Lattice construction of mixed ’t Hooft anomaly with higher form symmetry

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Lattice 2023 @ Fermilab
2023/07/31～08/04

Topology of SU(N) lattice gauge theories coupled with $\mathbb{Z}_N$ 2-form gauge fields
M. Abe, O. Morikawa, S. Onoda, H. Suzuki and Y. Tanizaki
arXiv:2303.10977[hep-th]

Fractional topological charge in lattice Abelian gauge theory
M. Abe, O. Morikawa and H. Suzuki
PTEP 2023 (2023) 2, 023B03 [arXiv:2210.12967[hep-th]]

Symmetry and Anomaly Ⅰ

- Classical Theory：Conservation low $\longleftrightarrow$ Symmetry （Noether Theorem）
- Quantum Theory：The conservation low may be broken (Anomaly).
  - ➢ Focus on the Partition function

$$Z[A] = \int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A]}$$

  - ➢ How to distinguish the anomaly : Whether the Z is invariant or not under a transformation

$$Z' \overset{?}{=} Z$$

$$\rightarrow Z'[A+\partial\theta] = \int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A+\partial\theta]}$$

$$= \underbrace{e^{\mathcal{A}[\theta,A]}}_{\text{'t Hooft anomaly}} \underbrace{\int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A]}}_{=Z}$$

Symmetry and Anomaly II

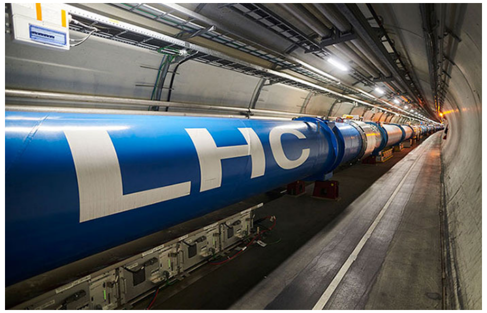
- ➢ We can predict low energy dynamics of the gauge theory.

  ※Gauge theory : Theory which describes the Standard Model of particles
- ✓ e. g., we decided the theory for the strong interaction is the SU(3) gauge theory because the theory and the experiment are well matched.

Particle Theory

Predict

Particle Experiment



https://www.icepp.s.u-tokyo.ac.jp/information/20220426.html

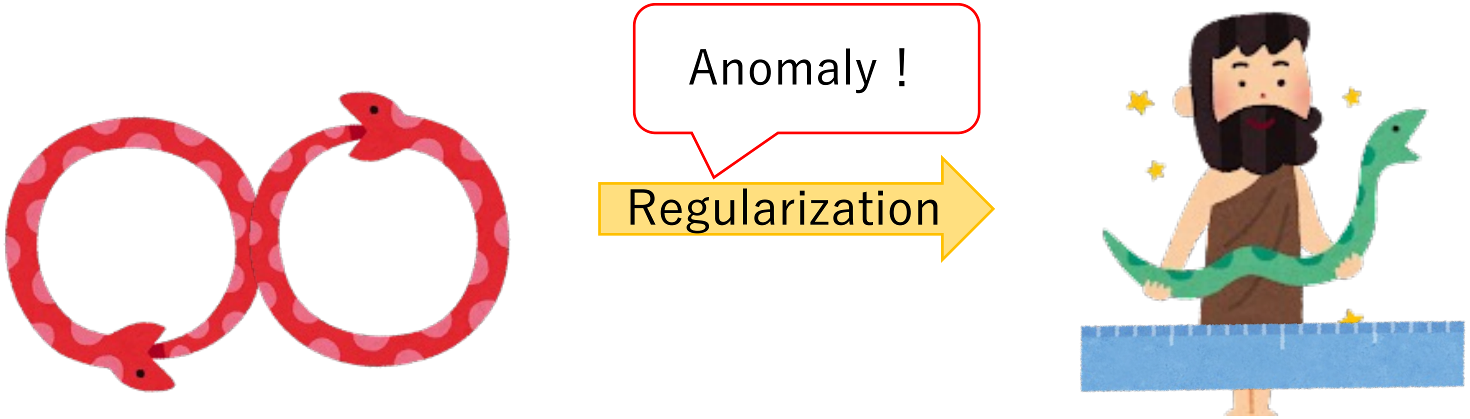
High

Low

Energy

Anomaly and Quantum Field Theory

- The anomaly is a peculiar phenomenon in quantum field theory (QFT).
  - ➢ QFT has the infinite degree of freedom.
  - ➢ To define QFT correctly, we let the infinite degree be finite (Regularization).
  - ➢ This regularization breaks the symmetry (Anomaly).



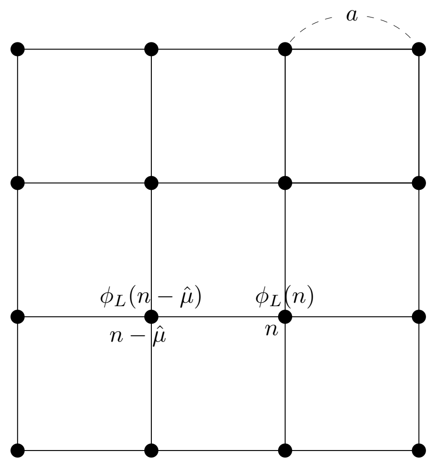
The infinite DOF　　　　　　The Finite DOF

Recent Developments in Anomalies

- Recently, Gaiotto et al. has expanded the concept of symmetry. : Higher Form Symmetry
  (Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148[hep-th])
  - ➢ By anomalies with higher form (and discrete) symmetries, the low energy dynamics of gauge theories has been discussed. (Gaiotto, Kapustin, Komargodski, Seiberg, arXiv:1703.00501)
  - ➢ Many new anomalies have been discovered and related studies has been done.
    - ✓ Yamaguchi, arXiv:1811.09390[hep-th]
    - ✓ Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389[hep-th]
    - ✓ Honda, Tanizaki, arXiv:2009.10183[hep-th]
    - ✓ etc.

☆Motivation : Understand these anomalies in the lattice field theory where we treat the regularization well.

Lattice Gauge Theory



Anomaly of the $SU(N)$ gauge theory with θ term

- The $SU(N)$ gauge theory with the θ term has the time reversal ($\mathcal{T}$) symmetry at $\theta = \pi$.

$$Z = \int \mathcal{D}a\ e^{S[a]} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\theta Q[a]}, \quad Q \in \mathbb{Z}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z' = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(-Q[a])} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(+Q[a])} \underbrace{e^{-i2\pi Q[a]}}_{=1} = Z$$

- Then, we construct the $SU(N)$ gauge theory with the higher form symmetry ($\mathbb{Z}_N$ 1-form gauge symmetry). This means we couple $\mathbb{Z}_N$ 2-form gauge field to the theory.
  - ➢ The topological charge (TC) becomes fractional, so it is not invariant under the $\mathcal{T}$ transformation.

Important!!

$$e^{-i2\pi Q} \neq 1$$

  - ➢ This theory at $\theta = \pi$ has the mixed anomaly between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry.

Topological Charge on the Lattice

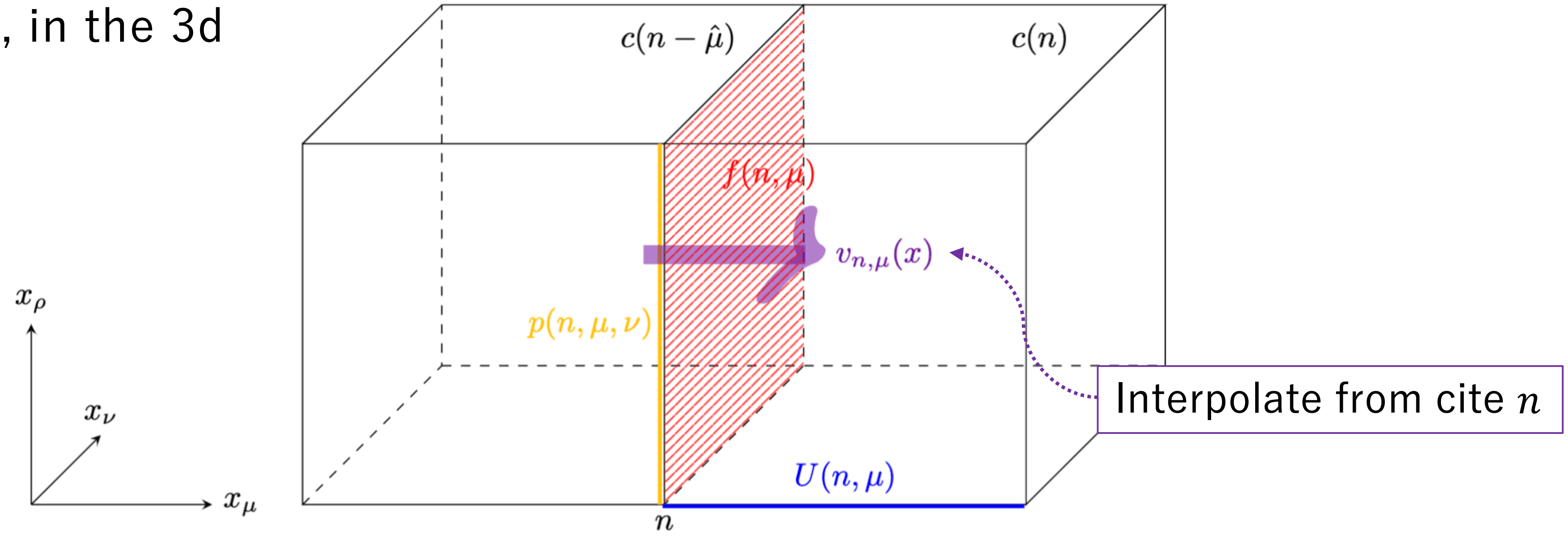
- How to calculate the topological charge Q,

$$Q = -\frac{1}{24\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{f(n,\mu)} \mathrm{d}^3x \, \mathrm{tr}\left[(\tilde{v}_{n,\mu}^{-1}\partial_\nu \tilde{v}_{n,\mu})(\tilde{v}_{n,\mu}^{-1}\partial_\rho \tilde{v}_{n,\mu})(\tilde{v}_{n,\mu}^{-1}\partial_\sigma \tilde{v}_{n,\mu})\right]$$
$$-\frac{1}{8\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{p(n,\mu,\nu)} \mathrm{d}^2x \, \mathrm{tr}\left[(\tilde{v}_{n,\mu}\partial_\rho \tilde{v}_{n,\mu}^{-1})(\tilde{v}_{n-\hat{\mu},\nu}^{-1}\partial_\sigma \tilde{v}_{n-\hat{\mu},\nu})\right].$$

  - $v_{n,\mu}(x)$ is the gauge translation function (transition function).
- On the lattice, topological values are ill-defined.
  - Restricting the size of plaquette (Admissibility condition), Lüscher constructed integer TC on the lattice (Lüscher, Commun. Math. Phys. 85 (1982)).
  - We aim to construct the fractional TC on the $SU(N)$ lattice by extended the Lüscher's topological charge.
    - ✓ Itou, arXiv:1811.05708[hep-th]
    - ✓ Anosova, Gattringer, Göschl, Sulejmanpasic, Törek, arXiv:1912.11685 [hep-lat]

Transition Function on the Lattice

- The manifold is divided by hyper cubes $c(n)$.
- Fiber bundle describes the gauge theory by transition function for gauge transformation.
- e. g., in the 3d



Transition Function for Fractional TC

- Coupling $\mathbb{Z}_N$ 2-form field to the theory, the structure of fiber bundle becomes rich.

$$v_{n-\hat{\nu},\mu}(n)v_{n,\nu}(n)v_{n,\mu}(n)^{-1}v_{n-\hat{\mu},\nu}(n)^{-1} = \mathbb{1}.$$

$$\tilde{v}_{n-\hat{\nu},\mu}(n)\tilde{v}_{n,\nu}(n)\tilde{v}_{n,\mu}(n)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(n)^{-1} = e^{\frac{2\pi i}{N}B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}\mathbb{1}.$$

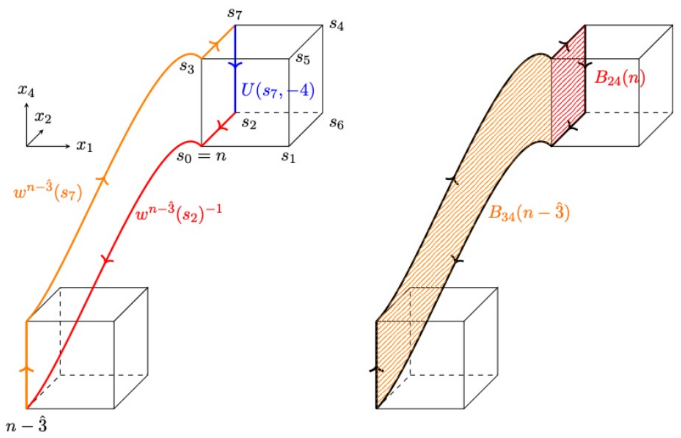
- We find that the $\mathbb{Z}_N$ 1-form gauge invariance plays the center role.

➢ Admissibility condition

$$\|\mathbb{1} - \tilde{U}_{\mu\nu}(n)\| < \varepsilon$$

$$\tilde{U}_{\mu\nu}(n) \equiv e^{-\frac{2\pi i}{N}B_{\mu\nu}(n)} \times U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}$$

➢ Components of transition function



Construction of $u^{n-\hat{3}}_{s_7 s_2}$ Construction of $\tilde{u}^{n-\hat{3}}_{s_7 s_2}$

Fractional TC

- By the $\mathbb{Z}_N$ 1-form invariant transition function, we calculate TC.

$$z_{\mu\nu} = \sum_{p \in (T^2)_{\mu\nu}} B_p \qquad \mod N.$$

$$Q_{\text{top}} = -\frac{1}{N}\int_{T^4} \frac{1}{2} P_2(B_p) \mod 1 \in -\frac{1}{N}\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8} + \mathbb{Z}$$

$$P_2(B_p) = B_p \cup B_p + B_p \cup_1 \mathrm{d}B_p$$

- In the $U(1)$ lattice gauge theory, we make sure that (cf. Abe, Morikawa, Suzuki, arXiv:2210.12967[hep-th])

$$Q_{\text{top}} = \frac{1}{32\pi^2} \sum_{n \in \Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \tilde{F}_{\mu\nu}(n) \tilde{F}_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) \in \frac{1}{N^2}\mathbb{Z} + \mathbb{Z}$$

Anomaly I

- The action on the lattice is

$$S[U_l, B_p] \equiv -S_W[U_l, B_p] + i\theta Q_{\mathrm{top}}[U_l, B_p]$$

- The topological charge is

$$Q_{\mathrm{top}} = -\frac{1}{N}\int_{T^4}\frac{1}{2}P_2(B_p) + \mathbb{Z} \equiv \mathrm{frac}[B_p] + \mathrm{int}[U_l, B_p]$$

  ✧ Invariant under the $\mathbb{Z}_N$ one-form gauge transformation

  ✧ Odd under the $\mathcal{T}$ transformation on the lattice,

$$Q_{\mathrm{top}} \xrightarrow{\mathcal{T}} -Q_{\mathrm{top}}$$

  ➢ We discuss the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry.

Anomaly II

- At $\theta = \pi$, the partition function is, under $\mathcal{T}$ transformation,

$$Z[B_p] = \int \mathcal{D}U_l \ e^{S[U_l, B_p]} = \int \mathcal{D}U_l \ e^{-S_W[U_l, B_p]} e^{i\theta Q_{\text{top}}[U_l, B_p]}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z'[B_p] = \int \mathcal{D}U_l \ e^{-S_W[U_l, B_p]} e^{i\pi(-Q_{\text{top}}[U_l, B_p])} = \int \mathcal{D}U_l \ e^{-S_W[U_l, B_p]} e^{i\pi Q_{\text{top}}[U_l, B_p]} \underbrace{e^{-i2\pi Q_{\text{top}}[U_l, B_p]}}_{= e^{-i2\pi \text{int}[U_l, B_p]} e^{-i2\pi \text{frac}[B_p]}}$$

$$= e^{-i2\pi \text{frac}[B_p]} \underbrace{\int \mathcal{D}U_l \ e^{-S_W[U_l, B_p]} e^{i\pi Q_{\text{top}}[U_l, B_p]}}_{=Z} \neq Z[B_p]$$

- This means that there is the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry.

Conclusion and Future Work

☆Conclusion

- We construct the fractional topological charge on the $SU(N)$ gauge theory.
- By this topological charge, we construct the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry.

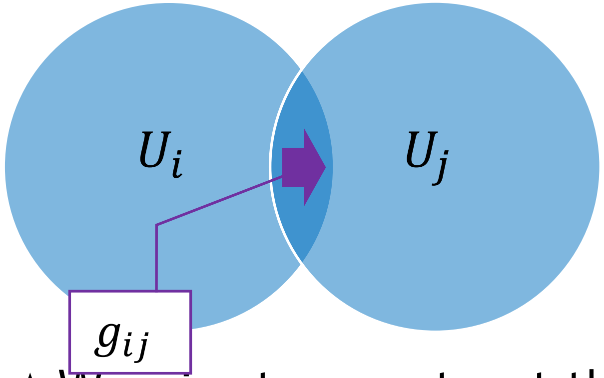
☆Future work

- Construct the magnetic operator under the admissibility condition on the lattice
    - ✓ cf. Abe, Morikawa, Onoda, Suzuki, Tanizaki, arXiv:2304.14815 [hep-lat]
- Construct non-invertible symmetries on the lattice

$U(1)$ Part

Fiber Bundle and Fractional Topological Charge

- We construct the fiber bundle which makes the topological charge fractional.
  ('t Hooft, Nucl. Phys. B 153 (1979), van Baal, Commun. Math. Phys. 85 (1982))



cocycle condition: $g_{ij}g_{jk}g_{ki} = \underbrace{\exp\left(\frac{2\pi i}{N} n_{ijk}\right)}_{\in \mathbb{Z}_N}$

$$(\text{non-trivial transition function}) \sim \omega_\mu \times (SU(N) \text{ transition function})$$

factor of fractionality

☆We aim to construct the fractional topological charge on the lattice.

- ➢ We utilize the formulation for the integer topological charge on the $SU(N)$ lattice gauge theory.
  (Lüscher, Commun. Math. Phys. 85 (1982))

➢We pay attention to the $\mathbb{Z}_N$ one form invariance.

New Transition Function on the Lattice

transition function with the factor of fractionality in the continuum theory

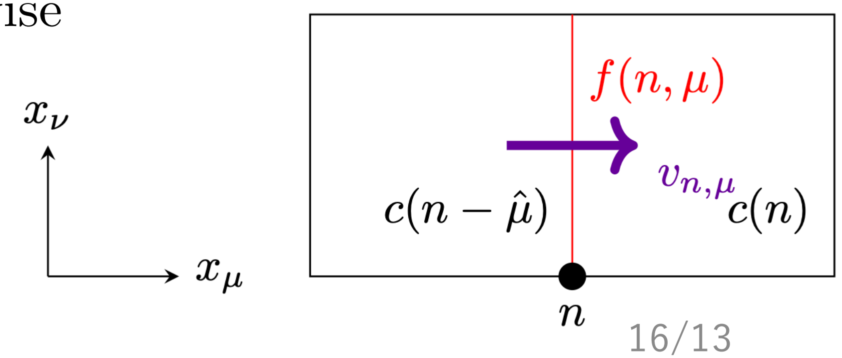
$$(\text{non-trivial transition function}) \sim \omega_\mu \times (SU(N)\ \text{transition function})$$

- We construct the transition function $v_{n,\mu}$ at $x \in f(n,\mu)$ in the $U(1)/\mathbb{Z}_q$ lattice gauge theory.
  - ω_μ is the factor of fractionality on the lattice.

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x)$$

$$\omega_\mu(x) \equiv \begin{cases} \exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right) & \text{for } x_\mu = 0 \bmod L \\ 1 & \text{otherwise} \end{cases}$$

  - $z_{\mu\nu} \in \mathbb{Z}$ and $z_{\mu\nu} = -z_{\nu\mu}$



Topological Charge on the Lattice

- We calculate the topological charge by the new transition function.

$$Q=-\frac{1}{8\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\int_{p(n,\mu,\nu)}\mathrm{d}^2x\,[v_{n,\mu}(x)\partial_\rho v_{n,\mu}(x)^{-1}][v_{n-\hat{\mu},\nu}(x)^{-1}\partial_\sigma v_{n-\hat{\mu},\nu}(x)]$$

- By the new transition function $v_{n,\mu}(x)=\omega_\mu(x)\check{v}_{n,\mu}(x)$

$\boxed{\text{factor of fractionality}}$

$$\omega_\mu(x)\sim\exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right)$$

$$Q=\underbrace{\frac{1}{8q^2}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}}_{\text{Fractional!!}}+\underbrace{\frac{1}{8\pi q}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}\sum_{n_\mu=0}\check{F}_{\rho\sigma}(n)}_{\text{cross term}}$$

$$+\underbrace{\frac{1}{32\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\check{F}_{\mu\nu}(n)\check{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})}_{\text{integer}}$$

Anomaly

- The action on the lattice is

$$S \equiv \overbrace{\frac{1}{4g_0^2}\sum_n \sum_{\mu,\nu} \check{F}_{\mu\nu}(n)\check{F}_{\mu\nu}(n) + S_{\text{matter}}}^{=S_0} \underbrace{-iq\theta Q}$$

By the Witten effect(Honda, Tanizaki, arXiv:2009.10183)

- The topological charge is

$$qQ = \frac{1}{8q}\sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \mathbb{Z} \equiv \mathrm{frac}[z] + \mathrm{int}[a, z]$$

✧ Invariant under the $\mathbb{Z}_q$ one-form gauge transformation

✧ Odd under the $\mathcal{T}$ transformation on the lattice, $qQ \xrightarrow{\mathcal{T}} -qQ$

➢ We discuss the anomaly at $\theta = \pi$ between the $\mathbb{Z}_q$-one form gauge and the $\mathcal{T}$ symmetry.

Anomaly

- Adding the local counter term, at $\theta = \pi$ the partition function is, under $\mathcal{T}$ transformaion,

$$Z[z] = \int \mathcal{D}a\ e^{S[a,z]} = \int \mathcal{D}a\ e^{S_0[a,z]} e^{i\theta q Q[a,z]}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z' = \int \mathcal{D}a\ e^{S_0[a,z]} e^{i\pi(-qQ[a,z])} = \int \mathcal{D}a\ e^{S_0[a,z]} e^{i\pi q Q[a,z]} \underbrace{e^{-i2\pi q Q[a,z]}}_{=e^{-i2\pi \mathrm{int}[a,z]} e^{-i2\pi \mathrm{frac}[z]}}$$

$$= e^{-i2\pi \mathrm{frac}[z]} \underbrace{\int \mathcal{D}a\ e^{S_0[a,z]} e^{i\pi q Q[a,z]}}_{=Z} \neq Z$$

$$\xrightarrow[\text{including counter term}]{} \exp\left[-\frac{2\pi i(2k+1)}{8q} \underbrace{\sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}_{=0,\pm 8,\pm 16,\cdots,}\right] Z$$

For $q \in 2\mathbb{Z}$, the anomaly exists!

Back Up

’t Hooft Anomaly

- ’t Hooft anomaly：
  Couple a background gauge field A_μ with the preserved current j_μ related to the symmetry

$$Z[A_\mu] = \left\langle \exp\left(i \int A_\mu j^\mu \right) \right\rangle \qquad Z[A_\mu + \partial_\mu \theta] = Z[A_\mu] \underbrace{\exp(i\mathcal{A}(\theta, A_\mu))}_{\text{Phase Gap}}$$

- ’t Hooft anomaly matching:
  The property of matching the ’t Hooft anomaly calculated respectively in both UV and IR theory

➢ Using the prediction of the low-energy physics of gauge theories

’t Hooft Anomaly Matching Condition

➢ Application example

✓ Restricting the low-energy effective theory of QCD, this condition requires lagrangian to have the Wess-Zumino-Witten term.

✓ Since a part of the background gauge field exists as the gauge field in Electro-Weak gauge theory, ’t Hooft anomaly can be observed in the collapse of neutral π meson. To match the experiment with this theory, the strong field theory is detected to $SU(3)$ gauge theory.

Topological Charge

- Topology：

Same under the continuum transformation

- Normal quantum physics : Wave functions are characterized by electron‘s charge.
  - ➢ There are physics which is characterized by “topological charge”.
  - ✓ e.g., superconductor, topological soliton,⋯

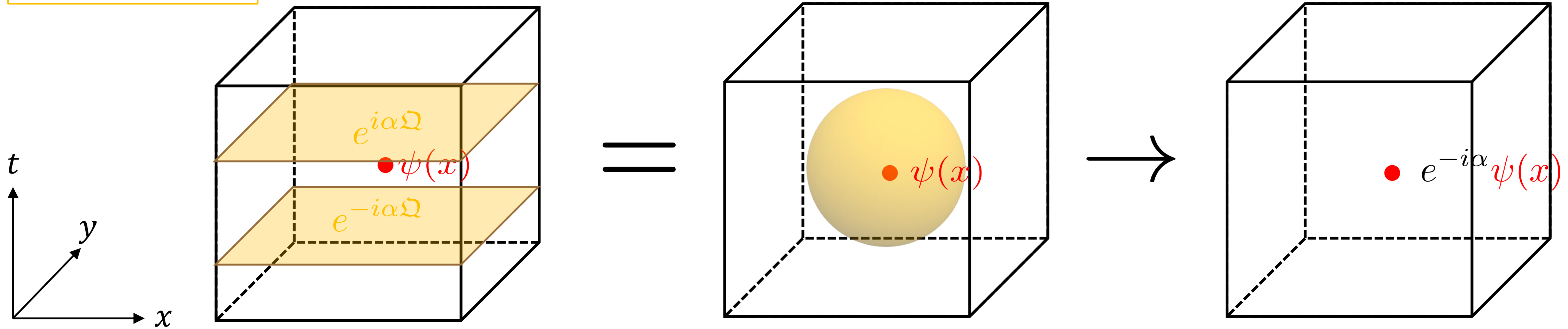
Higher Form Symmetry Ⅰ

- First, we look normal symmetry (0-form symmetry)

➢ (2+1)d
By the field $\psi(x)$'s transformation, the charge $\mathfrak{Q}$ spreads in 2d.

$$e^{i\alpha\mathfrak{Q}}\psi(x)e^{-i\alpha\mathfrak{Q}} = e^{-i\alpha}\psi(x), \quad \mathfrak{Q} = \int_{M^2} \mathrm{d}^2x\, j^0(x), \quad j^\mu(x) = i\bar{\psi}(x)\gamma^\mu\psi(x)$$

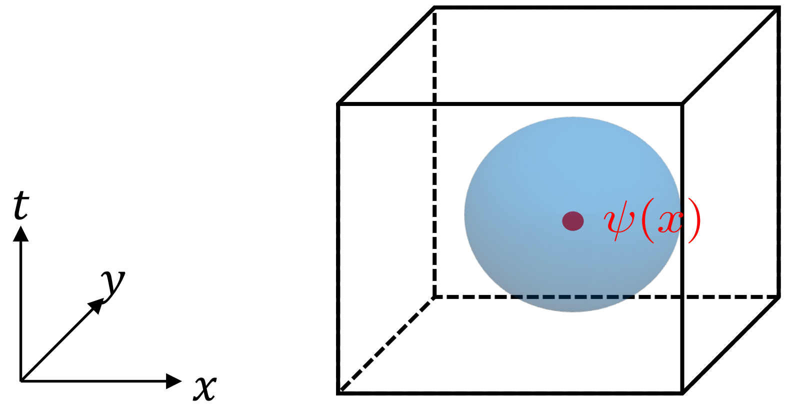
graphical view

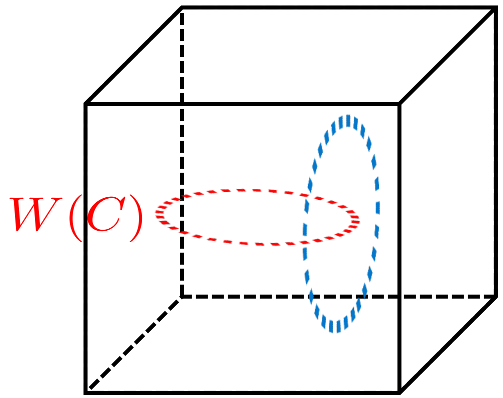


Higher Form Symmetry Ⅱ

- Normal symmetry (0-form symmetry) : transform the point $\psi(x)$
  - ✓ e.g., global $U(1)$ symmetry $\psi(x) \to e^{i\alpha}\psi(x)$
- Extend the point to 2d, 3d,⋯ objects

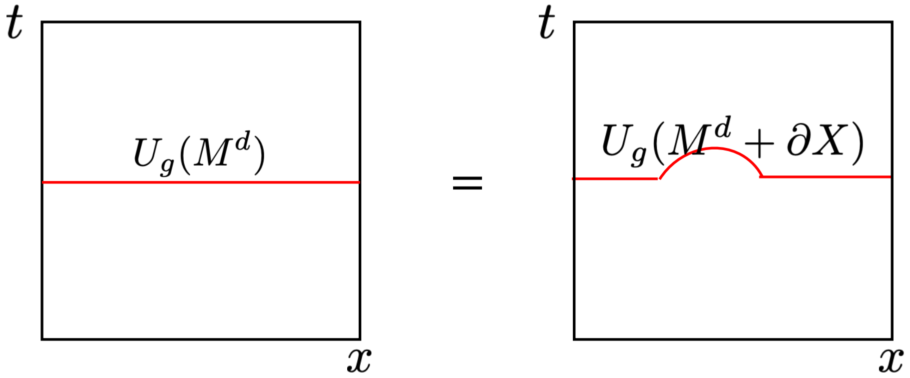
- zero form symmetry
  - ➢ transform point $\psi(x)$
- one form symmetry
  - ➢ transform loop $W(C)$





Symmetry Operator's Topological Invariance

- Infinitisimal transformation of M^d,



$$\delta Q = \int_{M^d+\delta M^d} j - \int_{M^d} j = \int_{\partial X^d} j = \int_{X^d} \mathrm{d}j = 0$$

$\mathbb{Z}_N$ one-form gauge symmetry

$\mathbb{Z}_N$ zero-form gauge symmetry	$\mathbb{Z}_N$ one-form gauge symmetry

$\mathbb{Z}_N$ zero-form gauge symmetry

- $U(1)$ gauge field A_μ and scalar field ϕ make a pair (A_μ, ϕ) and construct $\mathbb{Z}_N$ one-form gauge field.
- Constraint $NA_\mu = \partial_\mu \phi$
- $\mathbb{Z}_N$ zero-form gauge trans

$$\phi \mapsto \phi + N\lambda$$
$$A_\mu \mapsto A_\mu + \partial_\mu \lambda$$

$\mathbb{Z}_N$ one-form gauge symmetry

- $U(1)$ two-form gauge field $B_{\mu\nu}$ and $U(1)$ gauge field C_μ make a pair $(B_{\mu\nu}, C_\mu)$ and construct $\mathbb{Z}_N$ two-form gauge field.
- Constraint $NB_{\mu\nu} = \partial_\mu C_\nu$

Written like
$NB = \mathrm{d}C$

- $\mathbb{Z}_N$ one-form gauge trans

$$C_\mu \mapsto C_\mu + N\lambda_\mu$$
$$B_{\mu\nu} \mapsto B_{\mu\nu} + \partial_\mu \lambda_\nu$$

$\mathbb{Z}_N$ Zero-form Gauge Symmetry

- Introducing the $U(1)$ gauge field A_μ,

$$S = \int \mathrm{d}^4x \, D_\mu H^\dagger D_\mu H + \cdots \, , \quad D_\mu H = \partial_\mu H - iNA_\mu H$$

- Condense the Higgs H. ϕ is a scalar field.

$$H = he^{i\phi}, \quad \phi \sim \phi + 2\pi$$

$$S = \int \mathrm{d}^4x \, h^2(\partial_\mu \phi - NA_\mu)^2 + \cdots$$

- VEV $h \to \infty$, we get the constraint,

$$\partial_\mu \phi - NA_\mu = 0$$

$\mathbb{Z}_N$ Zero-form Gauge Symmetry

- Constraint: $\partial_\mu \phi = N A_\mu$
- If $N = 1$, A_μ is pure gauge by the constraint, $U(1)$ symmetry is broken completely. On the other hand, if $N > 1$, $\mathbb{Z}_N$ symmetry is remained. Wilson loop is

$$W^N = \left[\exp\left(i \int A_\mu\right)\right]^N = \exp\left(i \int \partial_\mu \phi\right) = 1$$

- By this constraint, a pair, (A_μ, ϕ), $U(1)$ gauge field A_μ and a scalar field ϕ, constructs $\mathbb{Z}_N$ one-form gauge field.
- This pair, (A_μ, ϕ), has the $\mathbb{Z}_N$ zero-form gauge symmetry, and the transformation is

$$\phi \mapsto \phi + N\lambda$$
$$A_\mu \mapsto A_\mu + \partial_\mu \lambda$$

$\mathbb{Z}_N$ One-form Gauge Symmetry

- An example of higher form symmetries, $\mathbb{Z}_N$ one-form gauge symmetry, is not familiar.

➢ Rough method of making $\mathbb{Z}_N$ one-form gauge symmetry

✓ Consider $\mathbb{Z}_N$ zero-form gauge symmetry

✓ Raise the rank of the derivative

✓ Consider $\mathbb{Z}_N$ one-form gauge symmetry

Couple with $SU(N)$ Gauge Theory with θ Term

- Action: $S = -\frac{1}{2g^2}\int \mathrm{tr}(F \wedge \star F) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(F \wedge F)$

$$S = -\frac{1}{2g^2}\int \mathrm{tr}\left[(\mathcal{F} - \mathbb{1}B) \wedge \star(\mathcal{F} - \mathbb{1}B)\right] + \frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\mathcal{F} - \mathbb{1}B) \wedge (\mathcal{F} - \mathbb{1}B)\right] + \frac{1}{2\pi}\int u \wedge (\mathrm{tr}\,\mathcal{F} - NB)$$

- Couple the pair, $(B_{\mu\nu}, C_{\mu})$, $\mathbb{Z}_N$ two-form gauge field, with $SU(N)$ gauge theory
- ➢ Extend the $SU(N)$ gauge theory to the $U(N)$ gauge theory
- ➢ $\mathcal{A}$: $U(N)$ gauge field, whose traceless part is $SU(N)$ gauge field A.
- ➢ Eliminate the trace-part by one-form gauge symmetry,

$$\mathcal{A} \mapsto \mathcal{A} + \lambda \mathbb{1}$$

- ➢ Imposing the constraint,

$$\mathrm{tr}(\mathcal{F}) = NB$$

$\mathbb{Z}_N$ One-form Gauge Transformation

$$C \mapsto C + N\lambda$$

$$B \mapsto B + \mathrm{d}\lambda$$

- ➢ With the gauge transformation of a pair (B, C), $\mathbb{Z}_N$ two-form gauge field, $F = \mathcal{F} - \mathbb{1}B$ becomes λ gauge invariant.
- ➢ By this F, we obtain the $SU(N)$ gauge action coupling with the $\mathbb{Z}_N$ two-form gauge field.

Anomaly in the $SU(N)$ Gauge Theory with θ Term

- Action :

$$S = -\frac{1}{2g^2}\int \mathrm{tr}(f \wedge \star f) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(f \wedge f)$$

time reversal sym at $\theta = 0, \pi$

➢ Replace $F = \mathcal{F} - B\mathbb{1}$, we couple $\mathbb{Z}_N$ two-form gauge field and $SU(N)$ gauge theory.

$$S = -\frac{1}{2g^2}\int \mathrm{tr}\left[(\mathcal{F} - \mathbb{1}B) \wedge (\mathcal{F} - \mathbb{1}B)\right] + \frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\mathcal{F} - \mathbb{1}B) \wedge (\mathcal{F} - \mathbb{1}B)\right] + \frac{1}{2\pi}\int u \wedge (\mathrm{tr}\,\mathcal{F} - NB)$$

➢ With $\mathbb{Z}_N$ one-form gauge sym, under T trans,

$$Z[B] \xrightarrow{\mathcal{T}} Z[B]\exp\left[i\frac{-1+N+2p}{4\pi N}\int NB \wedge NB\right]$$

$2\pi i\times$(fractional)

➢ Anomaly between $\mathbb{Z}_N$ one-form gauge and time reversal sym.

Fiber Bundle

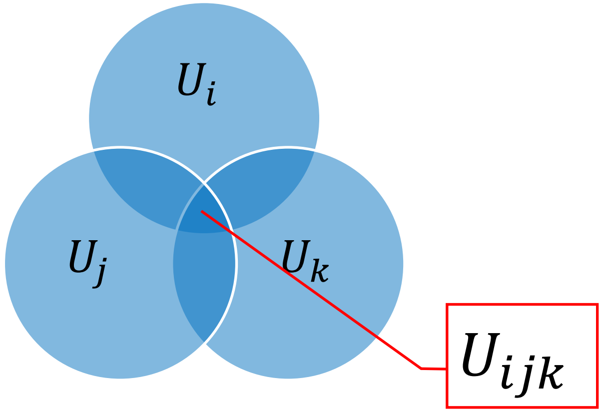
- The fiber bundle describes the gauge theory.
  - Covering a manifold M by patches U_i, each patch has the $SU(N)$ gauge field a_i and the matter field ϕ_i with the irreducible representation ρ.
- Gauge fields at $U_{ij} = U_i \cap U_j$ are connected by the gauge transformation function g_{ij}.

$$a_j = g_{ij}{}^{-1} a_i g_{ij} - i g_{ij}{}^{-1} \mathrm{d} g_{ij}$$

$$\phi_j = \rho(g_{ij}{}^{-1}) \phi_i$$

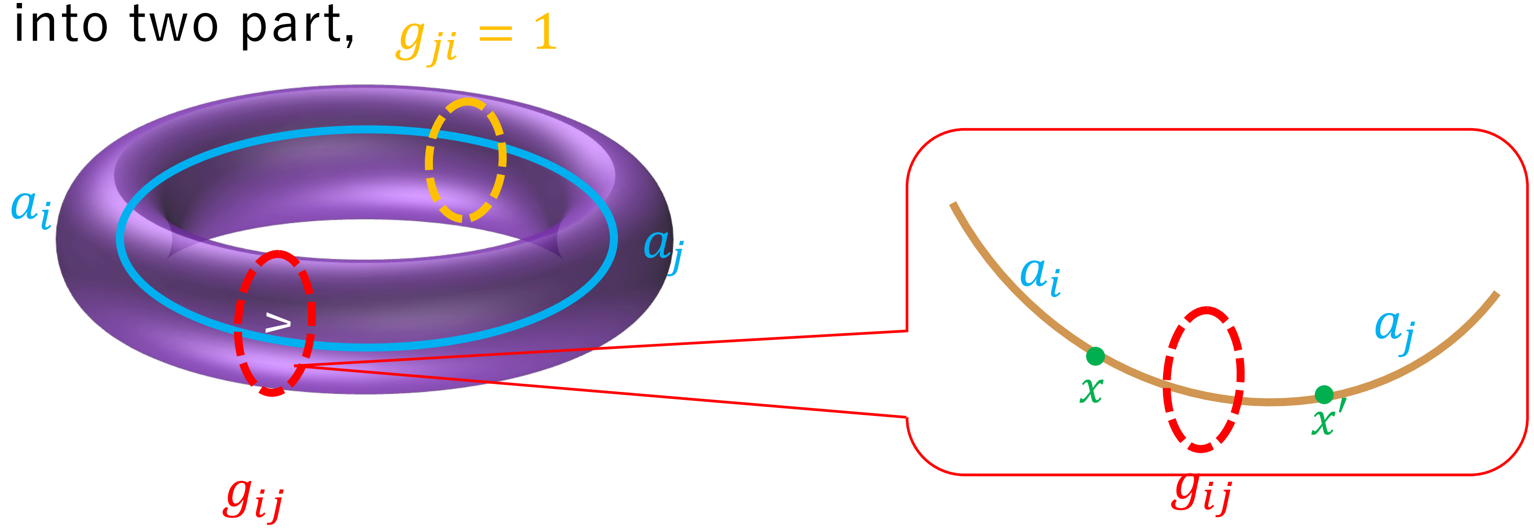
- At $U_{ijk} = U_i \cap U_j \cap U_k$, the cocycle condition is satisfied.

$$g_{ij} g_{jk} g_{ki} = 1$$



Wilson Loop and Transition Function

- Divided the torus into two part,



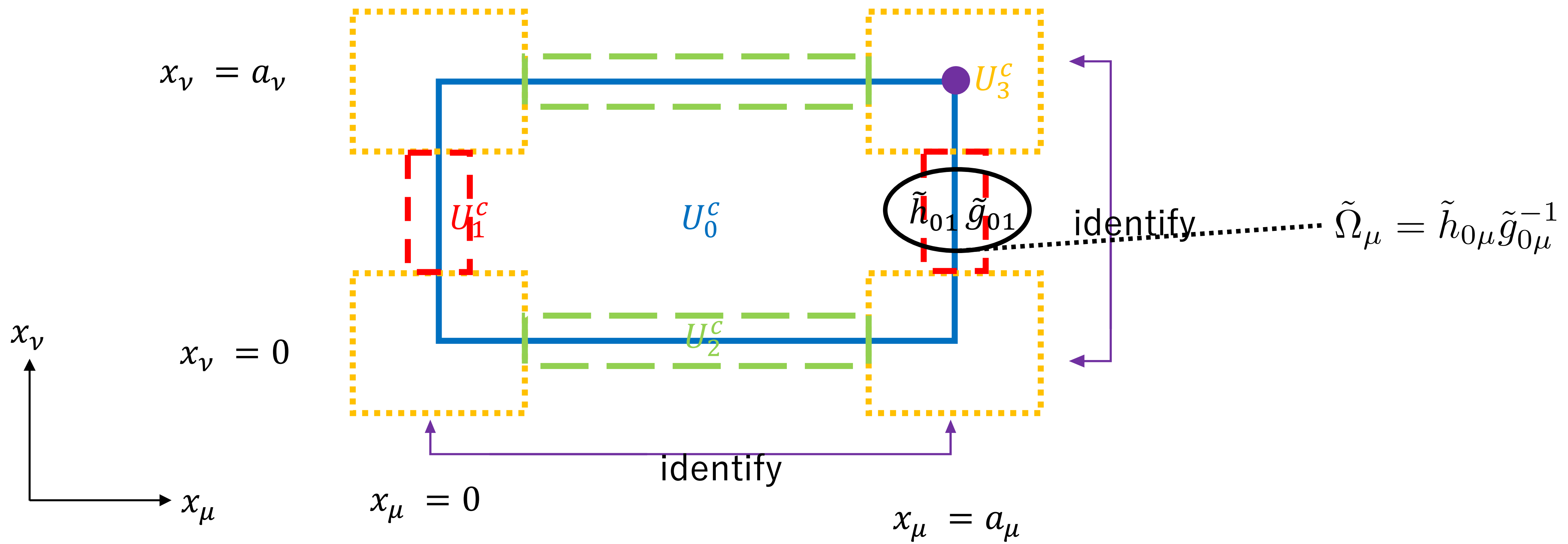
$$W(C) = e^{i\int_{y'}^{x'} a_j} e^{i\int_{y'}^{y} a_j} e^{i\int_{y}^{x} a_i} e^{i\int_{x}^{x'} a_j}$$

$$\xrightarrow[x\to x',\, y\to y']{} \quad g_{ji} e^{i\int_{x}^{y} a_i} g_{ij} e^{i\int_{y}^{x} a_i}$$

$$= g_{ij} e^{i\int_{C} a_i}$$

Transition Function in $SU(N)$ Gauge Theory

- Transition function is defined in nontrivial patches.
- In $2d$, the manifold T^2 is divided by four patches



$\mathbb{Z}_N$ One-form Gauge Sym and Fiber Bundle

- When the representation ρ is adjoint, cocycle condition becomes relaxed.

$$g_{ij}g_{jk}g_{ki} = \underbrace{\exp\left(\frac{2\pi i}{N}n_{ijk}\right)}_{\in \mathbb{Z}_N}$$

- Here, $\{n_{ijk}\}$ is mod N and antisymmetric.
  - $\{n_{ijk}\}$ has gauge redundancy.
  - Under trans of transition function,

$$g_{ij} \mapsto \underbrace{\exp\left(\frac{2\pi i}{N}\lambda_{ij}\right) g_{ij}}_{\text{Transition Function in } SU(N)/\mathbb{Z}_N \text{ Gauge Theory}}$$

    we want to be invariant of cocycle condition,

$$n_{ijk} \mapsto n_{ijk} + (\delta\lambda)_{ijk}$$

$$(\delta\lambda)_{ijk} \equiv \lambda_{ij} - \lambda_{ik} + \lambda_{jk}$$

  - We call this trans $\mathbb{Z}_N$ one-form gauge trans and $\{n_{ijk}\}$ $\mathbb{Z}_N$ two-form gauge field.

Transition Function in $SU(N)$ Gauge Theory

- By the transition function $\tilde{\Omega}_\mu$, the cocycle condition is

$$\tilde{\Omega}_\mu(x_\nu = a_\nu)\tilde{\Omega}_\nu(x_\mu = 0)\tilde{\Omega}_\mu^{-1}(x_\nu = 0)\tilde{\Omega}_\nu^{-1}(x_\mu = a_\mu) = 1$$

- To consider the fractional topological charge, we redefine the transition function Ω_μ. (Making $SU(N)/\mathbb{Z}_N$ bundle)

$$\Omega_\mu = \tilde{h}_{0\mu}\omega_\mu\tilde{g}_{0\mu}^{-1}$$

factor of fractionality

$$\omega_\mu = \exp\left(\frac{\pi i}{N}\sum_\nu \frac{n_{\mu\nu}x_\nu}{a_\nu}T_1\right)$$

$SU(N)$'s generator

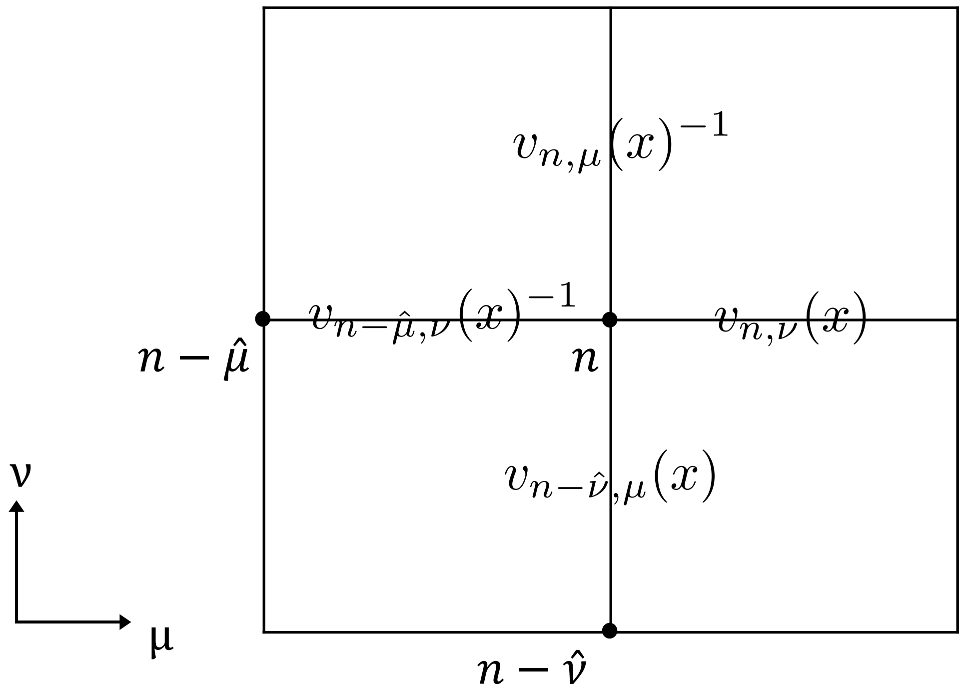
➢ The cocycle condition is relaxed,

$$\Omega_\mu(x_\nu = a_\nu)\Omega_\nu(x_\mu = 0)\Omega_\mu^{-1}(x_\nu = 0)\Omega_\nu^{-1}(x_\mu = a_\mu) = \exp\left(\frac{2\pi i}{N}n_{\mu\nu}\right)$$

Cocycle Condition on the Lattice

$$(\text{new transition function}) \sim \omega_\mu \times (\text{normal transition function})$$

- By the new transition function, the cocycle condition is



ordinary

$$\check{v}_{n-\hat{\mu},\nu}(x)\check{v}_{n,\mu}(x)\check{v}_{n,\nu}(x)^{-1}\check{v}_{n-\hat{\nu},\mu}(x)^{-1}=1$$

new

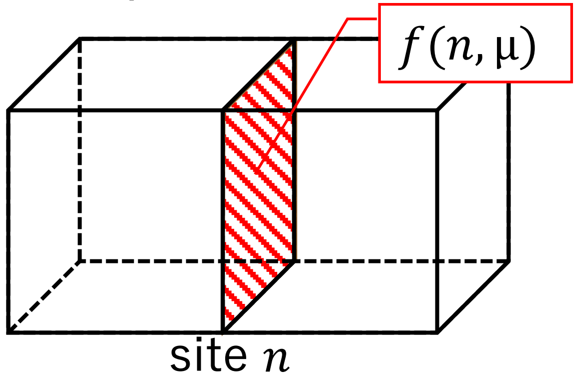
$$v_{n-\hat{\mu},\nu}(x)v_{n,\mu}(x)v_{n,\nu}(x)^{-1}v_{n-\hat{\nu},\mu}(x)^{-1}=\exp\left(\frac{2\pi i}{N}z_{\mu\nu}\right)$$

Lüscher's Idea

- Topological charge is defined by the continuum function: transition function $v_{n,\mu}$,

$$Q(v_{n,\mu}) = -\frac{1}{24\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\int_{f(n,\mu)} \mathrm{d}^3x\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\big((v_{n,\mu}^{-1}\partial_\nu v_{n,\mu})(v_{n,\mu}^{-1}\partial_\rho v_{n,\mu})(v_{n,\mu}^{-1}\partial_\sigma v_{n,\mu})\big)$$
$$+\frac{1}{8\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\int_{p(n,\mu,\nu)} \mathrm{d}^2x\,\varepsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big((v_{n,\mu}\partial_\rho v_{n,\mu}^{-1})(v_{n-\hat{\mu},\nu}^{-1}\partial_\sigma v_{n-\hat{\mu},\nu})\Big)$$

- By the interpolate function: "Parallel transporter", he defined the transition function $v_{n,\mu}$ on the face $f(n,\mu)$.

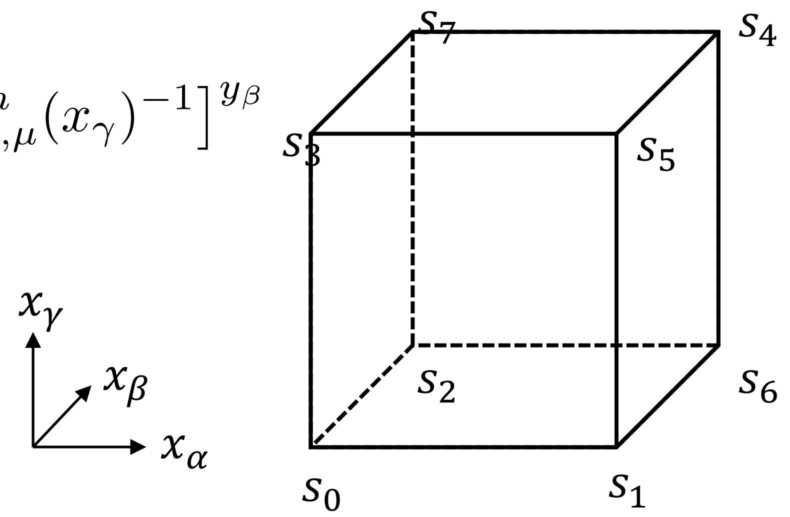


Interpolate Function in $SU(N)$ Gauge Theory

- In $x \in f(n, \mu)$,

$$
\begin{aligned}
f_{n,\mu}^{m}(x_\gamma) =& (u_{30})^{y_\gamma} (u_{03}^{m} u_{37}^{m} u_{72}^{m} u_{20}^{m})^{y_\gamma} u_{02}^{m} (u_{27}^{m})^{y_\gamma} \\
g_{n,\mu}^{m}(x_\gamma) =& (u_{51})^{y_\gamma} (u_{15}^{m} u_{54}^{m} u_{46}^{m} u_{61}^{m})^{y_\gamma} u_{16}^{m} (u_{64}^{m})^{y_\gamma} \\
h_{n,\mu}^{m}(x_\gamma) =& (u_{30})^{y_\gamma} (u_{03}^{m} u_{35}^{m} u_{51}^{m} u_{10}^{m})^{y_\gamma} u_{01}^{m} (u_{15}^{m})^{y_\gamma} \\
k_{n,\mu}^{m}(x_\gamma) =& (u_{72})^{y_\gamma} (u_{27}^{m} u_{74}^{m} u_{46}^{m} u_{62}^{m})^{y_\gamma} u_{26}^{m} (u_{64}^{m})^{y_\gamma} \\
l_{n,\mu}^{m}(x_\beta, x_\gamma) =& \left[f_{n,\mu}^{m}(x_\gamma)^{-1}\right]^{y_\beta} \left[f_{n,\mu}^{m}(x_\gamma) k_{n,\mu}^{m}(x_\gamma) g_{n,\mu}^{m}(x_\gamma)^{-1} h_{n,\mu}^{m}(x_\gamma)^{-1}\right]^{y_\beta} \\
& \cdot h_{n,\mu}^{m}(x_\gamma) \left[g_{n,\mu}^{m}(x_\gamma)\right]^{y_\beta} \\
S_{n,\mu}^{m}(x_\alpha, x_\beta, x_\gamma) =& (u_{03}^{m})^{y_\gamma} \left[f_{n,\mu}^{m}(x_\gamma)\right]^{y_\beta} \left[l_{n,\mu}^{m}(x_\beta, x_\gamma)\right]^{y_\alpha}
\end{aligned}
$$

$\boxed{\text{Difficult!!}}$



Parallel Transporter in the Lattice $U(1)$ Gauge Theory

- By the parallel transporter $w^m(x)$, we obtain the transition function $v_{n,\mu}$ in the continuum point x: $v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^n(x)^{-1}$

lattice

$$w^n(\bar{x}) = U(n,1)^{\sigma_1} U(n+\sigma_1\hat{1},2)^{\sigma_2} \cdots U(n+\sigma_1\hat{1}+\sigma_2\hat{2}+\cdots+\sigma_{D-1}\widehat{D-1}, D)^{\sigma_D}$$

$$\bar{x} = n + \sum_{\mu=1}^{D} \sigma_\mu \hat{\mu} \quad (\sigma_\mu = \{0,1\})$$

Interpolate

Continuum

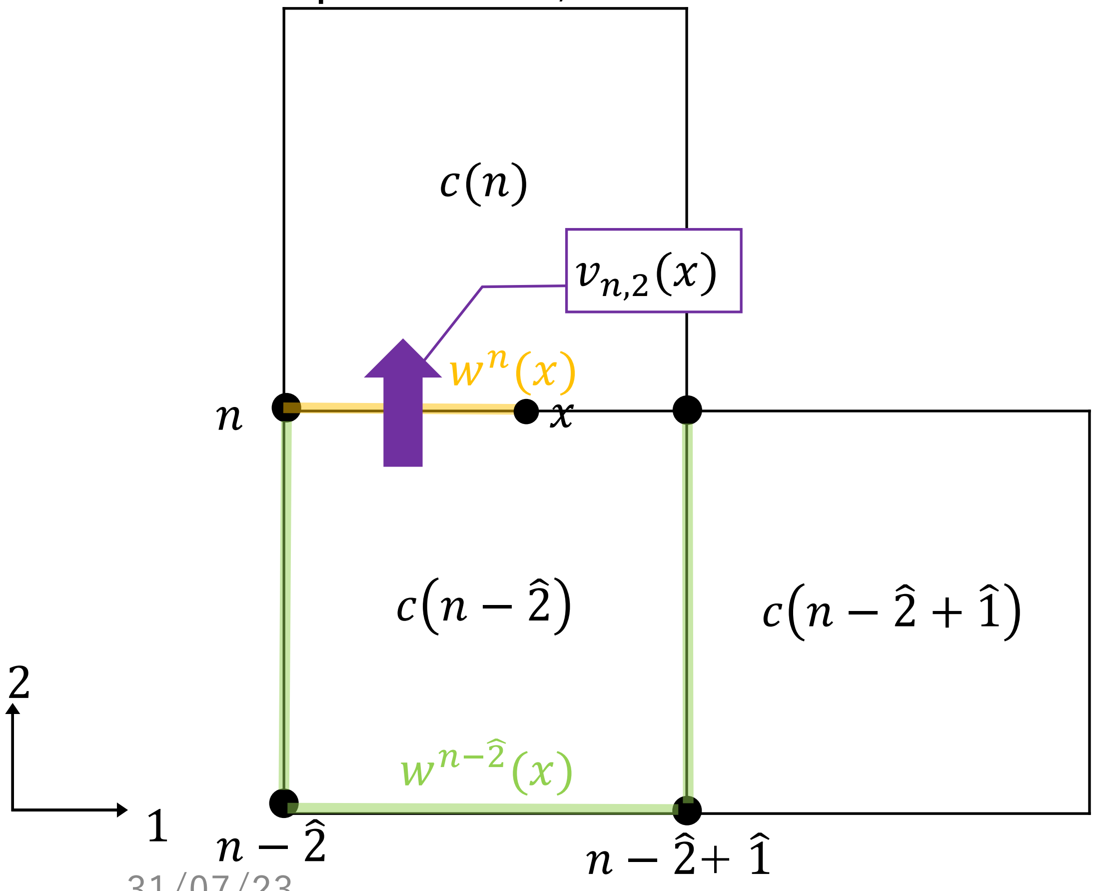
$$\underset{n,\,n-\hat{\mu}}{w^m}(x) = \prod_{\{\sigma_k=0,1\}_{k=1,\cdots,D-1}} w^m\left(n + \sum_{k=1}^{D-1} \sigma_k \hat{\mu}_k\right)^{\prod_{k=1}^{D-1}(\sigma_k y_k + (1-\sigma_k)(1-y_k))}$$

Interpolate Parameter y_k

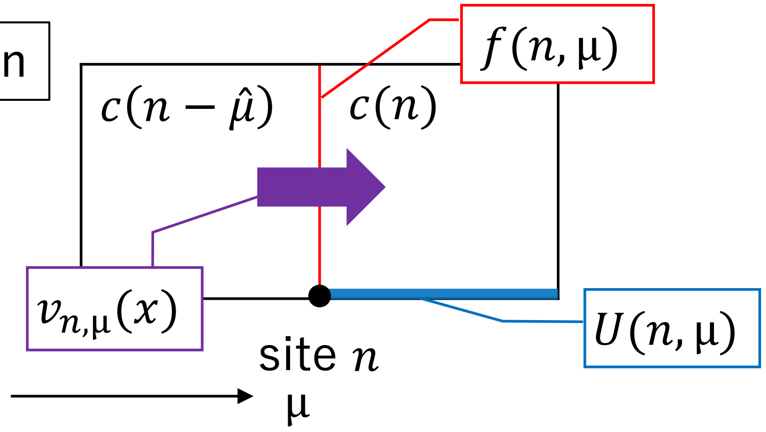
$$x = n + \sum_{k=1}^{D-1} y_k \hat{\mu}_k, \quad 0 \le y_k \le 1$$

Image of Parallel Transporter

➢ Example: in $2d$,



Notation



Parallel Transporter

$0 \leq y_1 \leq 1$

$$w^n(x) = U(n,1)^{y_1}$$

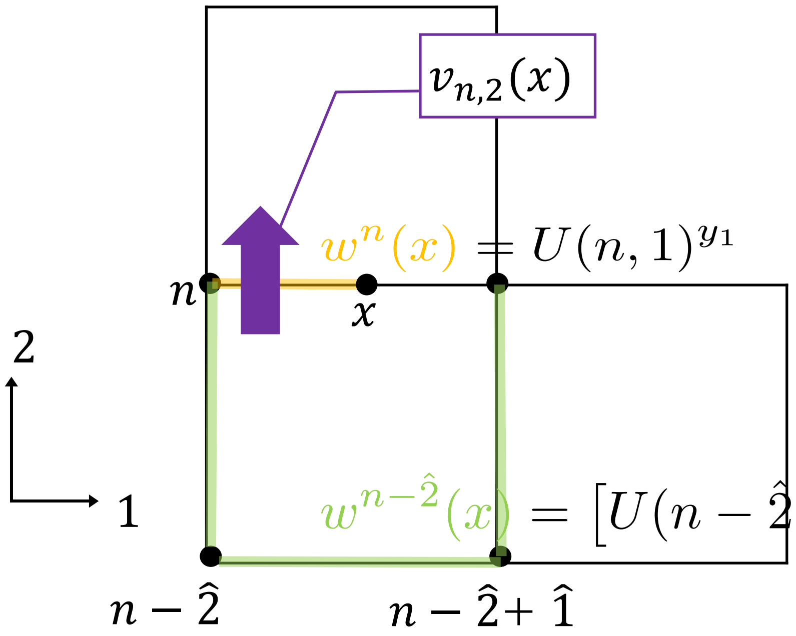
$$w^{n-\hat{2}}(x) = \left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)\right]^{y_1} U(n-\hat{2},2)^{1-y_1}$$

Transition Function on the Lattice in 2d

- By using the parallel transport function, the transition function is,

$$v_{n,\mu}(x) = w^{n-\hat{\mu}}(x) w^{n}(x)^{-1}$$

➢ Example: in 2d,



$$w^{n}(x) = U(n,1)^{y_1}$$

$$w^{n-\hat{2}}(x) = \left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)\right]^{y_1} U(n-\hat{2},2)^{1-y_1}$$

$$\begin{aligned} v_{n,2}(x) &= w^{n-\hat{2}}(x) w^{n}(x)^{-1} \\ &= U(n-\hat{2},2)\left[U(n-\hat{2},1)U(n-\hat{2}+\hat{1},2)U(n,1)^{-1}U(n-\hat{2},2)^{-1}\right]^{y_1} \\ &= U(n-\hat{2},2)\exp\left[i y_1 F_{12}(n-\hat{2})\right] \end{aligned}$$

$$F_{\mu\nu}(n) = \frac{1}{i}\ln\left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]$$

Transition Function on the Lattice in $4d$

In $2d$

$$v_{n,2}(x) = U(n-\hat{2},2)\exp\left[iy_1 F_{12}(n-\hat{2})\right]$$

- In $4d$,

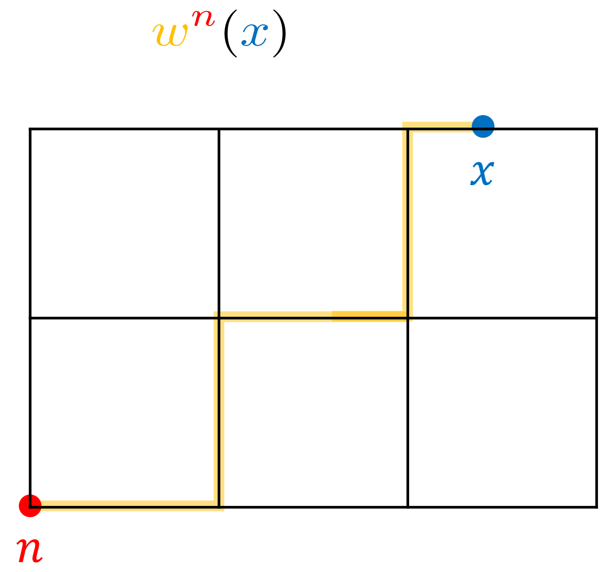
$$\begin{aligned}
v_{n,1}(x) &= U(n-\hat{1},1) \\
&\quad \times \exp\Big[iy_4F_{14}(n-\hat{1}) + iy_3y_4F_{13}(n-\hat{1}+\hat{4}) + iy_3(1-y_4)F_{13}(n-\hat{1}) \\
&\qquad + iy_2y_3y_4F_{12}(n-\hat{1}+\hat{3}+\hat{4}) + iy_2y_3(1-y_4)F_{12}(n-\hat{1}+\hat{3}) \\
&\qquad + iy_2(1-y_3)y_4F_{12}(n-\hat{1}+\hat{4}) + iy_2(1-y_3)(1-y_4)F_{12}(n-\hat{1})\Big], \\
v_{n,2}(x) &= U(n-\hat{2},2)\exp\left[iy_4F_{24}(n-\hat{2}) + iy_3y_4F_{23}(n-\hat{2}+\hat{4}) + iy_3(1-y_4)F_{23}(n-\hat{2})\right], \\
v_{n,3}(x) &= U(n-\hat{3},3)\exp\left[iy_4F_{34}(n-\hat{3})\right], \\
v_{n,4}(x) &= U(n-\hat{4},4)
\end{aligned}$$

- Field strength is

$$F_{\mu\nu}(n) = \frac{1}{i}\ln\left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]$$

Parallel Transport Function

- Parallel transport function's image is "by the interpolate parameter y, the transition function is defined as the function on an arbitrarily point x on the link".



Link Variables

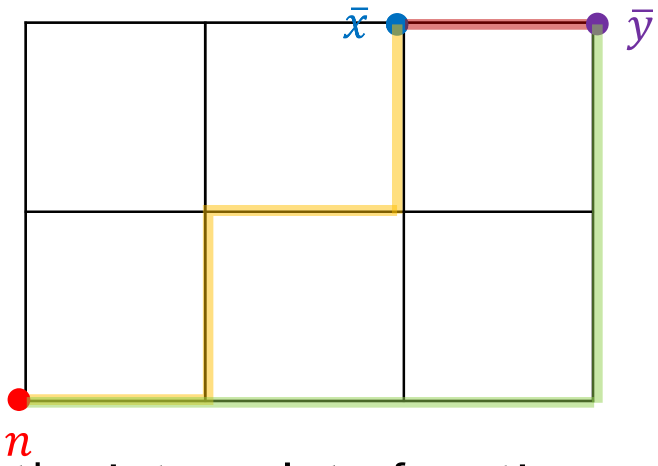
- In $SU(N)$ gauge field, this process is very complicated.

➢ By the parallel transport function, we defined the new link variable.

$$u_{xy}^{n} = w^{n}(\bar{x})U(\bar{x},\mu)w^{n}(\bar{y})^{-1} \qquad (\bar{y} = n + \hat{\mu})$$

$$u_{xy}^{n} = (u_{xy}^{n})^{-1} \qquad (\bar{y} = n - \hat{\mu})$$

Image of u_{xy}^{n}



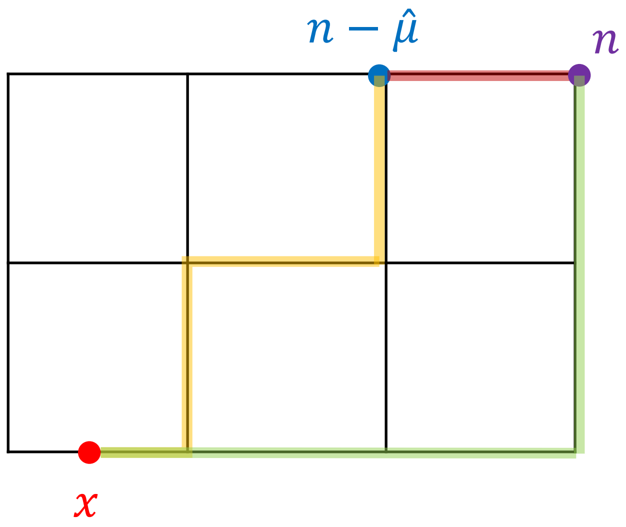
➢ By this link variable, we define the interpolate function.

Transition Function

- By the interpolate function made from the new link variable, we define the transition function as continuum function on the lattice .

$$\tilde{v}_{n,\mu}(x) = S_{n,\mu}^{n-\hat{\mu}}(x)^{-1}\tilde{v}_{n,\mu}(n)S_{n,\mu}^{n}(x)$$

Image of $\tilde{v}_{n,\mu}$



Cocycle Condition

- Check the cocycle condition by this new transition function

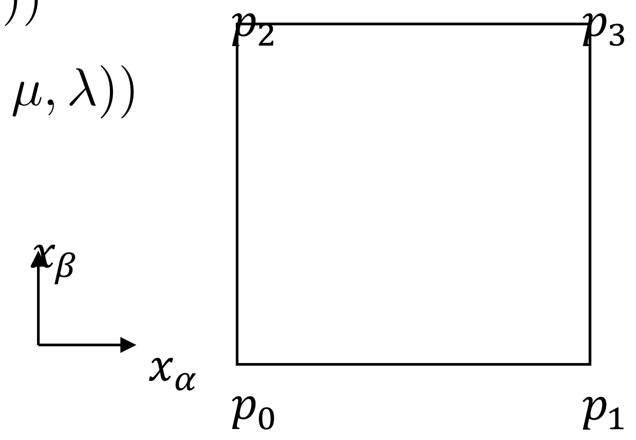
➢ In $x \in p(n, \mu, \nu)$, we define new function,

$$P^m_{n,\mu\nu}(x_\alpha, x_\beta) = (u^m_{p_0p_2})^{y_\beta}\left[(u^m_{p_2p_0})^{y_\beta}\left(u^m_{p_0p_2}u^m_{p_2p_3}u^m_{p_3p_1}u^m_{p_1p_0}\right)^{y_\beta}u^m_{p_0p_1}(u^m_{p_1p_3})^{y_\beta}\right]^{y_\alpha}$$

➢ The relation with $S^m_{n,\mu}(x)$ is

$$S^m_{n,\mu}(x) = P^m_{n,\mu\lambda}(x) \qquad (x \in p(n, \mu, \lambda))$$

$$S^m_{n,\mu}(x) = R^m_{n,\mu;\lambda}P^m_{n+\hat{\lambda},\mu\lambda}(x) \qquad (x \in p(n+\hat{\lambda}, \mu, \lambda))$$



Cocycle Condition on the Lattice

New Transition Function

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x) \quad \text{at } x \in f(n,\mu)$$

- By original transition function $\check{v}_{n,\mu}$, cocycle condition is

$$\check{v}_{n-\hat{\mu},\nu}(x)\check{v}_{n,\mu}(x)\check{v}_{n,\nu}(x)^{-1}\check{v}_{n-\hat{\nu},\mu}(x)^{-1} = 1$$

cocycle condition on the lattice

$g_{ij}g_{jk}g_{ki} = 1$

- By new transition function $v_{n,\mu}$, ω_μ causes

$$v_{n-\hat{\mu},\nu}(x)v_{n,\mu}(x)v_{n,\nu}(x)^{-1}v_{n-\hat{\nu},\mu}(x)^{-1}$$

$$= \begin{cases} \exp\left(\frac{2\pi i}{q} z_{\mu\nu}\right) & \text{for } x_\mu = x_\nu = 0 \text{ mod } L \\ 1 & \text{otherwise} \end{cases}$$

$\mathbb{Z}_q$の元

$\in \mathbb{Z}_q$

Cocycle Condition

- R^m is

$$
\begin{aligned}
R^m_{n,\mu;\alpha}(x_\beta, x_\gamma) =& \left[\left(u^m_{03}u^m_{37}u^m_{72}u^m_{20}\right)^{y_\gamma} u^m_{02} \right. \\
& \cdot \left(u^m_{27}u^m_{74}u^m_{46}u^m_{62}\right)^{y_\gamma} u^m_{26}u^m_{61} \left(u^m_{16}u^m_{64}u^m_{45}u^m_{51}\right)^{y_\gamma} \\
& \left. \cdot \; u^m_{10} \left(u^m_{01}u^m_{15}u^m_{53}u^m_{30}\right)^{y_\gamma} \right]^{y_\beta} \left(u^m_{03}u^m_{35}u^m_{51}u^m_{10}\right)^{y_\gamma} u^m_{01} \\
R^m_{n,\mu;\beta}(x_\alpha, x_\gamma) =& \left(u^m_{03}u^m_{37}u^m_{72}u^m_{20}\right)^{y_\gamma} u^m_{02} \\
R^m_{n,\mu;\gamma}(x_\alpha, x_\beta) =& u^m_{03}
\end{aligned}
$$

Cocycle Condition

- By the new interpolate function, in $x \in p(n, \mu, \nu)$, the cocycle condition is

$$\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x) = \left(P^{n-\hat{\nu}-\hat{\mu}}_{n,\mu\nu}(x)^{-1} v_{n-\hat{\mu},\nu}(n) P^{n-\hat{\mu}}_{n,\mu\nu}(x)\right)\left(P^{n-\hat{\mu}}_{n,\mu\nu}(x)^{-1} v_{n,\nu}(n) P^{n}_{n,\mu\nu}(x)\right)$$
$$= P^{n-\hat{\nu}-\hat{\mu}}_{n,\mu\nu}(x)^{-1} v_{n-\hat{\mu},\nu}(n) v_{n,\nu}(n) P^{n}_{n,\mu\nu}(x)$$

$$\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x) = P^{n-\hat{\nu}-\hat{\mu}}_{n,\mu\nu}(x)^{-1} v_{n-\hat{\nu},\mu}(n) v_{n,\mu}(n) P^{n}_{n,\mu\nu}(x)$$

- When (cocycle condition)=1 is satisfied at each site,

$$\tilde{v}_{n-\hat{\mu},\nu}(x)\tilde{v}_{n,\mu}(x)\tilde{v}_{n,\nu}(x)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(x)^{-1} = 1$$

Topological Charge

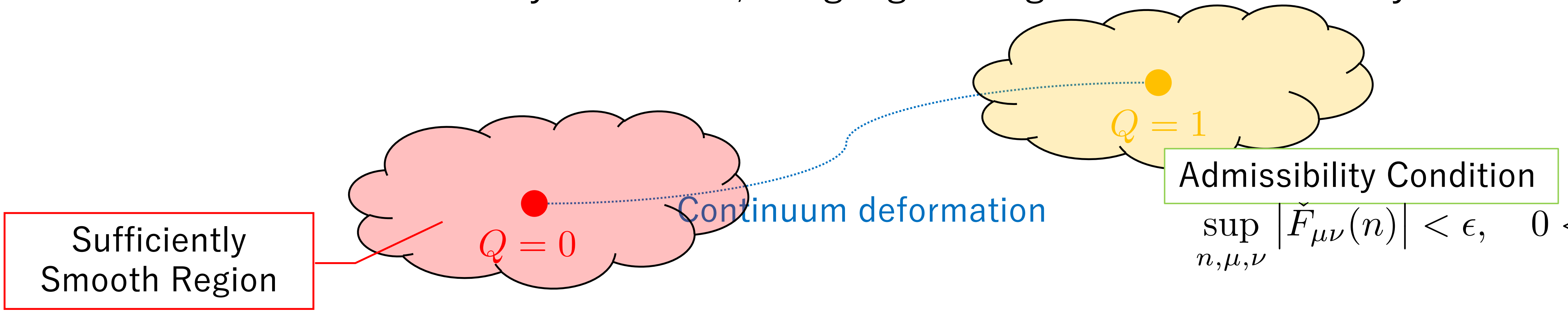
- By the new transition function, the topological charge is

$$
\begin{aligned}
P(\tilde{v}_{n,\mu}) =& \frac{1}{24\pi^2} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \left\{ 3 \int_{p(n+\hat{\mu}+\hat{\nu},\mu,\nu)} \mathrm{d}^2 x \, \mathrm{Tr} \left[P^n_{n+\hat{\mu}+\hat{\nu},\mu\nu} \partial_\rho (P^n_{n+\hat{\mu}+\hat{\nu},\mu\nu})^{-1} (R^n_{n+\hat{\mu},\mu;\nu})^{-1} \partial_\sigma R^n_{n+\hat{\mu},\mu;\nu} \right] \right. \\
& - 3 \int_{p(n+\hat{\nu},\mu,\nu)} \mathrm{d}^2 x \, \mathrm{Tr} \left[P^n_{n+\hat{\nu},\mu\nu} \partial_\rho (P^n_{n+\hat{\nu},\mu\nu})^{-1} (R^n_{n,\mu;\nu})^{-1} \partial_\sigma R^n_{n,\mu;\nu} \right] \\
& - \int_{f(n+\hat{\mu},\mu)} \mathrm{d}^3 x \, \mathrm{Tr} \left[S^n_{n+\hat{\mu},\mu} \partial_\nu (S^n_{n+\hat{\mu},\mu})^{-1} S^n_{n+\hat{\mu},\mu} \partial_\rho (S^n_{n+\hat{\mu},\mu})^{-1} S^n_{n+\hat{\mu},\mu} \partial_\sigma (S^n_{n+\hat{\mu},\mu})^{-1} \right] \\
& \left. + \int_{f(n,\mu)} \mathrm{d}^3 x \, \mathrm{Tr} \left[S^n_{n,\mu} \partial_\nu (S^n_{n,\mu})^{-1} S^n_{n,\mu} \partial_\rho (S^n_{n,\mu})^{-1} S^n_{n,\mu} \partial_\sigma (S^n_{n,\mu})^{-1} \right] \right\}
\end{aligned}
$$

Admissibility Condition

- It is impossible to define the topological charge which has intervals on the lattice.

➢ Under the “Admissibility condition”, the gauge configuration is sufficiently smooth.



$$\sup_{n,\mu,\nu}\left|\check{F}_{\mu\nu}(n)\right| < \epsilon, \quad 0<\epsilon<\frac{\pi}{3q}$$

- Field strength is

$$\check{F}_{\mu\nu}(n) = \frac{1}{iq}\ln\left[U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}\right]^q$$

※ q is needed for the invariance under the $\mathbb{Z}_q$ one-form transformation.

Topological Charge in the $SU(N)$ Gauge Theory

- By the new transition function, we calculate topological charge $Q(v_{n,\mu})$.
- In $4d$ continuum theory, (van Baal, Commun. Math. Phys. 85 (1982))

$$
\begin{aligned}
Q(v_{n,\mu}) =& \frac{1}{24\pi^2} \sum_{\mu} \int \mathrm{d}_3\sigma_\mu \varepsilon_{\mu\nu\alpha\beta} \operatorname{Tr}\Big((v_{n,\mu}\partial_\nu v_{n,\mu}^{-1})(v_{n,\mu}\partial_\alpha v_{n,\mu}^{-1})(v_{n,\mu}\partial_\beta v_{n,\mu}^{-1})\Big) \\
&+ \frac{1}{8\pi^2} \sum_{\mu,\nu} \int \mathrm{d}_2 S_{\mu\nu} \varepsilon_{\mu\nu\alpha\beta} \operatorname{Tr}\Big((v_{n,\nu}^{-1}\partial_\alpha v_{n,\nu})_{x_\mu=a_\mu}(v_{n,\mu}\partial_\beta v_{n,\mu}^{-1})_{x_\nu=0}\Big) \\
=& \underbrace{\mathbb{Z}}_{\text{integer}} + \underbrace{\frac{N-1}{N} \cdot \frac{1}{8} \varepsilon_{\mu\nu\alpha\beta} z_{\mu\nu} z_{\alpha\beta}}_{\text{fractional}}
\end{aligned}
$$

Differential Calculus on the Lattice

- k-form function: $f(n) \equiv \frac{1}{k!} \sum_{\mu_1, \cdots, \mu_k} f_{\mu_1 \cdots \mu_k}(n) \, \mathrm{d}x_{\mu_1} \cdots \mathrm{d}x_{\mu_k}$
- The definition of extra derivative: $\mathrm{d}x_\mu \, f_{\mu_1 \cdots \mu_k}(n) = f_{\mu_1 \cdots \mu_k}(n + \hat{\mu}) \, \mathrm{d}x_\mu$

➢ By this extra derivative on the lattice, the Leibniz rule on the lattice is

$$\mathrm{d}[f(n)g(n)] = \mathrm{d}f(n) \cdot g(n) + (-1)^k f(n) \cdot \mathrm{d}g(n)$$

➢ Example:

$$f(n) = \frac{1}{2} \sum_{\mu,\nu} f_{\mu\nu}(n) dx_\mu dx_\nu \quad \Rightarrow \quad \begin{aligned} f(n)f(n) &= \frac{1}{4} \sum_{\mu,\nu,\rho,\sigma} f_{\mu\nu}(n) f_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) dx_\mu dx_\nu dx_\rho dx_\sigma \\ &= \frac{1}{4} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} f_{\mu\nu}(n) f_{\rho\sigma}(n + \hat{\mu} + \hat{\nu}) dx_1 dx_2 dx_3 dx_4 \end{aligned}$$

$\mathbb{Z}_q$ One-form Global Symmetry and Gauge Symmetry

- $\mathbb{Z}_q$ one-form symmetry is corresponding to multiplying the $\mathbb{Z}_q$ element by the transition function from the point of fiber bundle.

➢ Consider the transformation of the transition function on the lattice

➢ Firstly, consider the $\mathbb{Z}_q$ one-form global symmetry

Admissibility Condition

- Field strength is

$$\check{F}_{\mu\nu}(n) = \frac{1}{iq} \ln \left[U(n,\mu) U(n+\hat{\mu},\nu) U(n+\hat{\nu},\mu)^{-1} U(n,\nu)^{-1} \right]^q \quad |F_{\mu\nu}(n)| < \pi$$

➢ Invariant under the $\mathbb{Z}_q$ one-form gauge transformation

➢ We require the admissibility condition to the field strength,

$$\sup_{n,\mu,\nu} \left| \check{F}_{\mu\nu}(n) \right| < \epsilon, \quad 0 < \epsilon < \frac{\pi}{3q}$$

➢ Under this condition, the Bianchi identity is satisfied.

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu \check{F}_{\rho\sigma}(n) = 0$$

Proof of Admissibility Condition

- Field strength is

$$
\begin{aligned}
F_{\mu\nu}(n) =& \frac{1}{iq}\ln\left[e^{i(a_\mu(n)+a_\nu(n)+\hat{\mu}-a_\mu(n+\hat{\nu})-a_\nu(n))}\right]^q \\
=& \frac{1}{iq}\left[i\left(a_\mu(n)+a_\nu(n)+\hat{\mu}-a_\mu(n+\hat{\nu})-a_\nu(n)\right)\cdot q+2\pi i N_{\mu\nu}(n)\right] \\
=& \Delta_\nu a_\mu(n)-\Delta_\mu a_\nu(n)+\frac{2\pi}{q}N_{\mu\nu}(n)
\end{aligned}
$$

➢ $N_{\mu\nu}$ is the function for taking $F_{\mu\nu}$ back to the range $[-\pi,\pi]$.

Proof of Admissibility Condition

- By the admissibility condition,

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu F_{\mu\nu}(n) < 6\epsilon$$

➢ By definition,

$$\sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu \left(\Delta_\rho a_\sigma(n) - \Delta_\rho a_\sigma(n) + \frac{2\pi}{q} N_{\rho\sigma}(n) \right) = \frac{2\pi}{q} \sum_{\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \Delta_\nu N_{\rho\sigma}(n)$$

➢ By $\varepsilon_{\mu\nu\rho\sigma} \Delta_\nu N_{\rho\sigma}(n) < 1$

$$0 < 6\epsilon < \frac{2\pi}{q} \quad \Rightarrow \quad 0 < \epsilon < \frac{\pi}{3q}$$

$\mathbb{Z}_q$ Two-form Gauge Field

- $\mathbb{Z}_q$ two-form gauge field is defined by

$$z_{\mu\nu}(n) = z_{\mu\nu}\delta_{n_\mu,L-1}\delta_{n_\nu,L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$

➢ To protect the antisymmetric value,

$$\begin{cases} 0 \leq z_{\mu\nu}(n) < q & \text{for } \mu < \nu, \\ z_{\mu\nu}(n) \equiv -z_{\nu\mu}(n) & \text{for } \mu > \nu \end{cases}$$

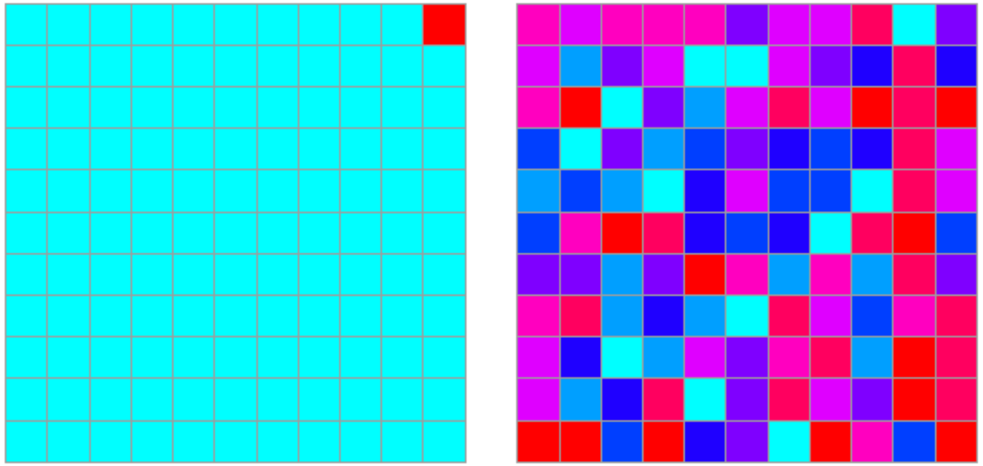
➢ Under the $\mathbb{Z}_q$ one-form gauge transformation, $\mathbb{Z}_q$ two-form field is

$$z_{\mu\nu}(n) \to z_{\mu\nu}(n) + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n)$$

$\mathbb{Z}_q$ Two-form Gauge Field

- This $\mathbb{Z}_q$ two-form gauge field is connected to an arbitrary gauge configuration by the $\mathbb{Z}_q$ one-form gauge transformation.

$$z_{\mu\nu}(n) = {\color{red} z_{\mu\nu}\delta_{n_\mu,L-1}\delta_{n_\nu,L-1}} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$



Fractional Topological Charge by $\mathbb{Z}_q$ Two-form Gauge Field

$$Q = \frac{1}{32\pi^2} \sum_{n\in\Lambda} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \left[F_{\mu\nu}(n) + \frac{2\pi}{q} z_{\mu\nu}(n) \right] \left[F_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) + \frac{2\pi}{q} z_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) \right]$$

$$z_{\mu\nu}(n) = z_{\mu\nu}\delta_{n_\mu, L-1}\delta_{n_\nu, L-1} + \Delta_\mu z_\nu(n) - \Delta_\nu z_\mu(n) + qN_{\mu\nu}(n) \quad \in \mathbb{Z}$$

$$Q = \frac{1}{8q^2} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \frac{1}{8\pi q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} \sum_{n_\mu=0} \check{F}_{\rho\sigma}(n)$$
$$+ \frac{1}{32\pi^2} \sum_{n} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \check{F}_{\mu\nu}(n) \check{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})$$

$\mathbb{Z}_q$ One-form Global Symmetry on the Lattice

- The factor of fractionality ω_μ is related to the $\mathbb{Z}_q$ one-form transform.

➢ Link variable

$$U(n,\mu) \to \underbrace{\exp\left(\frac{2\pi i}{q} z_\mu\right)}_{\in \mathbb{Z}_q} U(n,\mu) \qquad n_\mu = 0$$

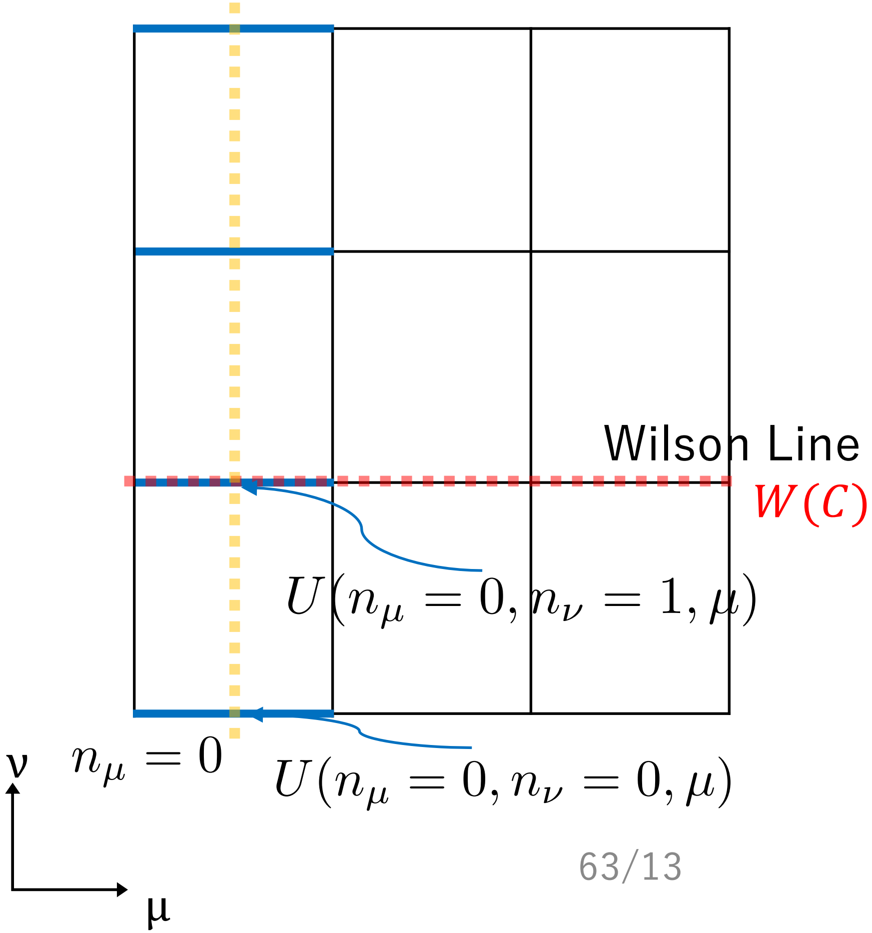
➢ Transition function

$$\check{v}_{n,\mu}(x) \to \begin{cases} \exp\left(\frac{2\pi i}{q} z_\mu\right) \check{v}_{n,\mu}(x) & \text{for } x_\mu = 1 \\ \check{v}_{n,\mu}(x) & \text{otherwise} \end{cases}$$

➢ Cocycle condition

Not $\mathbb{Z}_q$ "Relax"

$$\check{v}_{n-\hat{\nu},\mu}(x)\check{v}_{n,\nu}(x)\check{v}^{-1}_{n,\mu}(x)\check{v}^{-1}_{n-\hat{\mu},\nu}(x) = 1$$



$\mathbb{Z}_q$ One-form Gauge Symmetry on the Lattice

- The factor of fractionality ω_μ is related to the $\mathbb{Z}_q$ one-form transform.

➢ Link variable

$$U(n,\mu) \to \underbrace{\exp\left[\frac{2\pi i}{q} z_\mu(n)\right]}_{\in \mathbb{Z}_q} U(n,\mu)$$

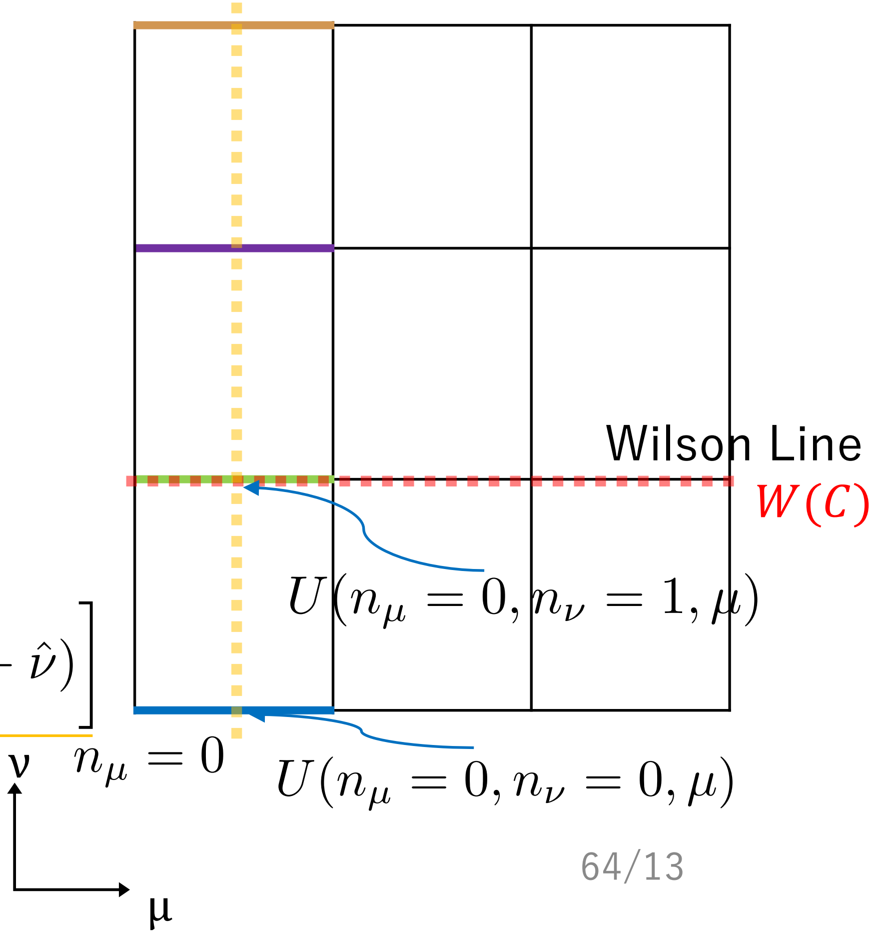
➢ Transition function

$$v_{n,\mu}(x) \to \exp\left[\frac{2\pi i}{q} z_\mu(n-\hat{\mu})\right] v_{n,\mu}(x) \qquad x \in f(n,\mu)$$

➢ Cocycle condition

$\mathbb{Z}_q$ “relax”

$$v_{n-\hat{\nu},\mu}(x) v_{n,\nu}(x) v_{n,\mu}(x)^{-1} v_{n-\hat{\mu},\nu}(x)^{-1} \equiv \underbrace{\exp\left[\frac{2\pi i}{q} z_{\mu\nu}(n-\hat{\mu}-\hat{\nu})\right]}_{\in \mathbb{Z}_q}$$



Mixed 't Hooft Anomaly

- e^{iS} is ,under the $\mathcal{T}$-transformation,

$$e^{i\pi q Q} \xrightarrow{\mathcal{T}} e^{-i\pi q Q} = e^{-2\pi i q Q} \cdot e^{i\pi q Q}$$

$$= \exp\left(-\frac{2\pi i}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right) e^{i\pi q Q}$$

➢ Introducing a local counter term which is invariant under the $\mathbb{Z}_q$ one-form gauge transformation,

$$e^{-S_{\text{counter}}} \equiv \exp\left[\frac{2\pi i k}{4q} \sum_{n} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}(n) z_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})\right]$$

$$= \exp\left(\frac{2\pi i k}{4q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right)$$

Mixed 't Hooft Anomaly

- e^{iS} is ,under the $\mathcal{T}$-transformation, when $\theta = \pi$,

$$e^{i\pi q Q} \xrightarrow{\mathcal{T}} e^{-i\pi q Q} = e^{-2\pi i q Q} \cdot e^{i\pi q Q}$$

$$= \exp\left(-\frac{2\pi i}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right) e^{i\pi q Q}$$

➢ Introducing a local counter term which is invariant under the $\mathbb{Z}_q$ one-form gauge transformation,

$$e^{-S_{\mathrm{counter}}} \equiv \exp\left(\frac{2\pi i k}{8q} \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}\right)$$

Time Reversal Symmetry

$$U(n,\mu) \xrightarrow{\mathcal{T}} \begin{cases} U(\bar{n},\mu) & \text{for } \mu \neq 4, \\ U(\bar{n}-\hat{4},4)^{-1} & \text{for } \mu = 4, \end{cases}$$

$$\check{F}_{\mu\nu}(n) \xrightarrow{\mathcal{T}} \begin{cases} \check{F}_{\mu\nu}(\bar{n}) & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -\check{F}_{4\nu}(\bar{n}-\hat{4}) & \text{for } \mu = 4, \\ -\check{F}_{\mu 4}(\bar{n}-\hat{4}) & \text{for } \nu = 4. \end{cases}$$

$$z_{\mu\nu}(n) \xrightarrow{\mathcal{T}} \begin{cases} z_{\mu\nu}(\bar{n}) & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -z_{4\nu}(\bar{n}+\hat{4}) & \text{for } \mu = 4, \\ -z_{\mu 4}(\bar{n}+\hat{4}) & \text{for } \nu = 4, \end{cases}$$

$$z_{\mu\nu} \xrightarrow{\mathcal{T}} \begin{cases} z_{\mu\nu} & \text{for } \mu \neq 4,\ \nu \neq 4, \\ -z_{4\nu} & \text{for } \mu = 4, \\ -z_{\mu 4} & \text{for } \nu = 4. \end{cases}$$

Witten Effect

- Setting magnetic monopole with magnetic charge g, electric charge q is induced by θ term.

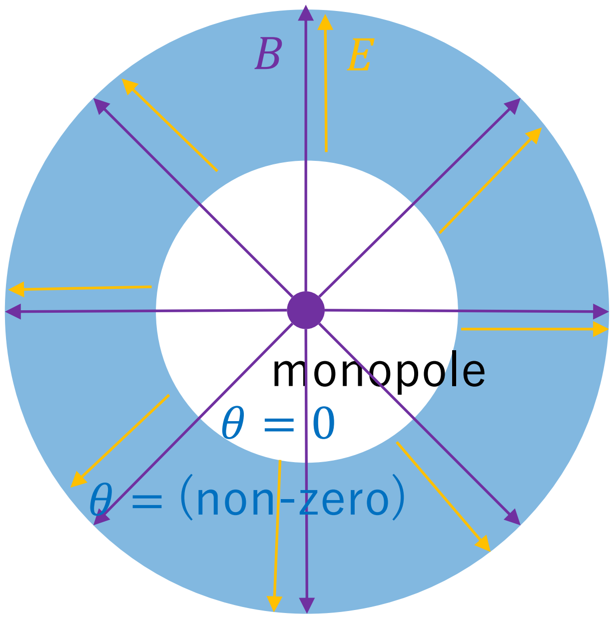
$$S = -\frac{1}{2g^2}\int \mathrm{tr}(f \wedge \star f) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(f \wedge f)$$

➢ In the abelian gauge theory, EOM is

$$\partial_\mu F^{\mu\nu} = \frac{g^2}{4\pi^2}\varepsilon_{\mu\nu\rho\sigma}\partial_\mu\left(\theta\partial_\rho A_\sigma\right)$$

$$\nabla \cdot \mathbf{E} = -\frac{g^2}{4\pi^2\epsilon_0}\nabla\theta \cdot \mathbf{B}$$

ρ/ϵ_0



➢ Dirac quaternization is condition: $gq = \theta$

Cardy-Rabinovici model

$$
\begin{aligned}
S[\tilde{a}_\mu, s_{\mu\nu}, n_\mu] =& S_{\mathrm{kin}}[\tilde{a}_\mu, s_{\mu\nu}] + S_{\mathrm{matter}}[\tilde{a}_\mu, s_{\mu\nu}, n_\mu] \\
=& \frac{1}{2g^2} \sum_{(x,\mu,\nu)} f_{\mu\nu}(x)^2 + iN \sum_{(x,\mu)} \left(n_\mu(x) + \frac{\theta}{2\pi} \sum_{\tilde{x}} F(x-\tilde{x}) m_\mu(\tilde{x}) \right) \tilde{a}_\mu(x)
\end{aligned}
$$