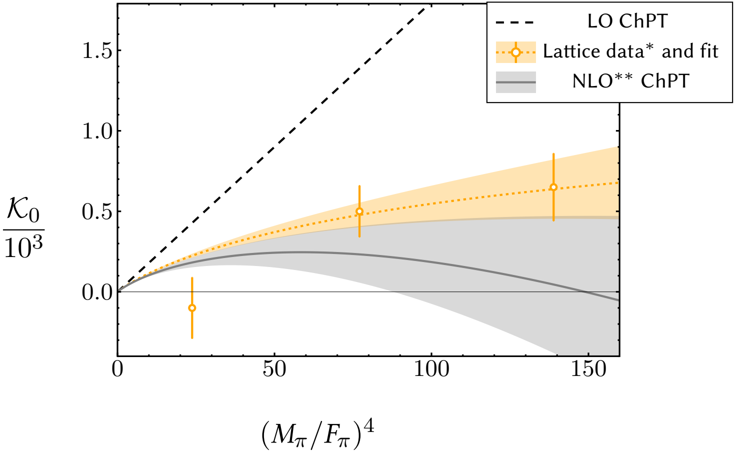

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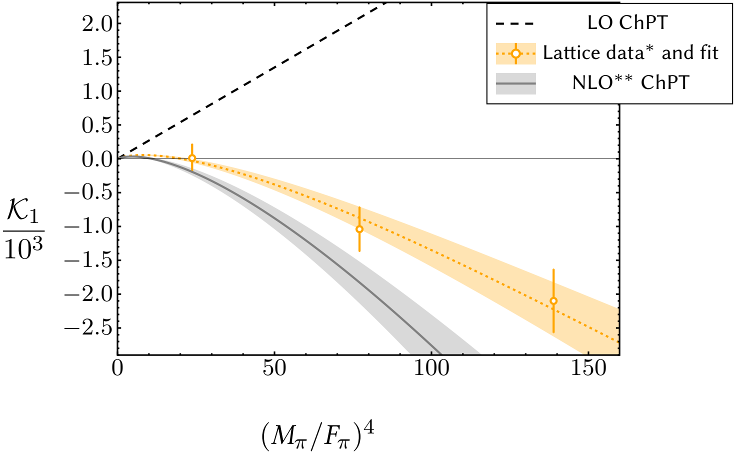
** using LECs from FLAG and Colangelo, Gasser & Leutwyler, *"ππ scattering"* *Nucl.Phys.B*, hep-ph/0103088

Ditto: Subleading order



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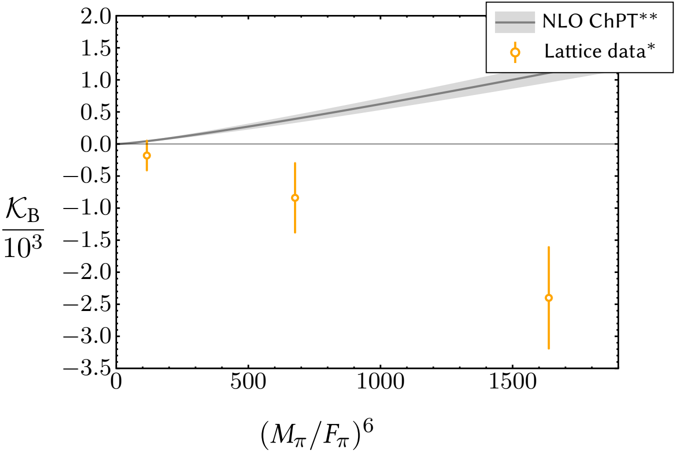
Higher orders?



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** using LECs from FLAG and Colangelo, Gasser & Leutwyler, "*$\pi\pi$ scattering*" *Nucl.Phys.B*, `hep-ph/0103088`

Next steps: All isospin channels

B.-B., Bijnens, Husek, R.-L., Sharpe & **Sjö**, *"The three-particle K-matrix at NLO in ChPT for general isospin"* (in preparation, title preliminary)

Isospin channels

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Done

$\pi^\pm\pi^\pm\pi^\pm$: Purely $I = 3$

Not done (for example)

$\pi^0\pi^0\pi^0$: Mixture of $I = 0, 1, 2, 3$

Principles already mapped out:

Hansen, R.-L. & Sharpe, *"Generalizing the relativistic quantization condition to include all three-pion isospin channels"* *JHEP*, `2003.10974[hep-lat]`

Nontrivial structures

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Representations of particle exchange

$I = 3$	Singlet	
$I = 2$	Doublet	
$I = 1$	Singlet	Doublet
$I = 0$	Antisymmetric singlet	

Much more complicated threshold expansion!

Teaser: LO results

$$I=0: \qquad \mathcal{K}=0\,,$$

$$I=1,\text{S-S}: \qquad \mathcal{K}=-\left[\frac{111}{8}+\frac{1137}{64}\Delta\right]\left(\frac{M_\pi}{F_\pi}\right)^4,$$

$$I=1,\text{S-D}: \qquad \mathcal{K}=-\frac{3\sqrt{30}}{8}\left(\frac{M_\pi}{F_\pi}\right)^4,$$

$$I=1,\text{D-D}: \qquad \mathcal{K}=\frac{1}{2}\left(\frac{M_\pi}{F_\pi}\right)^4,$$

$$I=2: \qquad \mathcal{K}=\frac{9}{2}\left(\frac{M_\pi}{F_\pi}\right)^4,$$

$$I=3: \qquad \mathcal{K}=[18+27\Delta]\left(\frac{M_\pi}{F_\pi}\right)^4.$$

Summary

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Complete

$\pi\pi\pi \to \pi\pi\pi$

Complete

$\pi\pi\pi$ K-matrix, $I = 3$

Summary

Complete

$\pi\pi\pi \to \pi\pi\pi$

Complete

$\pi\pi\pi$ K-matrix, $I = 3$

Ongoing

$\pi\pi\pi$ K-matrix, $I = 0, 1, 2$

Summary

Complete

$\pi\pi\pi \to \pi\pi\pi$

Complete

$\pi\pi\pi$ K-matrix, $I = 3$

Ongoing

$\pi\pi\pi$ K-matrix, $I = 0, 1, 2$

Future endeavor?

$\pi\pi K, KK\pi$ K-matrix

Backup slides

Simplification & shortening by $\mathcal{O}(6!)$

Flavour structures

$$\mathcal{F}_{\{6\}}(a_1,\dots,a_6) = \langle t^{a_1}\cdots t^{a_6}\rangle$$
$$\mathcal{F}_{\{2,4\}}(a_1,\dots,a_6) = \langle t^{a_1}t^{a_2}\rangle\langle t^{a_3}\cdots t^{a_6}\rangle$$
$$\mathcal{F}_{\{3,3\}}(a_1,\dots,a_6) = \langle t^{a_1}t^{a_2}t^{a_3}\rangle\langle t^{a_4}t^{a_5}t^{a_6}\rangle$$
$$\mathcal{F}_{\{2,2,2\}}(a_1,\dots,a_6) = \langle t^{a_1}t^{a_2}\rangle\langle t^{a_3}t^{a_4}\rangle\langle t^{a_5}t^{a_6}\rangle$$

$$\mathcal{M}(p_1,a_1;p_2,a_2;\dots;p_6,a_6) = \sum_R\sum_\sigma \mathcal{M}_R(\sigma[p_1,\dots,p_6])\mathcal{F}_R(\sigma[a_1,\dots,a_6])$$

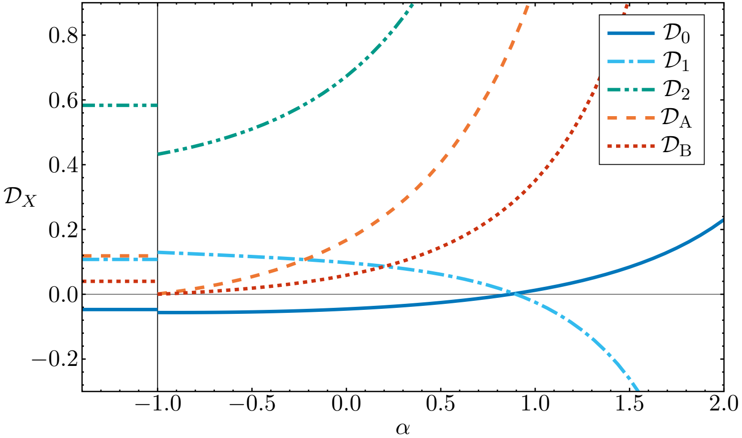
Stripping

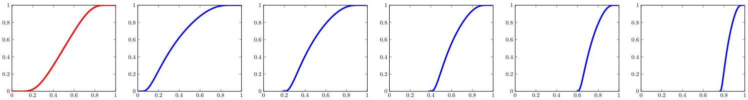
$\sigma \notin$ symmetries of $\mathcal{F}_R$
$\to$ well-known, unique

Deorbiting

$\sigma \in$ symmetries of $\mathcal{F}_R$
$\to$ novel, non-unique!

Cutoff dependence

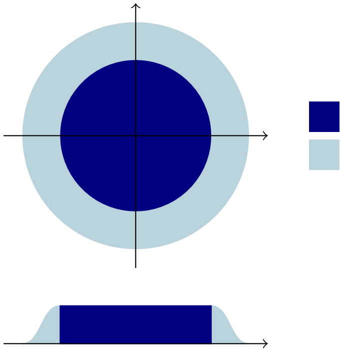




Semi-numerical bull's head

How to integrate?

$$\int \frac{d^3 \boldsymbol{r}}{2\omega_r} \frac{[\text{Non-analytic}]}{[\text{Complicated}]}$$

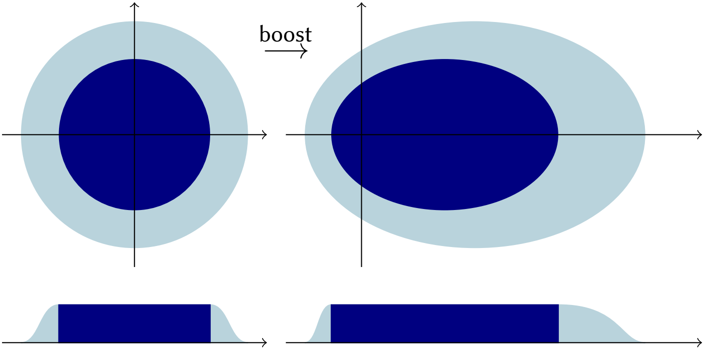


analytic, has poles
non-analytic, smooth

Semi-numerical bull's head

How to integrate?

$$\int \frac{d^3\boldsymbol{r}}{2\omega_r}\frac{[\text{Complicated angular dependence}]}{[\text{Much simpler}]}$$



Semi-numerical bull’s head

How to integrate?

$$\int \frac{d^3\boldsymbol{r}}{2\omega_r} \frac{\text{[Simple] - [Smooth]}}{\text{[Much simpler function]}}$$