2023 LATTICE

More (on) Minimal Renormalon Subtraction

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Lattice 2023 | Theoretical Developments
3 August 2023

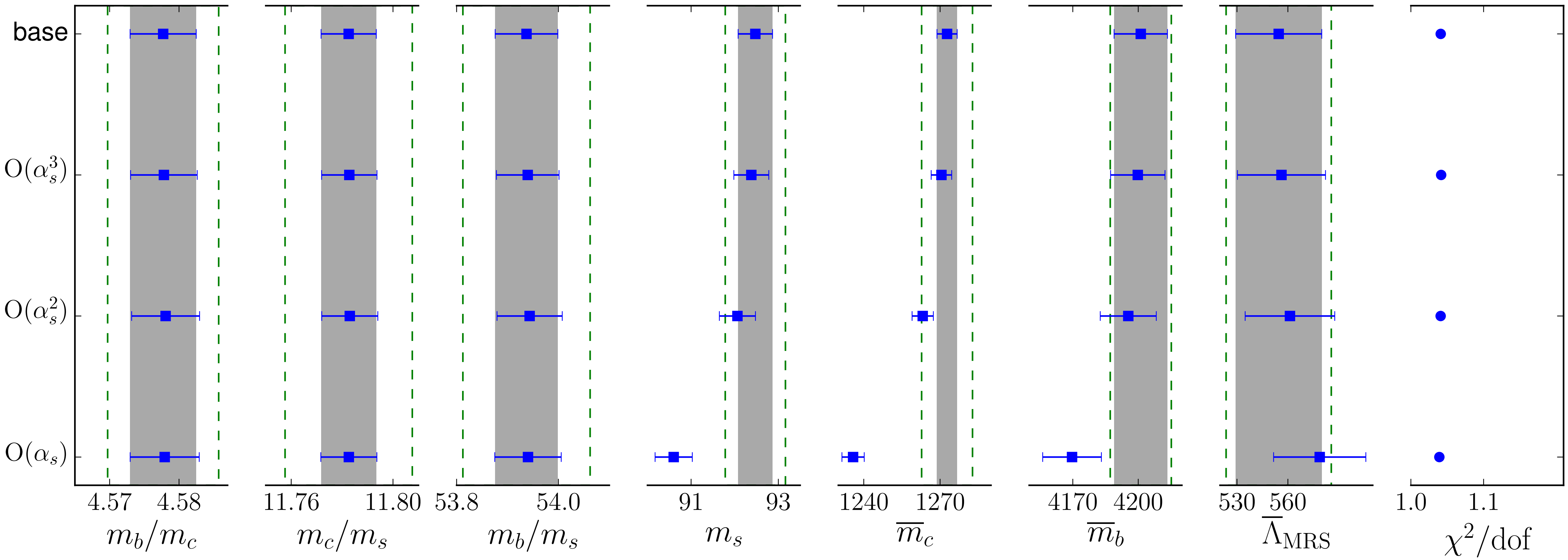
IAS TUM Institute for Advanced Study

No Perturbative Truncation Uncertainty!?!

- Quark masses in $\overline{\text{MS}}$ scheme with small uncertainties:
  - total < 1% for bottom, charm, strange;
  - and 1–2% for up and down. [arXiv:1802.04248].
- Negligible uncertainty for truncating perturbation theory:
  - order $\alpha_s{}^4$ “matching”, but still 🤨;
  - could whatever wizardry was used be generalized?

Perturbative Stability

Fermilab Lattice, MILC, & TUMQCD [arXiv:1802.04248]



meson mass

brown muck

heavy quark “pole” mass

$$M_B = m_b + \bar{\Lambda} + O(1/m_b),$$

$m_{\overline{\mathrm{MS}}}(\bar{m})$

$$m_b = \bar{m}\left(1 + \sum_{l=0} r_l \alpha_s^{l+1}(\bar{m})\right).$$

$$\{0.42, 1.03, 3.69, 17.4\}$$

Factorial Growth

- The r_l grow factorially (known for a long time):

$$r_l \sim R_0 (2\beta_0)^n \frac{\Gamma(l+1+b)}{\Gamma(1+b)} \equiv R_l$$

for $l \gg 1$. Here $b = \beta_1/2\beta_0^2 \stackrel{n_f=3}{=} 32/81 \approx 0.4\,.$

- Komijani [arXiv:1701.00347] found an expression for renormalon strength:

$$R_0 = \sum_{k=0}^{\infty} (k+1) \frac{\Gamma(1+b)}{\Gamma(k+2+b)} (2\beta_0)^{-k} f_k$$

obtained from r_j, β_j, $j \le k$; see below.

Minimal Renormalon Subtraction

- Brambilla, Komijani, & ASK, Vairo [arXiv:1712.04983] proposed minimal renormalon subtracted (MRS) mass:

$$m_b = \bar{m} + \bar{m}\sum_{l=0}^{\infty}[r_l - R_l]\alpha_s^{l+1}(\bar{m}) + \bar{m}\sum_{l=0}^{\infty}R_l\alpha_s^{l+1}(\bar{m})$$

  and, in practice, first sum is truncated at some order.

- Second sum can be carried out via Borel procedure, yielding two terms, one that has a convergent expansion in $1/\alpha_s$, the other can be absorbed into $\bar{\Lambda}$.

- Dubbed "minimal renormalon subtraction".

Perturbation Theory

- First four r_l are known:
  - one loop [NPB 183 (1981) 384]: $r_0 = 0.4244$
  - 2 loops [ZPC 48 (1990) 673]: $r_1 = 1.0351$ $(n_f = 3)$
  - 3 loops [2+1 papers]: $r_2 = 3.6932$ $(n_f = 3)$
  - 4 loops [arXiv:1606.06754]: $r_3 = 17.4358$ $(n_f = 3)$
- 5-loop Callan-Symanzik β function is known [arXiv:1606.08659].
- 5-loop mass anomalous dimension is known [arXiv:1402.6611].

Perturbation Theory

- First four r_l are known:

		R_n	
one loop [NPB 183 (1981) 384]:	$r_0 = 0.4244$	0.5350	
2 loops [ZPC 48 (1990) 673]:	$r_1 = 1.0351$	1.0691	$(n_f = 3)$
3 loops [2+1 papers]:	$r_2 = 3.6932$	3.5966	$(n_f = 3)$
4 loops [arXiv:1606.06754]:	$r_3 = 17.4358$	17.4195	$(n_f = 3)$

- 5-loop Callan-Symanzik β function is known [arXiv:1606.08659].
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Talk TOC

- Motivation
- Generalizations:
  - Λ^p instead of single power $\bar{\Lambda}$;
  - subtract subleading factorial growth, i.e., series with more than one renormalon;
  - scale dependence $\alpha_s(Q) \to \alpha_s(sQ)$:
    - static energy, “same” renormalon as m_b, Wilson line.

Higher-power Power Corrections

Notation & Setup

- Consider (dimensionless)

$$\mathscr{V}(Q) = v_{-1} + V(Q) + C_p \frac{\Lambda^p}{Q^p}, \qquad V(Q) = \sum_{l=0} v_l(\mu/Q)\alpha_s(\mu)^{l+1},$$

- RGE: oefficients' μ dependence must cancel that of α_s.
- $\therefore$ RGE sets Q dependence of $V(Q)$.

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physical quantity

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power correction

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Notation & Setup

- Consider (dimensionless)

power $p \leftrightarrow$ factorial growth

physical quantity

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- RGE: oefficients' μ dependence must cancel that of α_s.

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Renormalon Removal

- Define $$v_{-1}+F(Q)\equiv\frac{1}{pQ^{p-1}}\frac{\mathrm{d}\,Q^p\mathscr{V}}{\mathrm{d}Q}\equiv\hat{Q}_p\mathscr{V}$$
- This operation eliminates Λ^p/Q^p.
- As a series $F(Q)=\sum_{k=0} f_k\alpha_s^{k+1}$.
$$f_k=v_k-\frac{2}{p}\sum_{j=0}^{k-1}(j+1)\beta_{k-1-j}v_j$$
- Differential equation $V(\alpha)+\frac{2}{p}\beta(\alpha)V'(\alpha)=F(\alpha)$, which Komijani solves via developed asymptotic expansions.
- NB: solution of homogeneous equation is $C_p\Lambda^p$.

My Solution

- The relation between the coefficients is a matrix equation

$$f_k = v_k - \frac{2}{p}\sum_{j=0}^{k-1}(j+1)\beta_{k-1-j}v_j$$

$$\boldsymbol{f} = \left[\mathbf{1} - \frac{2}{p}\mathbf{D}\right]\cdot\boldsymbol{v} \equiv \mathbf{Q}_p\cdot\boldsymbol{v}$$

and $\mathbf{D}$ is on the lower triangle.

- Although the “matrix” is infinite, the triangular form makes an iterative solution straightforward.

- Let $\mathbf{P}_p = \text{diag}[1, p/2\beta_0, (p/2\beta_0)^2, (p/2\beta_0)^3, \ldots]$.
- Then $\mathbf{Q}_p = \mathbf{P}_p^{-1} \cdot \mathbf{B}_p \cdot \mathbf{P}_p$

$$\mathbf{B}_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -pb & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -(pb)^2 & -2pb & -3 & 1 & 0 & 0 & 0 & \cdots \\ -(pb)^3 & -2(pb)^2 & -3pb & -4 & 1 & 0 & 0 & \cdots \\ -(pb)^4 & -2(pb)^3 & -3(pb)^2 & -4pb & -5 & 1 & 0 & \cdots \\ -(pb)^5 & -2(pb)^4 & -3(pb)^3 & -4(pb)^2 & -5pb & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$b = \beta_1/2\beta_0^2 \overset{n_f=3}{=} 32/81 \approx 0.4$$

- (a certain scheme for α_s is chosen to simplify algebra)

- Inverse reveals that factorial growth begins at low orders:

$$\mathbf{B}_p^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{\Gamma(3+pb)}{\Gamma(2+pb)} & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{\Gamma(4+pb)}{\Gamma(2+pb)} & 2\frac{\Gamma(4+pb)}{\Gamma(3+pb)} & 3 & 1 & 0 & 0 & 0 & \cdots \\ \frac{\Gamma(5+pb)}{\Gamma(2+pb)} & 2\frac{\Gamma(5+pb)}{\Gamma(3+pb)} & 3\frac{\Gamma(5+pb)}{\Gamma(4+pb)} & 4 & 1 & 0 & 0 & \cdots \\ \frac{\Gamma(6+pb)}{\Gamma(2+pb)} & 2\frac{\Gamma(6+pb)}{\Gamma(3+pb)} & 3\frac{\Gamma(6+pb)}{\Gamma(4+pb)} & 4\frac{\Gamma(6+pb)}{\Gamma(5+pb)} & 5 & 1 & 0 & \cdots \\ \frac{\Gamma(7+pb)}{\Gamma(2+pb)} & 2\frac{\Gamma(7+pb)}{\Gamma(3+pb)} & 3\frac{\Gamma(7+pb)}{\Gamma(4+pb)} & 4\frac{\Gamma(7+pb)}{\Gamma(5+pb)} & 5\frac{\Gamma(7+pb)}{\Gamma(6+pb)} & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

- As a formula:

$$v_l = \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k$$

$$+ f_l$$

larger $p \Rightarrow$ slower growth

namely, the well-known factorial growth, a sum similar to Komijani's renormalon strength factor, and an extra term.

- Suppose we have L orders. For $l < L$, this equation returns the v_l we started with.

- For $l \geq L$, it suggests truncating on the f_k not the v_l.

- As a formula:

$$v_l = \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k$$

$$+ f_l$$

larger $p \Rightarrow$ slower growth

$$v_l \approx V_l \equiv V_0 \left(\frac{2\beta_0}{p}\right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)}$$

- $l \geq L$

$$V_0 \equiv \sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left(\frac{p}{2\beta_0}\right)^k f_k$$

- For $l \geq L$, it suggests truncating on the f_k not the v_l.

Comparing Truncations

- Standard
$$\sum_{l=0}^{\infty} v_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} v_l \alpha_s^{l+1}$$

- arXiv:1701.00347 + arXiv:1712.04983:

$$\sum_{l=0}^{\infty} v_l \alpha_s^{l+1} \to \sum_{l=0}^{\infty} [v_l - V_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} V_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} [v_l - V_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} V_l \alpha_s^{l+1}$$

- This analysis:

$$\sum_{l=0}^{\infty} v_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} v_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} V_l \alpha_s^{l+1} \to \sum_{l=0}^{L-1} [v_l - V_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} V_l \alpha_s^{l+1}$$

Sequence of Power Corrections

Next Approximation

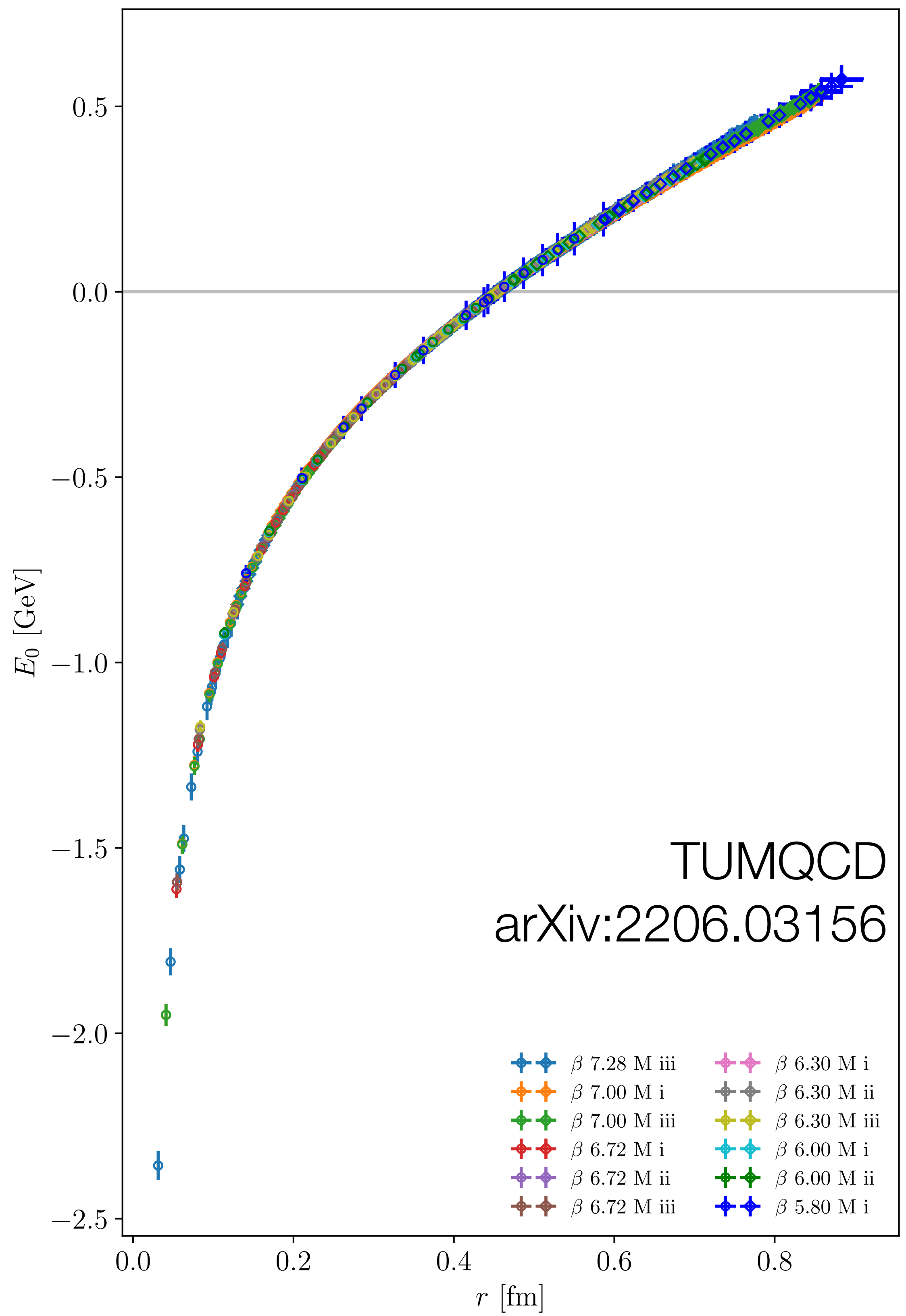
- If there is another power correction with $p_2 > p_1 = p$, then f_k will grow in a similar but slower fashion.

- Apply previous procedure with p_1; then repeat with p_2:

$$\begin{aligned} \boldsymbol{f}^{\{p_1,p_2\}} &\equiv \mathbf{Q}_{p_2} \cdot \mathbf{Q}_{p_1} \cdot \boldsymbol{v} \\ \Rightarrow \boldsymbol{v} &= \mathbf{Q}_{p_1}^{-1} \cdot \mathbf{Q}_{p_2}^{-1} \cdot \boldsymbol{f}^{\{p_1,p_2\}} \\ &= \left[\frac{p_2}{p_2 - p_1} \mathbf{Q}_{p_1}^{-1} + \frac{p_1}{p_1 - p_2} \mathbf{Q}_{p_2}^{-1} \right] \cdot \boldsymbol{f}^{\{p_1,p_2\}} \end{aligned}$$

- Extension to any sequence of higher powers by induction.

Scale Dependence



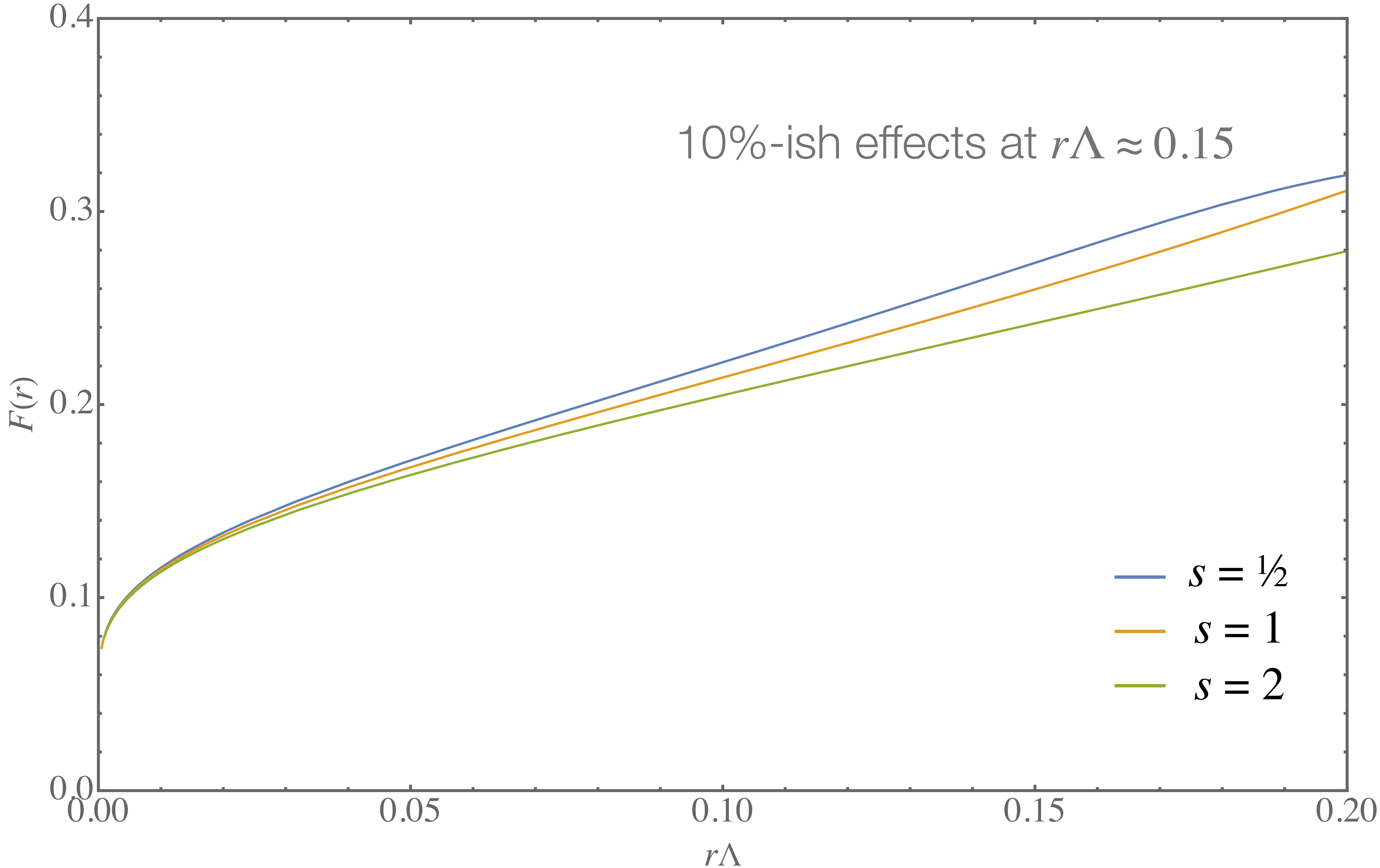
Static Energy

- The quantity we extract from oblong Wilson loops, aka "static potential" or "heavy-quark potential":
  - perturbative potential has IR divergences starting at 3 loops, compensated by multipole (retardation) term.

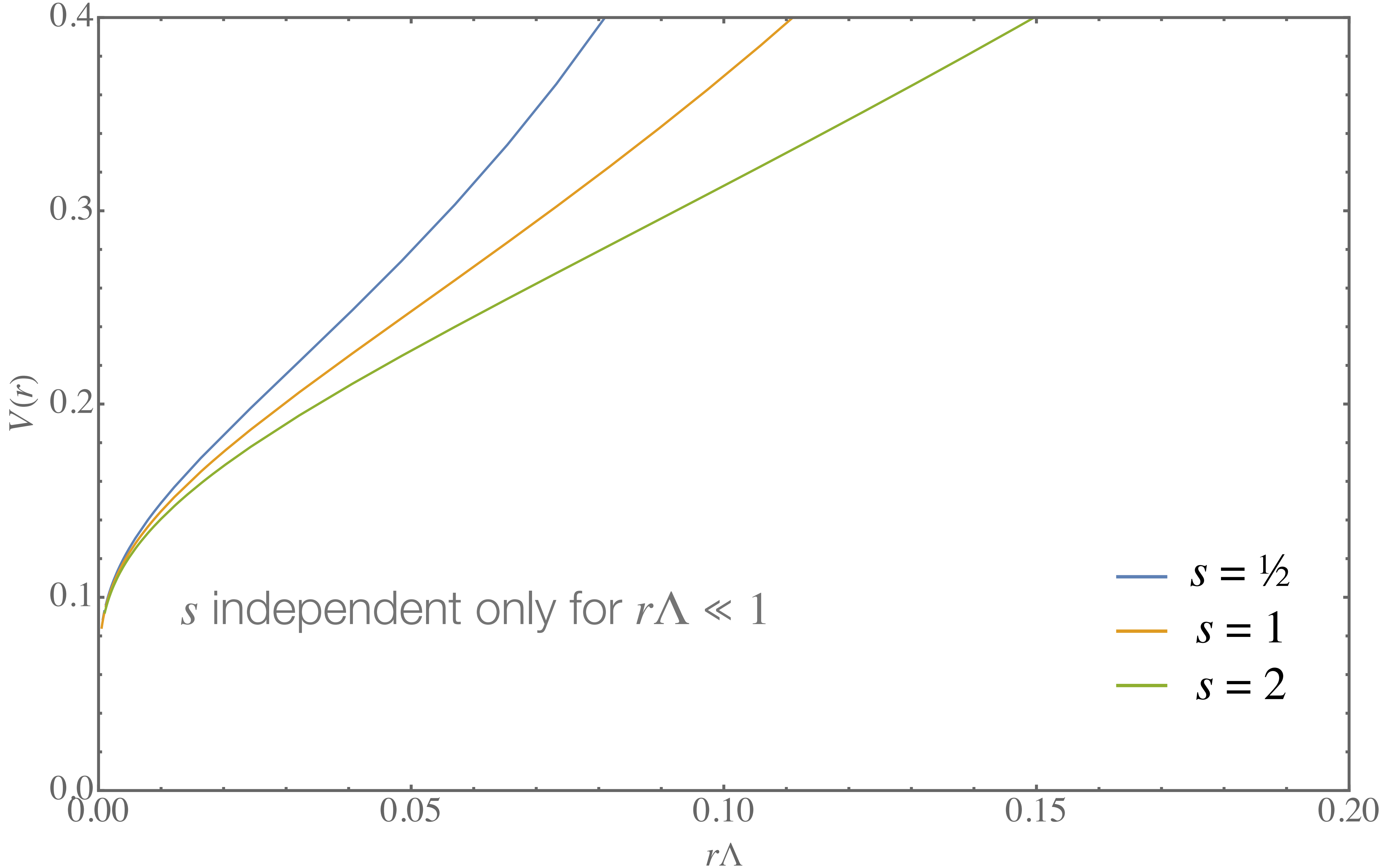
$$E_0(r) = -\frac{C_R}{r}\sum_{l=0} v_l(\mu r)\alpha_s(\mu)^{l+1} + \Lambda_0$$

  - In notation used above, $Q = 1/r$, $\mathscr{V}(1/r) = rE_0(r)$, $p = 1$.
  - Plots set $\mu = s/r$, and study r dependence for various s, using an algebra-simplifying scheme for .

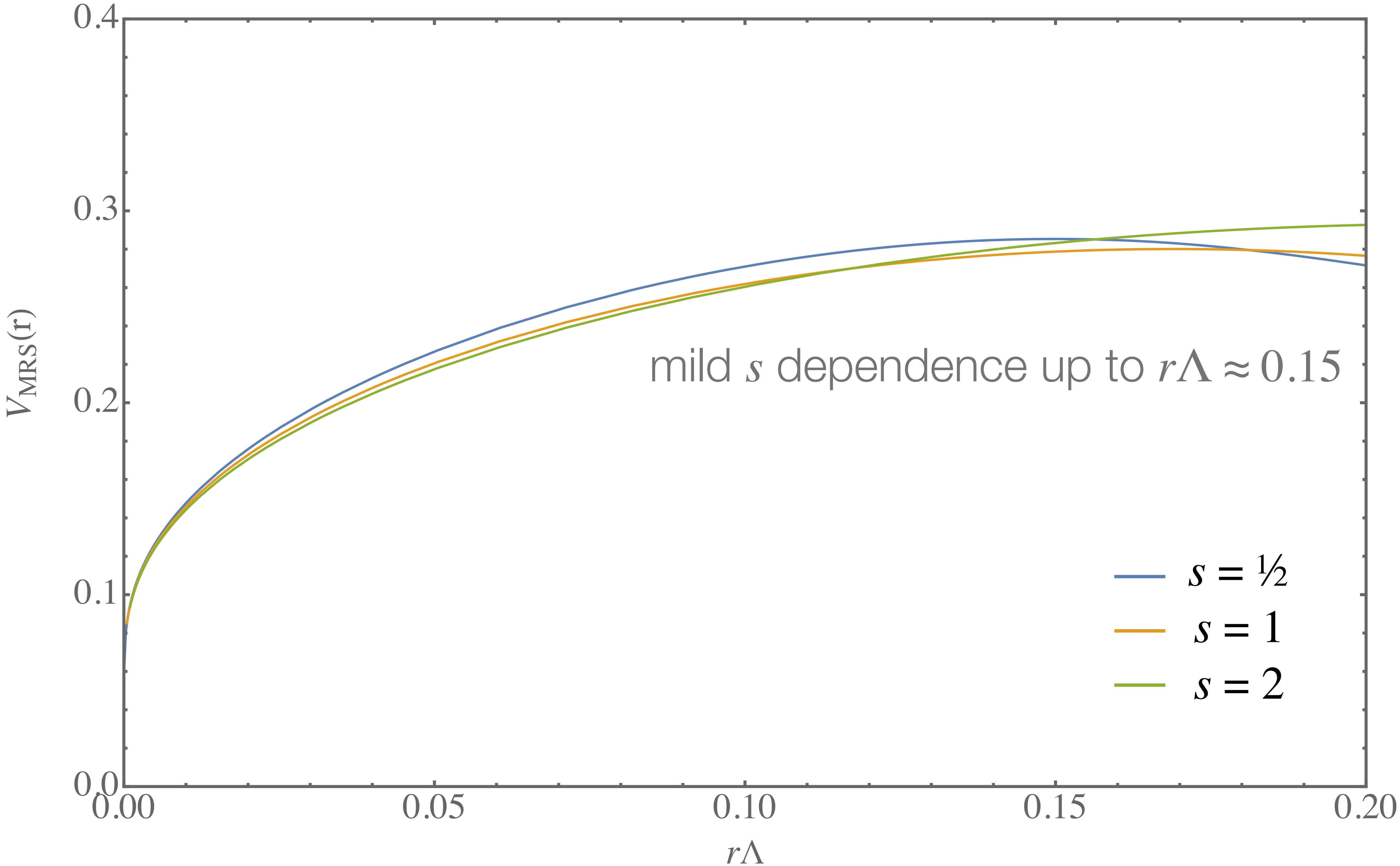
A good series (at most subleading growth)



The bad series, $rE_0(r)$



MRS applied to $rE_0(r)$



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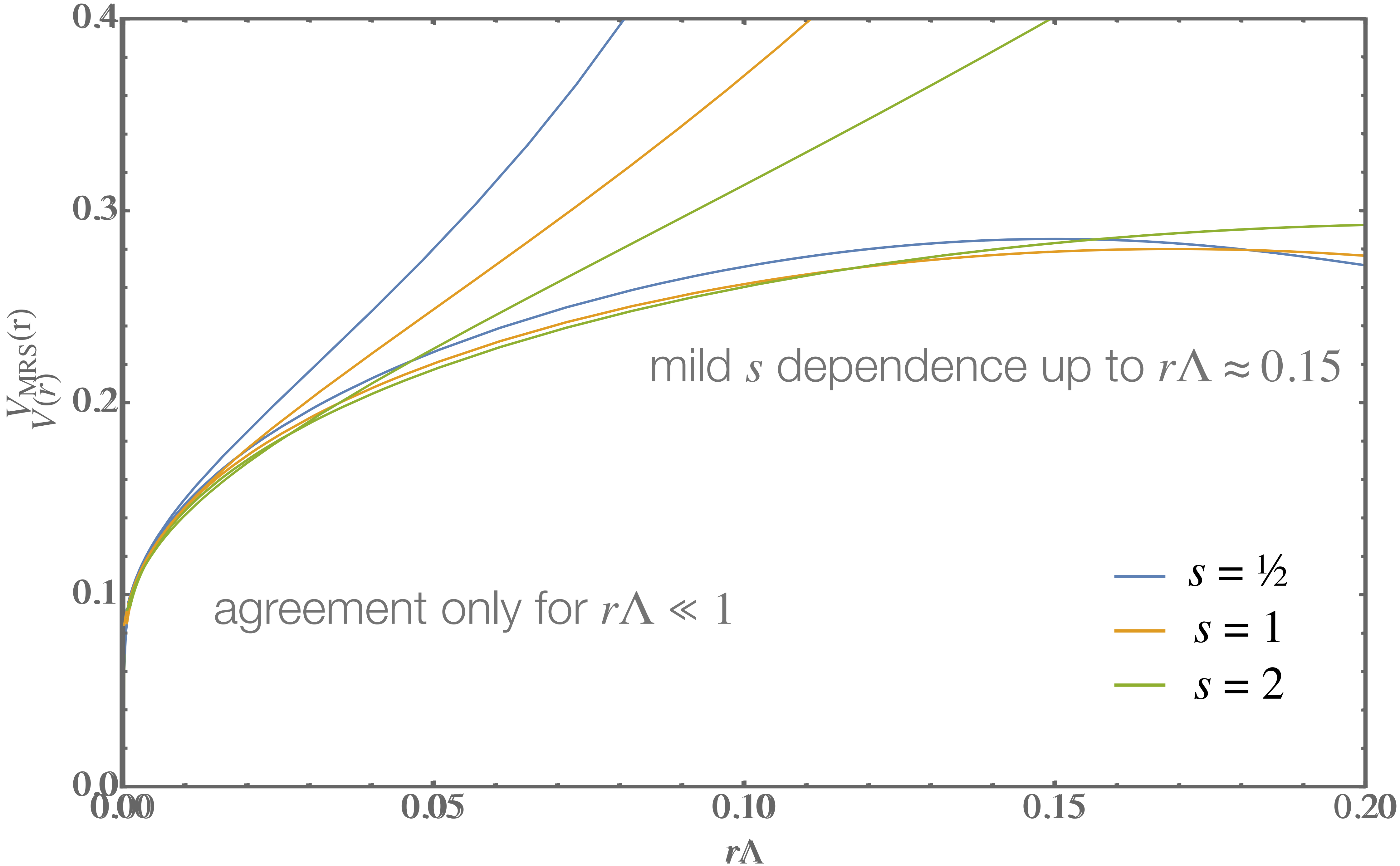
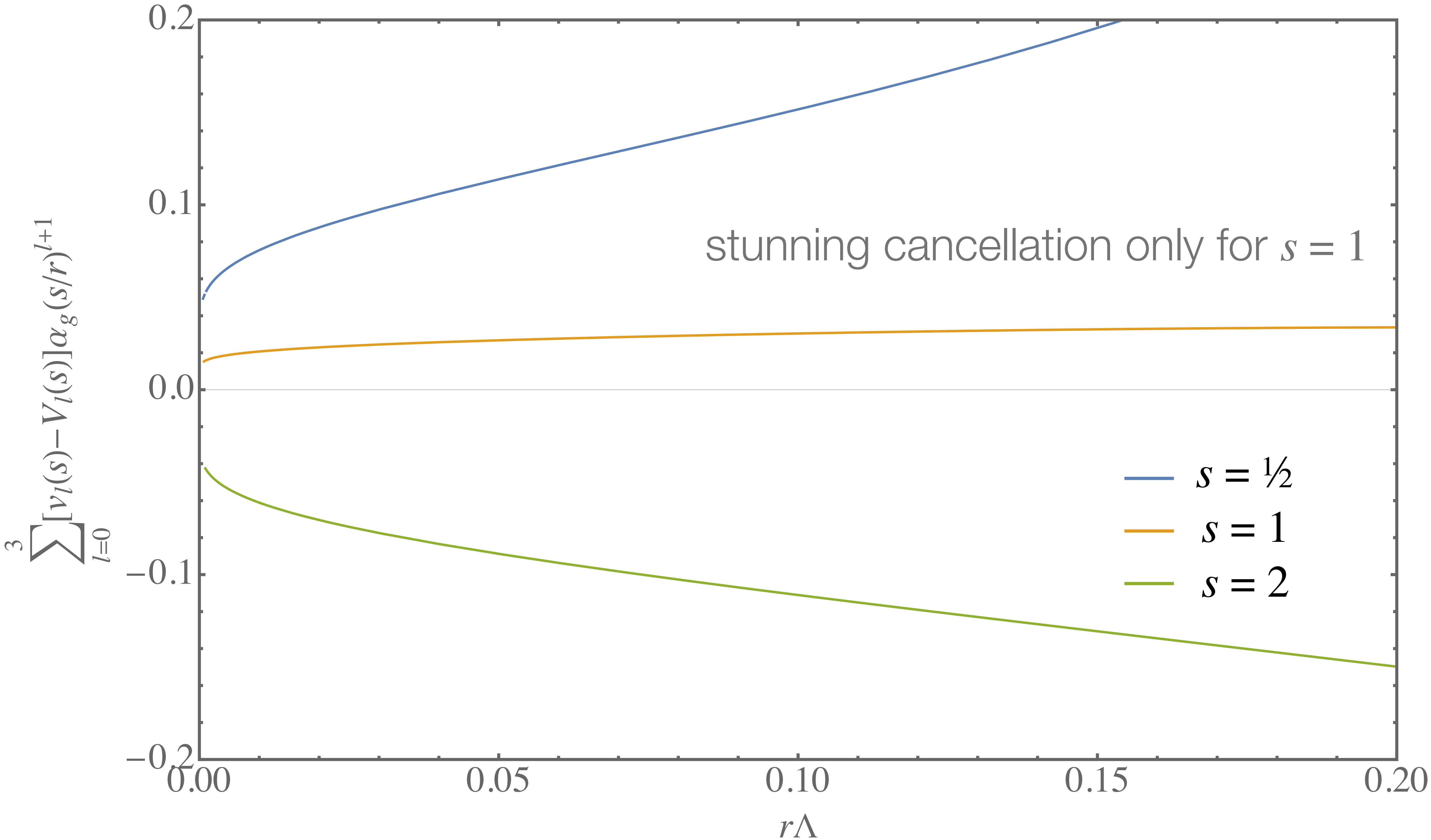
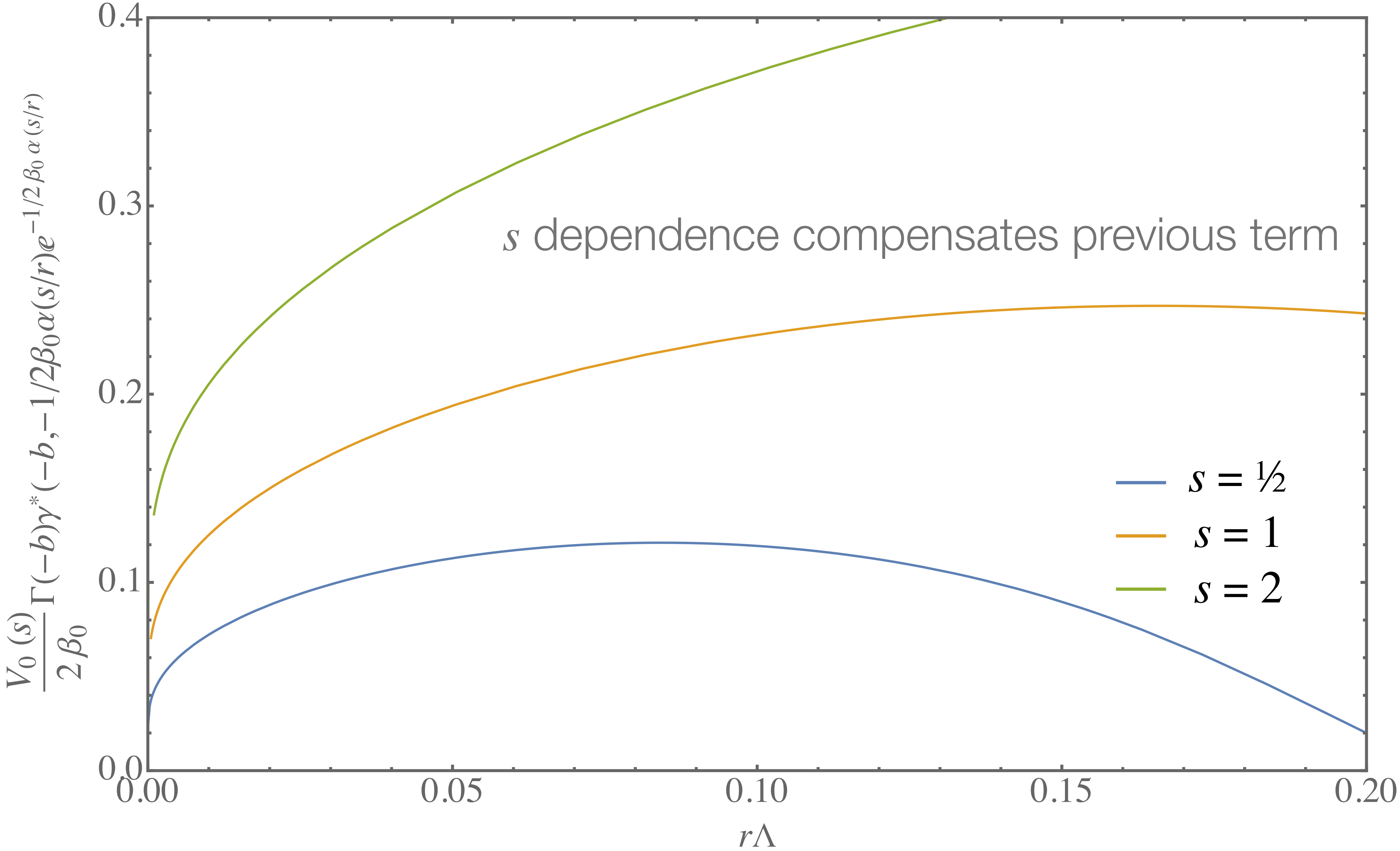


The subtracted series



The part that is a convergent series in $1/\alpha_s$



Summary

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- Minimal renormalon subtraction revisited:
  - formalism for any sequence of power corrections ↔ dominant, subdominant, sub-subdominant, … growth.
- Formulas for growth and normalization both follow from RGE and hold exactly at low orders.
- Cancellation scale dependent, but total is not.
- Scale dependence is mild.

Thank you for your attention

Questions?