Recent Developments of Euclidean Dynamical Triangulations with Non-Trivial Measure Term

Lattice 2023
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Marc Schiffer, Perimeter Institute

In collaboration with
M. Dai, W. Freeman, J. Laiho, and J. Unmuth-Yockey: to appear

see also [Mingwei Dai, today 4:20 pm, WH3NE], [Jack Laiho, today 4:40 pm, WH3NE]
Asymptotically Safe Quantum Gravity

- Perturbative quantum gravity: loss of predictivity
- Key idea of asymptotic safety: Quantum realization of scale symmetry
  - Imposes infinitely many conditions on theory space
  - Relevant directions: need measurement
  - Irrelevant directions: predictions of theory
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• Key idea of asymptotic safety: Quantum realization of scale symmetry
  ▶ imposes infinitely many conditions on theory space
  ▶ relevant directions: need measurement
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\[ k \partial_k \alpha_s = -\frac{11}{2\pi} \alpha_s^2 + \mathcal{O}(\alpha_s^4) \]
\[ k \partial_k G = \epsilon G - \frac{50}{3} G^2 + \mathcal{O}(G^3) \]
Dynamical triangulations

- Discretization of spacetime in terms of triangulations

\[ Z_{Dg e}^{-S}[g] \rightarrow X_{T1C T e -S E R} \]

with Euclidean Einstein-Regge action

\[ S_{ER} = -\kappa^2 N^2 + \kappa^4 N^4 \]

[Regge, 1961]
Dynamical triangulations

- Discretization of spacetime in terms of triangulations

\[
\sum_{g} g 
\rightarrow
\sum_{X} T
\]

with Euclidean Einstein-Regge action

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S_{ER} = -\kappa^2 N^2 + \kappa^4 N^4
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[Regge, 1961]
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\[ \int \mathcal{D}g \, e^{-S[g]} \rightarrow \sum_{\mathcal{T}} \frac{1}{C_{\mathcal{T}}} e^{-S_{\text{ER}}} \]

with Euclidean Einstein-Regge action \( S_{\text{ER}} = -\kappa_2 N_2 + \kappa_4 N_4 \) [Regge, 1961]
Lattice quantum gravity in $d = 4$

- Discretization of spacetime in terms of triangulations

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- in \( d = 4 \): no physical phase, no indications for higher-order transition in \( \kappa_2-\kappa_4 \) - space

[Ambjørn, Jain, Jurkiewicz, Kristjansen, 1993], [Bakker, Smit, 1994]
[Ambjørn, Jurkiewicz, 1995], [Bialas, Burda, Krzywicki, Petersson, 1996]
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  . . .

- CDT: impose causal structure


Taken from [Loll, 2020]
Lattice quantum gravity in $d = 4$

- Discretization of spacetime in terms of triangulations

$$\int \mathcal{D}g e^{-S[g]} \to \sum_T \frac{1}{C_T} \left[ \prod_{j=1}^{N_2} \mathcal{O}(t_j) \right] \beta e^{-S_{ER}}$$

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- CDT: impose causal structure

[Ambjørn, Loll, 1998], [Ambjørn, Jurkiewicz, Loll, 2000]

- EDT: include local measure term

[Bruegmann, Marinari, 1993], ...
[Laiho, Coumbe, 2011], [Ambjørn, Glaser, Goerlich, Jurkiewicz, 2013]

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Phase diagram of EDT

- Idea:
  follow AB-line towards large $\kappa_2$ and negative $\beta$

[Ambjorn, Glaser, Goerlich, Jurkiewicz, 2013]
[Coumbe, Laiho, 2014]
Phase diagram of EDT

- Idea:
  follow AB-line towards large $\kappa_2$ and negative $\beta$

- Challenge: Metropolis acceptance rate $p$ drops:

  $\kappa_2 = 1.7, \ p \sim 10^{-3}$;  
  $\kappa_2 = 2.45, \ p \sim 10^{-4}$;  
  $\kappa_2 = 3.0, \ p \sim 3 \cdot 10^{-5}$;  
  $\kappa_2 = 3.8, \ p \sim 5 \cdot 10^{-6}$;  
  $\kappa_2 = 4.5, \ p \sim 1 \cdot 10^{-6}$;

Need efficient algorithm for low acceptance rates.

[Ambjorn, Glaser, Goerlich, Jurkiewicz, 2013]
[Coumbe, Laiho, 2014]
Rejection-free algorithm

- Used in studies of dynamical systems (e.g., growth of crystals)
- Rejection-free algorithm follows the steps

1. Initialize:
   1.1 Compute probability $p_{A \rightarrow B}$ at each lattice site $i$, save them in a list $p_i = p_{A \rightarrow B}(i)$.
   1.2 Save a list of summed probabilities $P_i = P_{i \leq l} p_l$.

2. Generate a random number $r \in (0, \text{Max}(P_i))$.

3. Find the lattice site $j$ such that $P_j - 1 < r \leq P_j$ (with $P_0 = 0$).

4. Perform the move at lattice site $j$.

5. Update the entries of $p_i$ and $P_i$.

6. Repeat from 2.

If step (5.) is fast, compared to low Metropolis acceptance: $\Rightarrow$ potential speedup of simulations.

Problem: $\text{SER} = -\kappa^2 N^2 + \kappa^4 N^4$, purely global!
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Detailed balance and probabilities

- Including local measure term: $S_{\text{tot}} = S_{\text{loc}} + S_{\text{glob}}$
- Detailed balance:

$$\frac{p_{A \rightarrow B}}{p_{B \rightarrow A}} = \frac{e^{-S_B}}{e^{-S_A}},$$

Monte-Carlo time from weight (“dwell time”): $\omega = \frac{1}{\sum_{i=1}^{N} p_{A \rightarrow B_i}}$. 
Detailed balance and probabilities

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- Detailed balance:
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- Metropolis accept probabilities:
  \[
  p_{A\rightarrow B} = \begin{cases} 
  1 & \text{if } S_B < S_A \\
  e^{(S_A - S_B)} & \text{if } S_B > S_A 
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  \[
  p_{A\rightarrow B} \neq (p_{A\rightarrow B})_{\text{loc}} (p_{A\rightarrow B})_{\text{glob}} .
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[Di, Freeman, Laiho, MS, Unmuth-Yockey; to appear]

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\mathcal{P}_{A\rightarrow B} = e^{\frac{1}{2}(S_A-S_B)},
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Use of ponderances allows to separate global and local part of the action.
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Use of ponderances allows to separate global and local part of the action.

Monte-Carlo time from weight ("dwell time"): \( \omega = \frac{1}{N \sum_{i=1}^{N} \mathcal{P}_{A \rightarrow B_i}} \).
Testing the new algorithm I

- Proof of principles: 2d Ising model
Testing the new algorithm I

• Proof of principles: 2d Ising model

• Simulate probability distribution of energy states at $T = 2$:
  
  $P_k(T) = \frac{g_k e^{-\frac{4 k J}{T}}}{\sum_{k=0}^{N} g_k e^{-\frac{4 k J}{T}}}$, with multiplicity $g_k$ of states with energy $4k J$

Weight-accumulating Rejection-free algorithm reproduces correct distribution!
Testing the new algorithm I

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, with multiplicity $g_k$ of states with energy $4Jk$

At low acceptance rate: significant speedup!
Preliminary results!

- 4d EDT: compare order parameters and lattice observables
- $\delta \Lambda$: unphysical parameter; stabilizes lattice volume and tunes $\kappa_4$
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Metropolis and Rejection free algorithms seem to agree in physical limit $\delta \Lambda \to 0$
Preliminary results!

- 4d EDT: compare order parameters and lattice observables
- $\delta \Lambda$: unphysical parameter; stabilizes lattice volume and tunes $\kappa_4$

Some speedup, but plateau; Reason: higher connectivity.
Key object: the shelling function

- Shelling function $f(\tau)$: counts number of four-simplices at geodesic distance $\tau$ away from source-simplex.

- For de Sitter with $d_{\text{H}}=4$:

  \[ f(\tau) \sim N_4 \frac{1}{4} \cos^3 \tau + \delta N_1 \frac{1}{4} \]

- Peak-height: order parameter of AB-transition

- Lattice volume profiles:⇒ scale in agreement with $d_{\text{H}}=4$⇒ approximate de Sitter profile better for finer lattices
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\[\kappa_2 = 3.8\]

\( \# \text{ of 4-simplexes} \)

\( \rho \)

credits to Mingwei Dai
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EDT lattice spacings: $a/\ell$

- $a$: lattice spacing; edge length
- $\ell$: dual lattice spacing; simplex distance

Fit:

$$f(\tau) = A \cos^3(B \tau + C)$$

- $A$: measures volume $\Rightarrow$ units of $a$
- $B$: $\sim 1/\tau$ $\Rightarrow$ units of $\ell$

Scale factor for Euclidean de Sitter: $a_H = q_3 \Lambda \cos q_3 \Lambda 3 \tau$

Assume:

$$f(\tau) \sim (a_H)^3$$ $\Rightarrow$ $a/\ell \sim 1/(A^{1/3}B^{3/4})$  

Fit:

$$a/\ell = A + B V + C V^2$$

Example: "rather-fine" lattices: $a/\ell = 16.585(683)$
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  $$a_H = \sqrt{\frac{3}{A}} \cos \left( \sqrt{\frac{A}{3}} \tau \right)$$
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- Example: "rather-fine" lattices:
  $$\frac{a}{\ell} = 16.585(683)$$

credits to Mingwei Dai
Determining absolute and relative lattice spacings

\[ \langle N^4 \rangle \approx k^2 4(\kappa^4 - \kappa_c^4)^2 \Rightarrow k = |\kappa^4 - \kappa_c^4|^{1/4} N^4. \]

Use finite-volume scaling of \( \kappa^4 \) to test recovery of semi-classical limit

Match lattice saddle point approximation with continuum calculation:

\[ Z(\kappa_2^2, \kappa_4^2) \approx \exp k^2 4(\kappa^4 - \kappa_c^4) \]

Assumption: Continuum is dominated by de Sitter instanton [Hawking, Moss, 1987]

Extract \( G \) from lattice data:

\[ G_{\ell^2, \text{fid}} \sim a_{\ell^2} 2_{\ell^2, \text{rel}} |s|, \]
Determining absolute and relative lattice spacings

• Saddle point approximation:
  \[ \langle N_4 \rangle \simeq \frac{k^2}{4(\kappa_4 - \kappa_4^c)^2} \Rightarrow k = |\kappa_4 - \kappa_4^c| \sqrt{N_4} . \]

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Use finite-volume scaling of \( \kappa_4 \) to test recovery of semi-classical limit

• Match lattice saddle point approximation with continuum calculation:

\[ Z(\kappa_2, \kappa_4) \approx \exp \left( \frac{k^2(\kappa_2)}{4(\kappa_4 - \kappa_c^4)} \right) = \exp \left( \frac{3\pi}{G \Lambda} \right) \]

Assumption: Continuum is dominated by de Sitter instanton \[\text{[Hawking, Moss, 1987]}\]
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  [Ambjorn, Goerlich, Jurkiewicz, Loll, 2012]

\[ \langle N_4 \rangle \simeq \frac{k^2}{4(\kappa_4 - \kappa_4^c)^2} \Rightarrow k = |\kappa_4 - \kappa_4^c| \sqrt{N_4}. \]

Use finite-volume scaling of \( \kappa_4 \) to test recovery of semi-classical limit

• Match lattice saddle point approximation with continuum calculation:

\[ Z(\kappa_2, \kappa_4) \approx \exp \left( \frac{k^2(\kappa_2)}{4(\kappa_4 - \kappa_4^c)} \right) = \exp \left( \frac{3\pi}{G \Lambda} \right) \]

Assumption: Continuum is dominated by de Sitter instanton  [Hawking, Moss, 1987]

Extract \( G \) from lattice data:

\[ \frac{G}{\ell_{\text{fid}}^2} \sim \left( \frac{a}{\ell} \right)^2 \frac{\ell_{\text{rel}}^2}{|s|}. \]
Numerical result: finite volume scaling

Example at $\beta = -0.575$, $\kappa_2 = 2.245$:

![Graph showing finite volume scaling](image)

Extract slope $s$: $\kappa_4(N_4) = A + s \frac{1}{\sqrt{N_4}}$
Fit result: $s = -0.268 \pm 0.011$
$\chi^2/d.o.f = 0.82$. 
Numerical result: the Newton coupling

- Extract slope for all ensembles
- Compute $G$ for each of the ensembles

At given lattice spacing, perform infinite volume extrapolation:

$$G = H_G V + I_G V^2 + G_0,$$

- Fine lattices: $G_0 = 64.4 \pm 4.4; \chi^2/d.o.f = 0.81 \Rightarrow \ell_{\text{fine}} \approx 0.125 \ell_{\text{Planck}}$
- Medium-Fine lattices: $G_0 = 30 \pm 20; \chi^2/d.o.f = 0.22 \Rightarrow \ell_{M\text{fine}} \approx 1.79 \ell_{\text{fine}}$

$\ell_{\text{rel}}$ for finer lattices: WIP; computationally intense; Marc Schiffer, Perimeter Institute
Numerical result: the Newton coupling

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- At given lattice spacing, perform infinite volume extrapolation:

$$G = \frac{H_G}{V} + \frac{I_G}{V^2} + G_0,$$

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Summary & Conclusion

- New algorithm: more and finer lattices
- Lattice geometries: resemble semi-classical de Sitter space with $d_H \approx 4$
- Extract $a/\ell$ from de Sitter volume profile
- Extract absolute and relative lattice spacing from semi-classical approximation

Stay tuned!

[Mingwei Dai, today 4:20 pm, WH3NE]

[Jack Laiho, today 4:40 pm, WH3NE]
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Thank you for your attention!