

Why magnetic monopole becomes dyon in topological insulators

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Based on collaborations with Shoto Aoki, Hidenori Fukaya, Mikito Koshino, Yoshiyuki Matsuki (Osaka U.), [arXiv:2304.13954](https://arxiv.org/abs/2304.13954) [cond-mat.mes-hall].

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What will happen to a magnetic monopole when it is put inside a topological insulator?

— We expect that the monopole is observed as a dyon with the electric charge $q_e = -1/2$, because of the Witten effect [Witten ('79)].

EFT of topological insulator and Witten effect

A topological insulator: the bulk is the insulator (gapped), but the edge is the gapless. The effective theory of the T-symmetric topological insulator is described by the $\theta = \pi$ vacuum. In the presence of the magnetic monopole, the θ -term is given by

$$L_\theta = \frac{\theta}{8\pi^2} \int d^3x \mathbf{E} \cdot \mathbf{B} = -\frac{q_m}{2} \int d^3x A^0 \delta^{(3)}(\mathbf{r}).$$

This implies that there is a particle with electric charge $q_e = -q_m/2$ which is coupled to the A^0 potential.

The monopole with $q_m = 1$ obtains the electric charge $q_e = -1/2$.

The effective theory description above is quite simple, but can't answer to the following questions:

- (1) what is the origin of the electric charge? (must be electrons)
- (2) if the origin is the electrons, why is it bound to monopole?
- (3) why is the electric charge fractional?

In this our work [[Aoki, Fukaya, Kan, Koshino Matsuki \('23\)](#)], we try to give answers to the questions from in terms of a microscopic description.

A microscopic description

Naively a UV description of the system is given by the Dirac Hamiltonian with a mass $m < 0$ [Yamagishi ('83)]:

$$H = \gamma_0 (\gamma_i (\partial_i - iA_i) + m),$$

where $\gamma_0 = \sigma_3 \otimes \mathbf{1}$ and $\gamma_i = \sigma_1 \otimes \sigma_i$.

In addition to J^2 and J_3 , there is an operator that commutes with H ; $[H, \sigma_3 \otimes D^{S^2}] = 0$, where we define the “spherical” operator

$$D^{S^2} := \sigma_i \left(L_i + \frac{n}{2} \frac{x_i}{r} \right) + 1, \quad L_i = -i\epsilon_{ijk} x_j (\partial_k - iA_k) - n \frac{x_i}{2r},$$

Physical meanings of the operator are explained later.

We find the normalizable zero-mode ($E = 0$) solution localized at the monopole ($r = 0$) with $j = |n/2| - 1/2$:

$$\psi_{j,j_3,0} = \frac{C_{j,j_3,0}}{r} \exp(-|m|r) \begin{pmatrix} 1 \\ \text{sign}(m)\text{sign}(n) \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

where $D^{S^2} \chi_{j,j_3,0}(\theta, \phi) = 0$.

1. No difference between the positive and negative mass in the solution. The Witten effect predicts the dyon appear only in the topological insulator ($m < 0$). The solution can't explain it w/o imposing "the chiral boundary condition" by hand.
2. Why does the electric charge become $q_e = -1/2$?

Regularized Dirac equation

We consider the “regularized” Dirac Hamiltonian:

$$H_{\text{reg}} = \gamma_0 \left(\gamma^i D_i + m + \frac{D_i^\dagger D^i}{M_{\text{PV}}} \right).$$

We can interpret the additional “Wilson term” as a reg. eff., since

$$\begin{aligned} Z &= \det \left(\frac{D + m}{D + M_{\text{PV}}} \right), \\ &= \det \left[\frac{1}{M_{\text{PV}}} \left(D + m + \frac{1}{M_{\text{PV}}} D_\mu^\dagger D^\mu \right. \right. \\ &\quad \left. \left. + \mathcal{O}(1/M_{\text{PV}}^2, m/M_{\text{PV}}, F_{\mu\nu}/M_{\text{PV}}) \right) \right]. \end{aligned}$$

in the Pauli–Villars reg., where $M_{\text{PV}}(> 0)$ is the PV mass.

Since the Laplacian $D_i^\dagger D^i$ is always positive, the mass shift due to the Wilson term is always positive when we take M_{PV} positive.

For $m < 0$ (or inside topological insulators), it is possible to locally flip the sign of the “effective” mass

$$m < 0 \quad \rightarrow \quad m_{\text{eff}} = m + \frac{D_i^\dagger D^i}{M_{\text{PV}}} \sim m + \frac{1}{M_{\text{PV}} r_1^2} > 0,$$

when the magnetic flux is concentrated in the region $r < r_1$.

It's implies that the inside region $r < r_1$ becomes a normal insulator, and the spherical domain-wall is dynamically created and the chiral edge-mode appears on it! (It doesn't happen in the normal insulator with $m > 0$.)

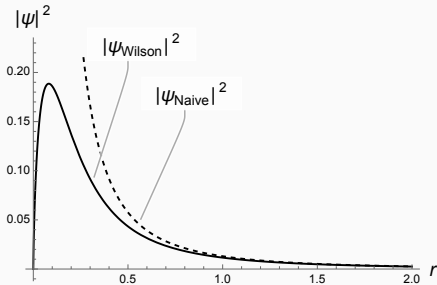
The regularized Hamiltonian is

$$H_{\text{reg}} = \begin{pmatrix} m - D_i D_i / M_{\text{PV}} & \sigma_i (\partial_i - i A_i) \\ -\sigma_i (\partial_i - i A_i) & -m + D_i D_i / M_{\text{PV}} \end{pmatrix},$$

The analytical solution of the zero-mode for $r_1 \rightarrow 0$ is given by

$$\psi_{j,j_3}^{\text{mono}} = \frac{B e^{-M_{\text{PV}} r / 2}}{\sqrt{r}} I_\nu(\kappa r) \begin{pmatrix} 1 \\ -\text{sign}(n) \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

where $\nu = \sqrt{2|n| + 1}/2$, and $\kappa = M_{\text{PV}} \sqrt{1 + 4m/M_{\text{PV}}}/2$.



The plot with $n = 1$, $m = -0.1$, $M_{\text{PV}} = 10$.

- The solution ψ_{Wilson} coincides with ψ_{Naive} for large r .
- A peak at $r = |n|/(2M_{\text{PV}}) \sim 1/M_{\text{PV}}$ is the (spherical) domain-wall.
- D^{S^2} is the Dirac operator on the spherical domain-wall created around the monopole!

The Atiyah–Singer index theorem and the half-integral charge

The 2D chirality is

$$\sigma_r \chi_{j,j_3,0}(\theta, \phi) = s \chi_{j,j_3,0}(\theta, \phi), \quad s := \text{sign}(n),$$

and # of the degeneracy is $2j + 1 = |n|$. Then the Dirac index is

$$\text{Ind } D^{S^2} = n.$$

On the other hand, the topological index is

$$\frac{1}{4\pi} \int_{S^2} d^2x \epsilon^{\mu\nu} F_{\mu\nu} = n.$$

Stability of the zero modes on the domain-wall is topologically protected by the AS index theorem.

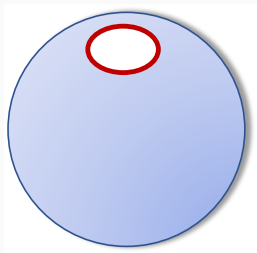
So far, we considered a \mathbb{R}^3 space, but in order to discuss topological feature of the fermion zero mode, we also need an IR regularization, such as the one-point compactification, S^3 .

Then the topological insulator region with ($m_{\text{eff}} < 0$) has topology of a disk with a small S^2 boundary at $r = r_1$.

However, due to the cobordism invariance of the AS index,

$$\int_{\partial M} F = \int_M dF = 0,$$

the disk is not possible.



A resolution is: to create another domain-wall at, say, $r = r_0$, outside of the topological insulator.

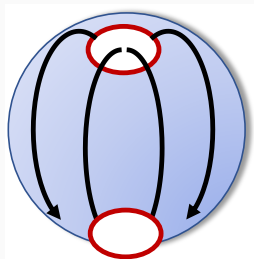
Another zero-modes are localized at the outside domain-wall, and the index is kept trivial:

$$0 = \int_M dF = \int_{\Sigma_{\text{mono}}} F + \int_{\Sigma_{\text{out}}} F,$$

where $\partial M = \Sigma_{\text{mono}} \cup \Sigma_{\text{out}}$.

Then the 50% of the zero-mode state is located at the monopole, while the other 50% is sit at the domain-wall.

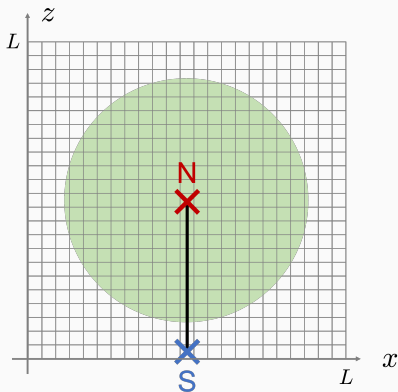
Thus the dressed electric charge of the monopole becomes $-1/2!$



Numerical analysis

Lattice setup

On a three-dimensional hyper-cubic lattice with size L with open boundary conditions, we put a monopole at $\mathbf{x}_m = (L/2, L/2, L/2)$ with a magnetic charge $n/2$. We also put an antimonopole at $\mathbf{x}_a = (L/2, L/2, 1/2)$ with the opposite charge $-n/2$.

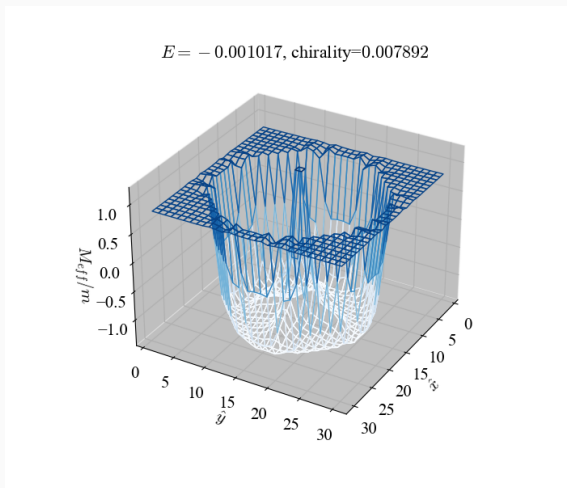


To directly confirm creation of the domain-wall near the monopole, we plot distribution of the “effective mass” (normalized by m_0),

$$m_{\text{eff}}(\mathbf{x}) = \phi(\mathbf{x})^\dagger \left[- \sum_{i=1,2,3} \frac{1}{2} \nabla_i^f \nabla_i^b + m(\mathbf{x}) \right] \phi(\mathbf{x}) / \phi(\mathbf{x})^\dagger \phi(\mathbf{x}),$$

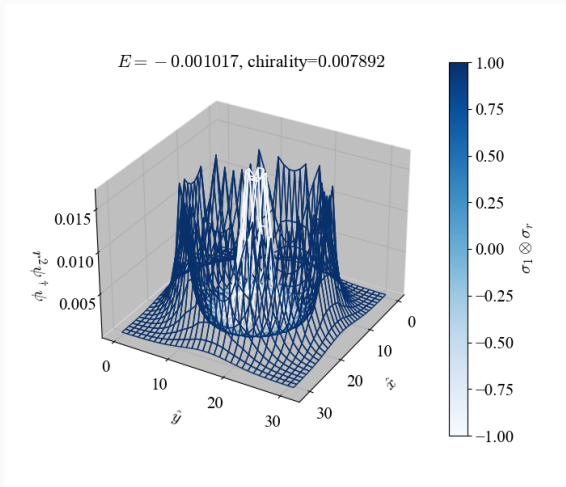
on the $z = (L + 1)/2$ slice.

The effective mass of the nearest zeromode with $n = 1$ on $z = 16$ slice:



Amplitude

The amplitude of the nearest-zero mode for $n = 1$ in $z = 16$ slice:



Summary

We discussed a microscopic description of the Witten effect with the Wilson term.

How do we distinguish between the normal insulator ($m > 0$) and topological insulator ($m < 0$)?

- It is the topological insulator if the mass is relatively negative compared to the PV mass.

Why are electrons bound to monopole?

- Because of the positive mass correction from the magnetic field of the monopole, the domain-wall is dynamically created (only for the negative mass).

Why do the stable chiral zero modes appear?

- Because the zero modes localized at the domain-wall are protected by the AS index.

Why is the electric charge fractional?

- Because the 50% of the wavefunction is located around the monopole (the other 50% is located at the surface of the topological insulator).