Higher-group symmetry in lattice gauge theories with restricted topological sectors

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31/7/2023 Lattice2023 @Fermilab

- N. Kan, OM, Y. Nagoya and H. Wada, EPJC **83**, no.6, 481 (2023) [arXiv:2302.13466].
- M. Abe, OM and S. Onoda, PRD **108**, 014506 (2023) [arXiv:2304.11813].
- Motokazu Abe, Next talk based on [2210.12967, 2303.10977]

Generalization of Symmetry

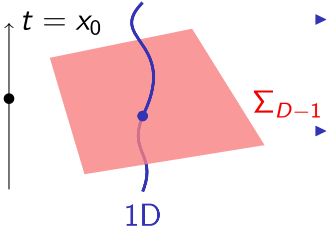
- Symmetry: fundamental tool in physics
  - Universal applications to hep, condensed matter and math
- 't Hooft anomaly matching for low-energy dynamics ['79]
  - Gauging global symmetry $\xrightarrow{\text{Anomalous}}$ 't Hooft anomaly

$$\text{'t Hooft anomaly} \xrightarrow{\text{RG inv.}} \text{Low energy scenario}$$

  - restriction on SSB, phase structure, SPT phase
- Recent generalization of symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]
  - Changing topological structure [Kapustin–Seiberg '14]:
    $\leftarrow$ Nontrivial information by using 't Hooft anomaly matching
  - Basic property: fractionality of topological charge
    - ⋆ Discrete higher-form gauge field $Q \sim \int B\tilde{B} \in \frac{1}{N}\mathbb{Z}$
      [Kapustin–Thorngren '13, Gaiotto–Kapustin–Komargodski–Seiberg '17]
    - $\Leftrightarrow$ 't Hooft twisted boundary condition $Q \sim \int F\tilde{F} \in \frac{1}{N}\mathbb{Z}$
      [Edwards-Heller-Narayanan, de Forcrand-Jahn,
      Fodor-Holland-Kuti-Nógrádi-Schroeder, Kitano-Suyama-Yamada, Itou, . . .]

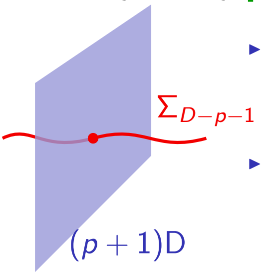
Generalization: higher-form symmetry

- (0-form) Symmetry



  - Charge (codim 1)
  $$Q \equiv \int_{\Sigma_{D-1}} j_0 dx_1 \wedge \cdots \wedge dx_{D-1}$$
  - Symmetry operator
  $$U_\alpha(\Sigma_{D-1}) = e^{i\alpha Q}$$
  Topological under deformation of Σ_{D-1}

- Higher-form symmetry [Gaiotto–Kapustin–Seiberg–Willet '14]



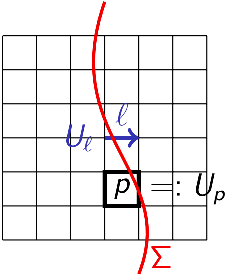
  - p-form symmetry $G^{[p]}$ (codim $p+1$)
  $$Q \equiv \int_{\Sigma_{D-p-1}} \star j^{(p+1)}, \quad U_\alpha(\Sigma) = e^{i\alpha Q}$$
  - Transforming a "loop" operator $W(C)$
  $$W(C) \mapsto U(\Sigma) W(C)$$
  $$= e^{i\alpha \#(\Sigma, C)} W(C) \quad \text{w/ linking } \#$$

Center symmetry in YM theory

- Lattice $SU(N)$ YM theory
  - link variable $U_\ell \in SU(N)$
- Center symmetry: $\mathbb{Z}_N^{[1]}$

  $e^{\frac{2\pi i}{N}k} \in \mathbb{Z}_N \subset SU(N); \quad U_\ell \mapsto e^{\frac{2\pi i}{N}k\#(\Sigma,\ell)} U_\ell$

  Intersection # of Σ & link ℓ; $U_p \mapsto U_p$

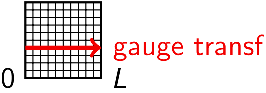


- Gauging the center symmetry

$$S \sim \sum \mathrm{Tr}\, e^{-\frac{2\pi i}{N} B_p} U_p \qquad B_p\text{: 2-form gauge field assoc. } \mathbb{Z}_N^{[1]}$$

invariant under $U_\ell \mapsto e^{\frac{2\pi i}{N}\lambda_\ell} U_\ell,\ B_p \mapsto B_p + (d\lambda)_p$

  - Recall 't Hooft twisted b.c. ['79]: $U_{n+L\hat{\nu},\mu} = g_{n,\nu}^{-1} U_{n,\mu} g_{n+\hat{\mu},\nu}$



$$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}} \in \mathbb{Z}_N$$

't Hooft flux $z_{\mu\nu} = \sum B_p \bmod N$

Aiming at transparent understanding

- Wise but *not transparent* understanding
  - Topological objects from lattice viewpoint as center sym
  - Formal discussion in continuum theory
    - ⋆ $\mathbb{Z}_N^{[q]}$ gauge field: $U(1)$ field $B^{(q)}$
      with constraint $NB^{(q)} = dB^{(q-1)}$ from charge-N Higgs
  - $Q \sim \int B \wedge B? \xrightarrow[\text{cohomological operations}]{\text{Swapping } \wedge \text{ with}} \int P_2(B) \sim -\frac{\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}{8N}$
    - ⋆ Global nature described by Čech cohomology (discrete group!)
  - Indicating mixed 't Hooft anomaly with chiral sym/θ-periodicity

$$\mathcal{Z}_{\theta+2\pi}[B_p] = e^{-2\pi i Q} \mathcal{Z}_\theta[B_p] \quad \text{not } 2\pi \text{ periodic [Abe's talk]}$$

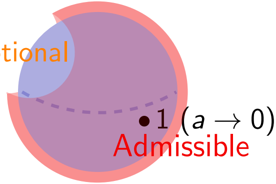
- Fully regularized framework:
  Topological approach in lattice gauge theory [Lüscher '84]
  coupled with *higher-form* lattice gauge fields
- Application to higher-group symmetry [this talk]
  - We hope this approach will be useful to other generalized symmetries as non-invertible (categorical) symmetry

Review on topology of lattice gauge theory

- No continuity for lattice gauge fields? Index theorem for finite a?

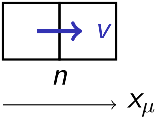
$$\mathrm{Index}(D) = -\frac{a}{2}\,\mathrm{Tr}\,\gamma_5 D_{\mathrm{ov}} = \underbrace{n_+ - n_-}_{\in\mathbb{Z}} \overset{?}{=} \frac{1}{32\pi^2}\int_x \varepsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}[F_{\mu\nu}F_{\rho\sigma}]$$

- Admissibility condition $\mathrm{tr}(1 - U_p) < \epsilon$ [Lüscher '84]



- Admissible lattice gauge fields: well-defined conf space $\sim$ disk
- Exceptional region
  - Topological freezing
  - Monopole as lattice artifact

- Smooth interpolation to principal fiber bundle on $\forall x$
  - Transition function $v_{n,\mu}(n) \to v_{n,\mu}(x)$
  - Q is written in terms of $v_{n,\mu}(x)^{-1}\partial_\nu v_{n,\mu}(x)$ then $Q \sim \int F\tilde{F} \in \mathbb{Z}$



- Fractional topological charge described by [Abe]

Generalization: higher-group structure

- In general, a naive direct product of symmetry groups?
  - Can each symmetry be gauged *individually*?
- Gauging $G^{[0]} \times H^{[1]}$ global symmetry, then gauge transf.:

$$A \mapsto A + d\lambda^{(0)}, \qquad B \mapsto B + d\lambda^{(1)} + A d\lambda^{(0)}$$

  2-group symmetry (cf. superstring theory [Green–Schwarz '84])
  - p-group symmetry: $G_0^{[0]} \tilde{\times} \cdots \tilde{\times} G_{p-1}^{[p-1]}$
- E.g., 4D $SU(N)$ gauge theory with instanton number $p\mathbb{Z}$
  - For any $p \in \mathbb{Z}$, local and unitary [Seiberg '10]
  - Global symmetry: $\underbrace{\mathbb{Z}_N^{[1]} \text{ center sym}}_{\cancel{\text{gauging}}} \times \mathbb{Z}_p^{[3]} \text{ sym} \overset{\text{gauging}}{\to}$ 4-group

    [Tanizaki–Ünsal '19]
- How to modify instanton sum & realize higher-group on lattice?

Modified instanton-sum: $\mathbb{Z}_N^{[1]} \times \mathbb{Z}_p^{[3]}$ gauge sym?

- Inserting the delta function (or introducing Lagrange multiplier)

$$\delta\left(q_n - pc_n\right) \text{ or } \left[\sum_n \chi_n(q - \dots)\right] \quad \rightarrow \quad Q = p \underbrace{\sum_n c_n}_{\in \mathbb{Z}}$$

where $Q = \sum_n q_n$, $U(1)$ 4-form field strength c_n

  - $c = dc^{(3)}$; Charged object under $\mathbb{Z}_p^{[3]}$

$$V^{(3)} = e^{\int_{M_3} c^{(3)}} \rightarrow e^{i\chi(x)} V^{(3)} = e^{\frac{2\pi i}{p} \#(x, M_3)} V^{(3)}$$

  - θ term, $i\theta Q + i\hat{\theta} \sum_n c_n$, indicates the $2\pi/p$ periodicity of θ

- Just by counting numbers, obviously no nontrivial configurations for B_p:

$$\frac{1}{8N} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} = \sum_n \frac{1}{8N} \varepsilon_{\mu\nu\rho\sigma} B_{n,\mu\nu} B_{n+\hat{\mu}+\hat{\nu},\rho\sigma} \bmod 1 \in \mathbb{Z}$$

$\mathbb{Z}_N^{[1]} \times \mathbb{Z}_p^{[3]}$ global symmetry $\rightarrow$ ~~gauge symmetry~~

Modified instanton-sum: higher-group symmetry

- Introducting new field Ω_n ($\Omega_n \in \mathbb{R}$ and $\sum_n \Omega_n \in \mathbb{Z}$)
  - Replacement: $c_n \to c_n - \frac{1}{Np}\Omega_n$: 3-form gauge inv

    $$q_n - pc_n + \frac{1}{N}\Omega_n = 0 \qquad : \text{fractionality allowed}$$

  - Redefine Ω_n as $\tilde{\Omega}_n \equiv \frac{1}{N}\Omega_n - \underbrace{\frac{1}{8N}\varepsilon_{\mu\nu\rho\sigma} B_{n,\mu\nu} B_{n+\hat{\mu}+\hat{\nu},\rho\sigma}}_{\text{fractional part of } Q}$

    Again $\sum_n \tilde{\Omega}_n \in \mathbb{Z}$

    $$\check{q}_n - pc_n + \tilde{\Omega}_n = 0 \quad \text{where } \check{q}_n\text{: integeral part of } Q$$

- 1-form and 3-form gauge transf with $\Omega_n^{(3)} \in \mathbb{R}$:

  $$B_p \mapsto B_p + (d\lambda)_p, \quad c_n \mapsto c_n + \frac{1}{p} d\Omega_n^{(3)}\,(+\mathbb{Z}),$$

  $$\tilde{\Omega}_n \mapsto \tilde{\Omega}_n + d\Omega_n^{(3)}\,(+p\mathbb{Z}) + \text{“}\frac{2}{N} B \wedge d\lambda + \frac{1}{N} d\lambda \wedge d\lambda\text{”}\,(+\mathbb{Z})$$

  Finally, $w_n \equiv$ integral part of $\tilde{\Omega}_n$
  - defined by using 1-form and *continuum* 3-form gauge transf
  - Theory possesses “mixed 1-form” $\times$ *discrete* $\mathbb{Z}_p^{[3]}$ gauge sym

Summary

- Generalized symmetries have been developed in this decade
  - Higher-form sym, higher-group sym
  - Through 't Hooft anomaly matching, new insights about *nontrivial dynamics & classification of phases*
- Standing on a fully regularized framework: lattice gauge theory
  - Maintaining locality, $SU(N)$ gauge inv & higher-form gauge inv
  - There exists interpolation to smooth enough bundle structure

  $Q \in \mathbb{Z} \xrightarrow{\text{Gauging } \mathbb{Z}_N^{[1]}} \frac{1}{N}\mathbb{Z}$ & mixed 't Hooft anomaly [Abe's talk]
- Robust discussion on higher-group from lattice
  - [Continuum case] $SU(N) \to U(N)$ & many $U(1)$ fields: $F \to \tilde{F}$, $\operatorname{tr} \tilde{F} = B^{(2)}$, $NB^{(2)} = dB^{(1)}$, $pD^{(4)} = dD^{(3)} + \frac{N}{4\pi} B^{(2)} \wedge B^{(2)}$

    What is the corresponding topological object? $\Rightarrow \int ND^{(4)} \in \frac{2\pi}{p}\mathbb{Z}$
  - [Lattice case] Counting integers and fractional numbers; mixture of symmetries
- Future works
  - Monopole, 't Hooft line (cf. [Abe-Morikawa-Onoda-Suzuki-Tanizaki '23])
  - Other kinds: subsystem sym & non-invertible (categorical) sym