Bond-weighting method for the Grassmann tensor renormalization group

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Based on SA, JHEP11(2022)030

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Tensor network & Lattice field theory

✔ **A method to investigate quantum many-body system expressing an objective function as a tensor contraction (= tensor network).**

Orús, APS Physics 1(2019)538-550
Bañuls-Cichy, Rep. Prog. Phys. 83(2020)024401
Meurice-Sakai-Unmuth–Yockey, Rev. Mod. Phys. 94(2022)025005
Okunishi-Nishino-Ueda, J. Phys. Soc. Jap. 91(2022)062001

✔ **TN allows us to study the lattice QFTs w/ and w/o the sign problem.**

- **w/ the Hamiltonian formalism**

  Describe a state vector as a TN, which is **variationally optimized**.

  Cf. **DMRG**, **TEBD**

  White, PRL69(1992)2863-2866, PRB48(1993)10345-10356
  Vidal, PRL91(2003)147902, PRL98(2007)070201
  Cf. Talks by Goksu+, Matsumoto+, Florio+, Hanqing+, David Lin+, …

- **w/ the Lagrangian formalism**

  Describe a path integral as a TN, which is **approximately contracted**.

  Cf. **TRG**, **TNR, Loop-TNR, GILT**

  Levin-Nave, PRL99(2007)120601
  Evenbly-Vidal, PRL115(2015)180405, Evenbly, PRB95(2017)045117
  Yang-Gu-Wen, PRL118(2017)110504
  Hauru-Delcamp-Mizera, PRB97(2018)045111
  Cf. Talks by Samlodia, Nakayama, Hite+, Judah+, Hostetler+, …

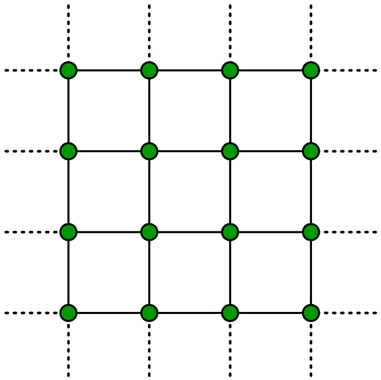
Advantages of the TRG approach

✔ Tensor renormalization group (TRG) approximately contracts a given TN based on the idea of real-space renormalization group.

- No sign problem
- **The computational cost scales logarithmically w. r. t. system size**
- **Direct evaluation of the Grassmann integrals**
- **Direct evaluation of the path integral**

✔ **Applicability to the higher-dimensional systems**

- If the system is translationally invariant on a lattice, we can easily apply the TRG to contract the TN.
- 4D LGTs have been investigated by the TRG.

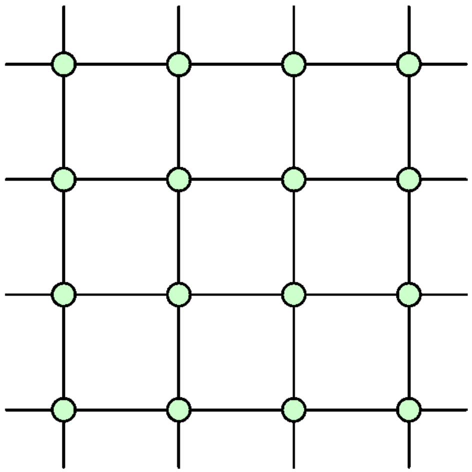


Cf. TRG study of 4D Z_n $(n = 2, 3)$ gauge-Higgs models at finite density

SA-Kuramashi, JHEP05(2022)102, arXiv:2304.07934

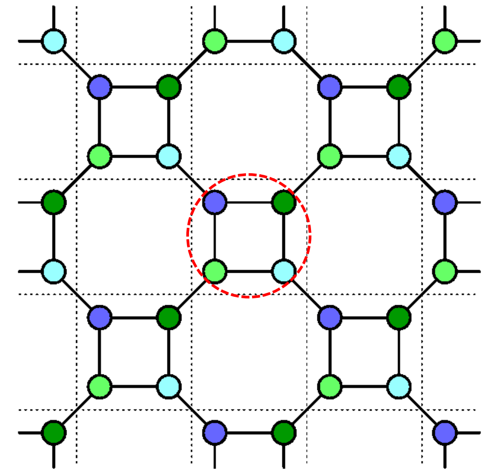
Levin-Nave TRG

Levin-Nave, PRL99(2007)120601

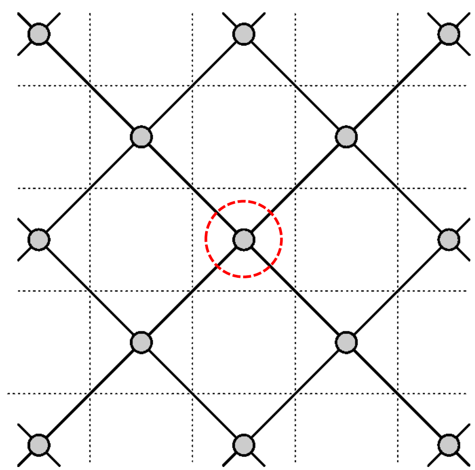


SVD defines three-leg tensors

$$T_{IJ}^{(n)} \approx \sum_{\alpha=1}^{D} U_{I\alpha}^{(n)} \sqrt{\sigma_{\alpha}^{(n)}} \sqrt{\sigma_{\alpha}^{(n)}} V_{J\alpha}^{(n)*}$$



Contraction



Iteration

Repeating this cycle n times, 2^n local tensors can be approximately contracted.

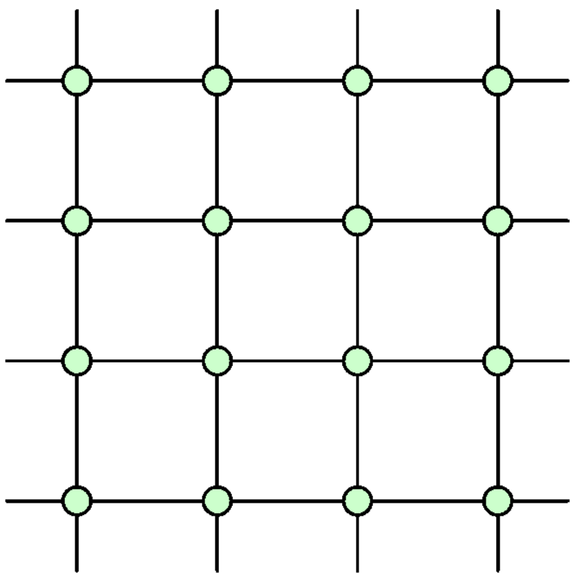
Bond-weighted TRG (BTRG)

Adachi-Okubo-Todo, PRB105(2022)L060402

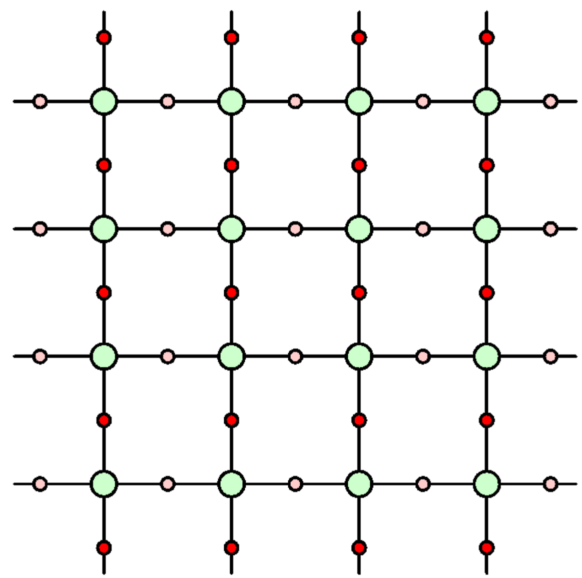
✔ Introduces some weight matrix on each bond in the tensor network.

✔ Considers a coarse-graining transformation including these bond matirces.

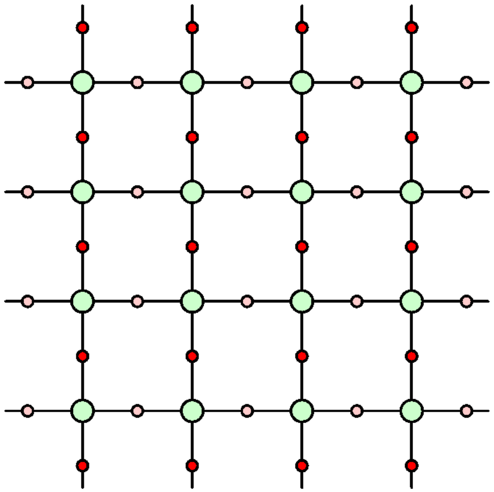
Levin-Nave TRG



BTRG



Schematic Picture of TRG w/ bond weights

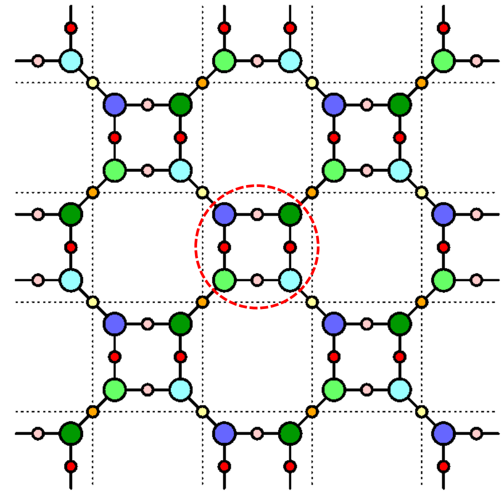


※ Initial weights are identity matrices

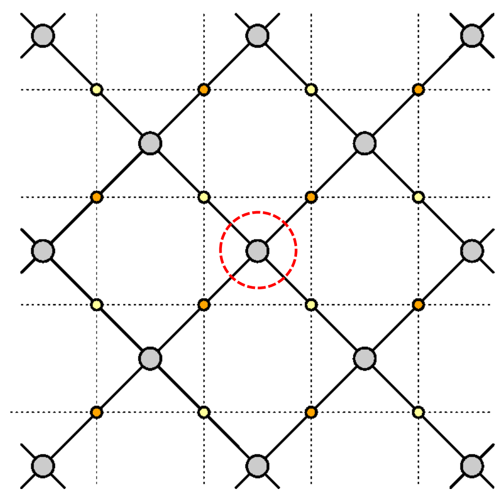
SVD defines three-leg tensors and new bond weights

$$T_{IJ}^{(n)} \approx \sum_{\alpha=1}^{D} U_{I\alpha}^{(n)} \left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} \left(\boldsymbol{\sigma_\alpha^{(n)}}\right)^{\boldsymbol{k}} \left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} V_{J\alpha}^{(n)*}$$

※ $k \in \mathbb{R}$ is a hyperparameter explained later



Contraction



Iteration

A good choice of the hyperparameter

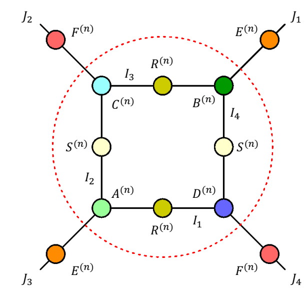
Adachi-Okubo-Todo, PRB105(2022)L060402

✔ **Introducing the hyperparameter $k \in \mathbb{R}$ in the SVD of local tensor.**

$$T_{IJ}^{(n)} \approx \sum_{\alpha=1}^{D} U_{I\alpha}^{(n)} \left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} \left(\sigma_\alpha^{(n)}\right)^{k} \left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} V_{J\alpha}^{(n)*}$$

n labels the renormalization steps

✔ A good choice of k? ⟶ **Power counting for the singular value.**



By the TRG renormalization, $T^{(n+1)} \sim \left[\left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} \left(\sigma_\alpha^{(n)}\right)^{k}\right]^4$

By the SVD of $T^{(n+1)}$, $T^{(n+1)} \sim \sigma_\alpha^{(n+1)}$

Suppose the singular-value spectrum becomes scale-invariant w/ sufficiently large n, we have

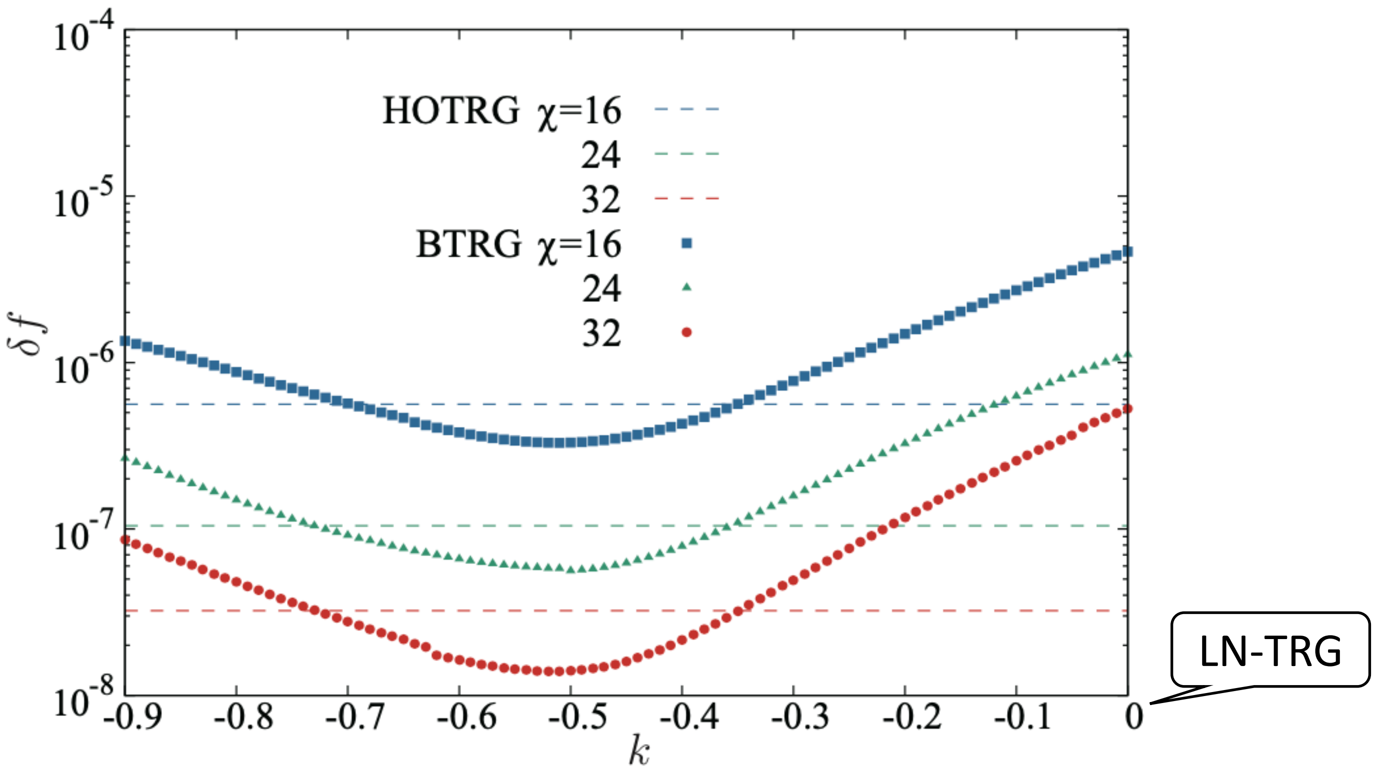
$$\left[\left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} \left(\sigma_\alpha^{(n)}\right)^{k}\right]^4 = \sigma_\alpha^{(n)} \Rightarrow \boxed{k = -0.5}$$

BTRG for the 2D classical Ising model

Adachi-Okubo-Todo, PRB105(2022)L060402

✔ $k = -0.5$ seems optimal and **the accuracy of the BTRG is higher than the Levin-Nave TRG and the HOTRG with the same bond dimension.**

✔ **Introduction of k does not increase the computational cost.** Therefore, the cost of the BTRG is same with the Levin-Nave TRG.



Extension of TRG to the lattice fermion

✔ **Any TRG algorithm can be used to evaluate the Grassmann path integral.**

Gu+,PRB88(2013)115139, Shimizu-Kuramashi,PRD90(2014)014508, Takeda-Yoshimura,PTEP2015(2015)043B01, Meurice,PoS LATTICE2018(2018)231, Bao's thesis,PhD,Uwaterloo

✔ The **Grassmann tensor** is useful to represent the Grassmann path integral.

※ Multi-linear combination of Grassmann numbers

SA-Kadoh, JHEP10(2021)188

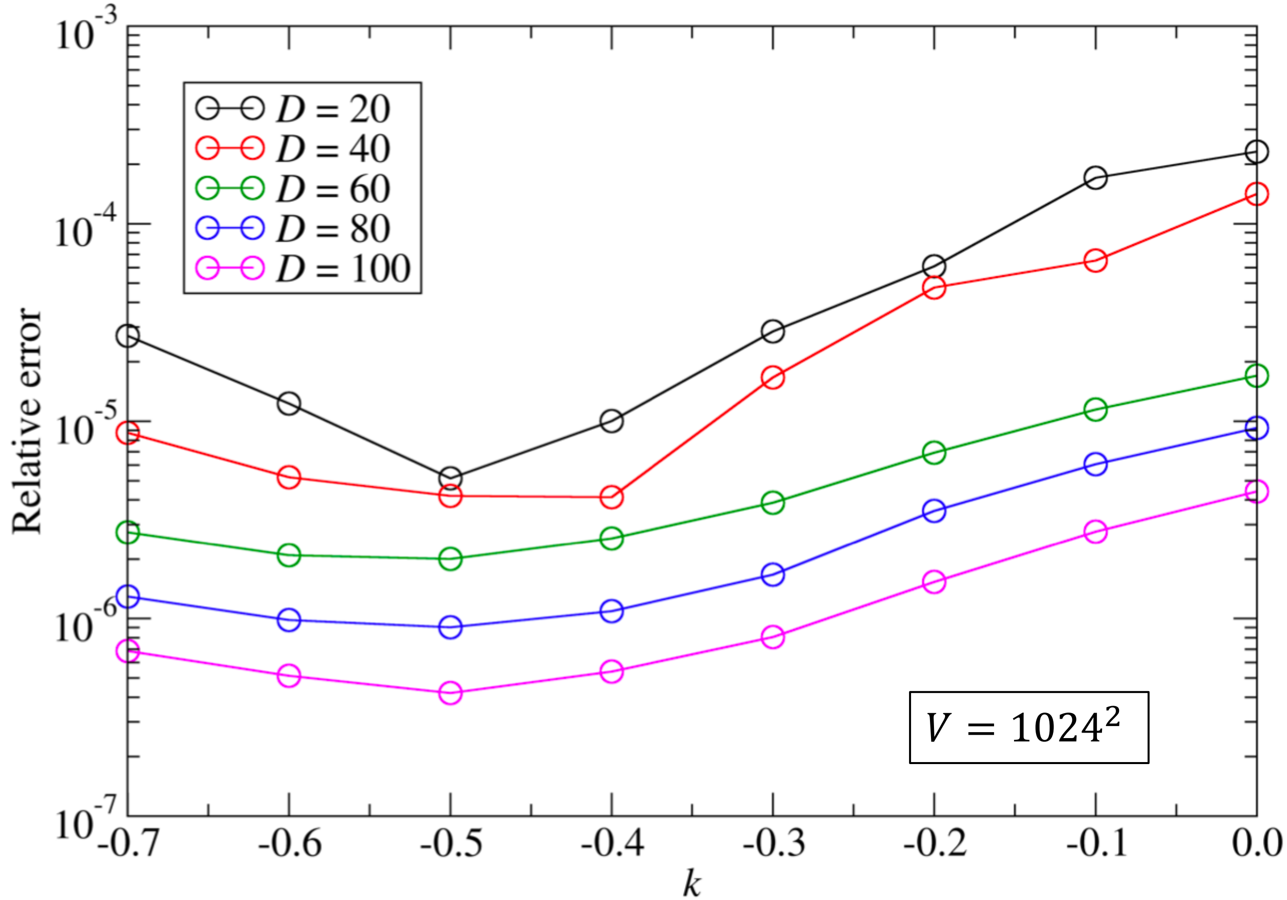
$$\mathcal{T}_{\eta_x\eta_t\bar{\eta}_x\bar{\eta}_t} = \sum_{x,t,x',t'} T_{xtx't'}\eta_x^x\eta_t^t\bar{\eta}_x^{x'}\bar{\eta}_t^{t'}$$

	Tensor	Grassmann Tensor
Index	Integer	Grassmann number
Contraction	$\Sigma_i \cdots$	$\int\int \mathrm{d}\bar{\eta}\mathrm{d}\eta \mathrm{e}^{-\bar{\eta}\eta} \cdots$
Path integral	$\mathrm{tTr}[\prod T]$	$\mathrm{gTr}[\prod \mathcal{T}]$

✔ **Does the bond weighting method improve the Grassmann TRG?**

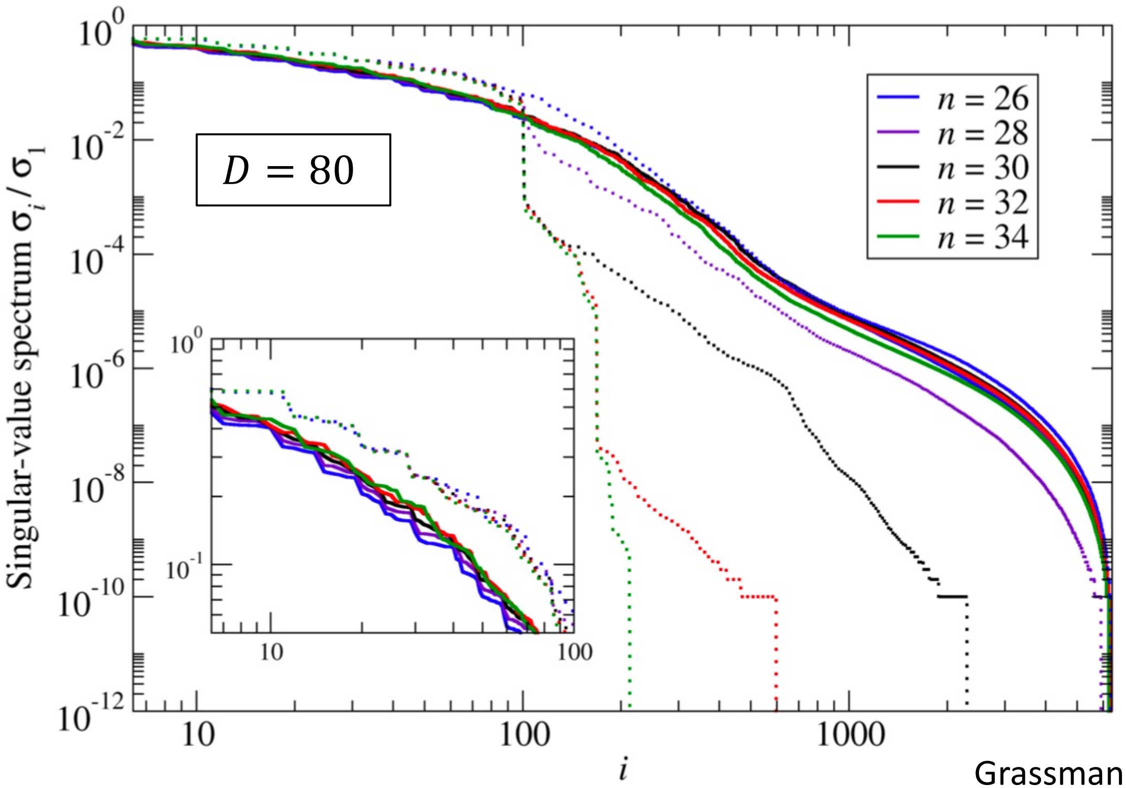
Benchmarking w/ the free massless Wilson fermion in 2D

✔ $k = -0.5$ **seems optimal and the bond-weighting method does improve the accuracy of the Grassmann Levin-Nave TRG.**



Hierarchy of the singular values

✔ The Levin-Nave algorithm does not reproduce the scale-invariant structure in the local Grassmann tensor, but the Grassmann BTRG does.



Grassmann BTRG (solid lines)
Grassmann Levin-Nave TRG (dotted lines)

n labels the renormalization steps or the lattice volume via $V = 2^n$

Summary

- ✔ Bond-weighting method is a new way to improve the TRG algorithm.
- ✔ The method was originally proposed for the spin system. We numerically confirmed **that the bond-weighting method is useful for the lattice fermions.**
- ✔ Benchmarking with the 2D free Wilson fermions, we have found that the accuracy of the TRG is highly improved. The optimal choice is $k = -0.5$, which suggests **the optimal bond weight be determined just by the geometry of TN.**
- ✔ **A sample code of the Grassmann BTRG is available on GitHub.**
  2D single-flavor Gross-Neveu-Wilson model at finite density as an example.
  https://github.com/akiyama-es/Grassmann-BTRG
- ✔ Several Grassmann BTRG studies of 2D LGTs are on-going.

Finite-entanglement scaling

✔ In 1+1D, we have the finite-entanglement scaling based on the Matrix Product State (MPS). The correlation length scales with $\xi_D \sim D^{\kappa}$, where

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}}+1\right)} \quad \longrightarrow \quad \kappa = 1.344\cdots \text{ w/ } c = 1$$

Tagliacozzo+, PRB78(2008)024410
Pollmann+, PRL102(2009)255701

✔ Assuming this, the relative error of the free energy should be fitted by $aD^{-2\kappa}$

$\boldsymbol{k = -0.5: a \approx 0.06, \kappa \approx 1.26}$

$k = 0 \quad : a \approx 0.4, \kappa \approx 1.22$