Bond-weighting method
for the Grassmann tensor renormalization group

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Based on SA, JHEP11(2022)030

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Tensor network & Lattice field theory

☑ A method to investigate quantum many-body system expressing an objective function as a tensor contraction (= tensor network).

Orús, APS Physics 1(2019)538-550
Meurice-Sakai-Unmuth–Yockey, Rev. Mod. Phys. 94(2022)025005

☑ TN allows us to study the lattice QFTs w/ and w/o the sign problem.

• w/ the Hamiltonian formalism
  Describe a state vector as a TN, which is variationally optimized.

Cf. DMRG, TEBD

Cf. Talks by Goksu+, Matsumoto+, Florio+, Hanqing+, David Lin+, ...

• w/ the Lagrangian formalism
  Describe a path integral as a TN, which is approximately contracted.

Cf. TRG, TNR, Loop-TNR, GILT

Levin-Nave, PRL99(2007)120601
Yang-Gu-Wen, PRL118(2017)110504
Hauru-Delcamp-Mizera, PRB97(2018)045111
Cf. Talks by Samlodia, Nakayama, Hite+, Judah+, Hostetler+, ...
Advantages of the TRG approach

✔ Tensor renormalization group (TRG) approximately contracts a given TN based on the idea of real-space renormalization group.
  
  - No sign problem
  - The computational cost scales logarithmically w. r. t. system size
  - Direct evaluation of the Grassmann integrals
  - Direct evaluation of the path integral

✔ Applicability to the higher-dimensional systems
  
  - If the system is translationally invariant on a lattice, we can easily apply the TRG to contract the TN.
  - 4D LGTs have been investigated by the TRG.

Cf. TRG study of 4D $Z_n$ ($n = 2, 3$) gauge-Higgs models at finite density

SA-Kuramashi, JHEP05(2022)102, arXiv:2304.07934
Levin-Nave TRG

SVD defines three-leg tensors

\[ T_{IJ}^{(n)} \approx \sum_{\alpha=1}^{D} U_{I\alpha}^{(n)} \sqrt{\sigma_{\alpha}^{(n)}} \sqrt{\sigma_{\alpha}^{(n)}} V_{J\alpha}^{(n)*} \]

Repeating this cycle \( n \) times, \( 2^n \) local tensors can be approximately contracted.

Iteration

Contraction

Levin-Nave, PRL99(2007)120601
Bond-weighted TRG (BTRG)

- Introduces some weight matrix on each bond in the tensor network.
- Considers a coarse-graining transformation including these bond matrices.

Adachi-Okubo-Todo, PRB105(2022)L060402
SVD defines three-leg tensors and new bond weights

\[
T_{ij}^{(n)} \approx \sum_{\alpha=1}^{D} U_{i\alpha}^{(n)} \left( \sigma_{\alpha}^{(n)} \right)^{1-k} \left( \sigma_{\alpha}^{(n)} \right)^{k} \left( \sigma_{\alpha}^{(n)} \right)^{1-k} V_{j\alpha}^{(n)*}
\]

※ \( k \in \mathbb{R} \) is a hyperparameter explained later

※ Initial weights are identity matrices
A good choice of the hyperparameter $k \in \mathbb{R}$ in the SVD of local tensor.

\[
T_{IJ}^{(n)} \approx \sum_{\alpha=1}^{D} U_{I\alpha}^{(n)} (\sigma_{\alpha}^{(n)}) \frac{1-k}{2} (\sigma_{\alpha}^{(n)})^k (\sigma_{\alpha}^{(n)}) \frac{1-k}{2} V_{J\alpha}^{(n)*}
\]

$n$ labels the renormalization steps

A good choice of $k$? $\rightarrow$ **Power counting for the singular value.**

By the TRG renormalization, $T^{(n+1)} \sim \left[ \left( \sigma_{\alpha}^{(n)} \right) \frac{1-k}{2} (\sigma_{\alpha}^{(n)})^k \right]^4$

By the SVD of $T^{(n+1)}$, $T^{(n+1)} \sim \sigma_{\alpha}^{(n+1)}$

Suppose the singular-value spectrum becomes scale-invariant w/ sufficiently large $n$, we have

\[
\left[ \left( \sigma_{\alpha}^{(n)} \right) \frac{1-k}{2} (\sigma_{\alpha}^{(n)})^k \right]^4 = \sigma_{\alpha}^{(n)} \Rightarrow k = -0.5
\]
BTRG for the 2D classical Ising model

✔ $k = -0.5$ seems optimal and the accuracy of the BTRG is higher than the Levin-Nave TRG and the HOTRG with the same bond dimension.

✔ Introduction of $k$ does not increase the computational cost. Therefore, the cost of the BTRG is same with the Levin-Nave TRG.

Adachi-Okubo-Todo, PRB105(2022)L060402
Extension of TRG to the lattice fermion

✔ Any TRG algorithm can be used to evaluate the Grassmann path integral.


✔ The Grassmann tensor is useful to represent the Grassmann path integral.

※ Multi-linear combination of Grassmann numbers

Sa-Kadoh, JHEP10(2021)188

\[ T_{x,t} x' t' \eta_\bar{x}_t \eta_\bar{x}' t' = \sum_{x,t,x',t'} T_{x,t} x' t' \eta_\bar{x}_t \eta_\bar{x}' t' \]

<table>
<thead>
<tr>
<th>Tensor</th>
<th>Grassmann Tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
<td>Integer</td>
</tr>
<tr>
<td>Contraction</td>
<td>( \Sigma_i \ldots \int \int d\eta d\bar{\eta} e^{-\bar{\eta} \eta} \ldots )</td>
</tr>
<tr>
<td>Path integral</td>
<td>( t\text{Tr}[\prod T] )</td>
</tr>
<tr>
<td></td>
<td>( g\text{Tr}[\prod T] )</td>
</tr>
</tbody>
</table>

✔ Does the bond weighting method improve the Grassmann TRG?
Benchmarking w/ the free massless Wilson fermion in 2D

✓ $k = -0.5$ seems optimal and the bond-weighting method does improve the accuracy of the Grassmann Levin-Nave TRG.

![Graph showing the relative error for different values of $D$. The graph has a logarithmic y-axis and a linear x-axis, with the x-axis ranging from -0.7 to 0.0 and the y-axis ranging from $10^{-7}$ to $10^{-3}$. The legend indicates different values of $D$: $D = 20$, $D = 40$, $D = 60$, $D = 80$, $D = 100$. The value of $V$ is $1024^2$.](image-url)
The Levin-Nave algorithm does not reproduce the scale-invariant structure in the local Grassmann tensor, but the Grassmann BTRG does.
Summary

✔ Bond-weighting method is a new way to improve the TRG algorithm.

✔ The method was originally proposed for the spin system. We numerically confirmed that the bond-weighting method is useful for the lattice fermions.

✔ Benchmarking with the 2D free Wilson fermions, we have found that the accuracy of the TRG is highly improved. The optimal choice is $k = -0.5$, which suggests the optimal bond weight be determined just by the geometry of TN.

✔ A sample code of the Grassmann BTRG is available on GitHub. 2D single-flavor Gross-Neveu-Wilson model at finite density as an example. [https://github.com/akiyama-es/Grassmann-BTRG](https://github.com/akiyama-es/Grassmann-BTRG)

✔ Several Grassmann BTRG studies of 2D LGTs are on-going.
Finite-entanglement scaling

In 1+1D, we have the finite-entanglement scaling based on the Matrix Product State (MPS). The correlation length scales with $\xi_D \sim D^\kappa$, where

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}} + 1\right)} \quad \rightarrow \quad \kappa = 1.344 \cdots \text{ w/ } c = 1$$

Assuming this, the relative error of the free energy should be fitted by $aD^{-2\kappa}$

$k = -0.5: a \approx 0.06, \kappa \approx 1.26$

$k = 0 \quad : a \approx 0.4, \kappa \approx 1.22$

Tagliacozzo+, PRB78(2008)024410
Pollmann+, PRL102(2009)255701