



## More (on) Minimal Renormalon Subtraction

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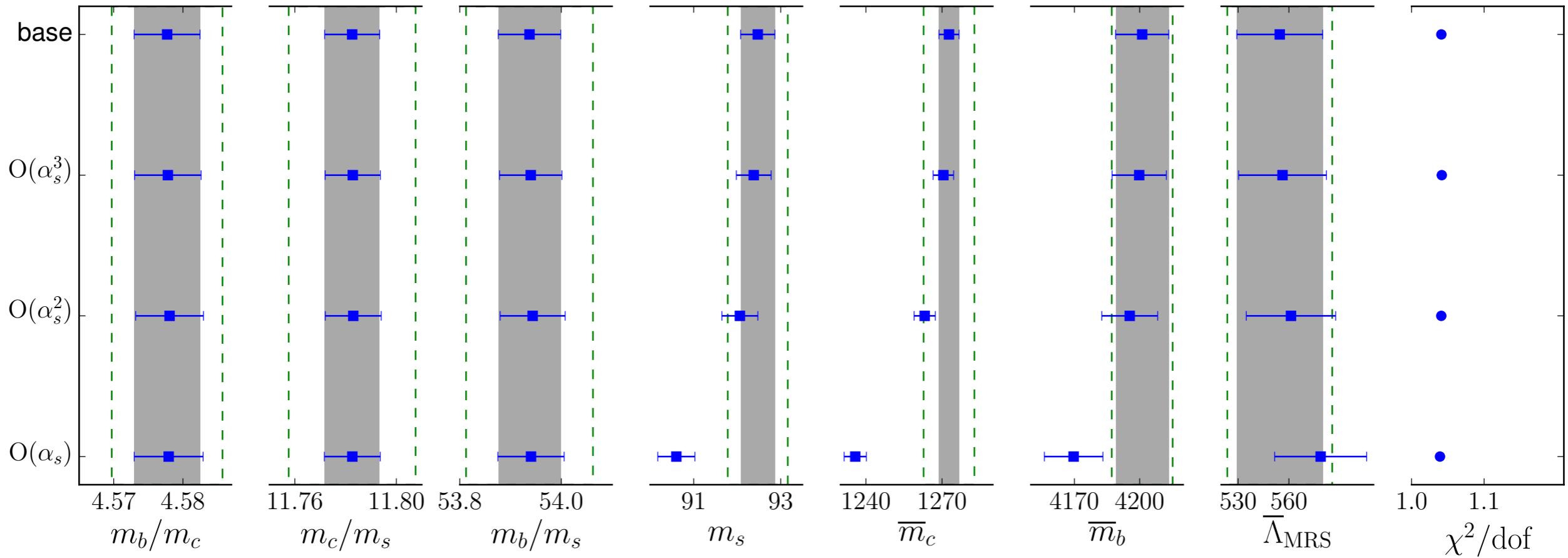
# No Perturbative Truncation Uncertainty!?

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- Quark masses in  $\overline{\text{MS}}$  scheme with small uncertainties:
  - total < 1% for bottom, charm, strange;
  - and 1–2% for up and down. [\[arXiv:1802.04248\]](#).
- Negligible uncertainty for truncating perturbation theory:
  - order  $\alpha_s^4$  “matching”, but still 😐;
  - could whatever wizardry was used be generalized?

# Perturbative Stability

Fermilab Lattice, MILC, & TUMQCD [[arXiv:1802.04248](https://arxiv.org/abs/1802.04248)]



meson mass      brown muck

$$M_B = m_b + \bar{\Lambda} + \mathcal{O}(1/m_b),$$

heavy quark “pole” mass

$m_{\overline{\text{MS}}}(\bar{m})$

$$m_b = \bar{m} \left( 1 + \sum_{l=0} \bar{r}_l \alpha_s^{l+1}(\bar{m}) \right).$$

$\{0.42, 1.03, 3.69, 17.4\}$

# Factorial Growth

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- The  $r_l$  grow factorially (known for a long time):

$$r_l \sim R_0 (2\beta_0)^n \frac{\Gamma(l+1+b)}{\Gamma(1+b)} \equiv R_l$$

for  $l \gg 1$ . Here  $b = \beta_1/2\beta_0^2 \stackrel{n_f=3}{=} 32/81 \approx 0.4$ .

- Komijani [[arXiv:1701.00347](https://arxiv.org/abs/1701.00347)] found an expression for renormalon strength:

$$R_0 = \sum_{k=0}^{\infty} (k+1) \frac{\Gamma(1+b)}{\Gamma(k+2+b)} (2\beta_0)^{-k} f_k$$



obtained from  $r_j, \beta_j, j \leq k$ ; see below.

# Minimal Renormalon Subtraction

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- Brambilla, Komijani, & ASK, Vairo [[arXiv:1712.04983](#)]  
proposed minimal renormalon subtracted (MRS) mass:

$$m_b = \bar{m} + \bar{m} \sum_{l=0}^{\infty} [r_l - R_l] \alpha_s^{l+1}(\bar{m}) + \bar{m} \sum_{l=0}^{\infty} R_l \alpha_s^{l+1}(\bar{m})$$

and, in practice, first sum is truncated at some order.

- Second sum can be carried out via Borel procedure,  
yielding two terms, one that has a convergent expansion  
in  $1/\alpha_s$ , the other can be absorbed into  $\bar{\Lambda}$ .
- Dubbed “minimal renormalon subtraction”.

# Perturbation Theory

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- First four  $r_l$  are known:
  - one loop [NPB 183 (1981) 384]:  $r_0 = 0.4244$
  - 2 loops [ZPC 48 (1990) 673]:  $r_1 = 1.0351$  ( $n_f = 3$ )
  - 3 loops [2+1 papers]:  $r_2 = 3.6932$  ( $n_f = 3$ )
  - 4 loops [arXiv:1606.06754]:  $r_3 = 17.4358$  ( $n_f = 3$ )
- 5-loop Callan-Symanzik  $\beta$  function is known [arXiv:1606.08659].
- 5-loop mass anomalous dimension is known [arXiv:1402.6611].

# Perturbation Theory

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- First four  $r_l$  are known:  $R_n$
- one loop [NPB 183 (1981) 384]:  $r_0 = 0.4244 \quad 0.5350$
- 2 loops [ZPC 48 (1990) 673]:  $r_1 = 1.0351 \quad 1.0691 \quad (n_f = 3)$
- 3 loops [2+1 papers]:  $r_2 = 3.6932 \quad 3.5966 \quad (n_f = 3)$
- 4 loops [arXiv:1606.06754]:  $r_3 = 17.4358 \quad 17.4195 \quad (n_f = 3)$
- 5-loop Callan-Symanzik  $\beta$  function is known [arXiv:1606.08659].
- 5-loop mass anomalous dimension is known [arXiv:1402.6611].

# Talk TOC

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- Motivation
- Generalizations:
  - $\Lambda^p$  instead of single power  $\bar{\Lambda}$ ;
  - subtract subleading factorial growth, i.e., series with more than one renormalon;
  - scale dependence  $\alpha_s(Q) \rightarrow \alpha_s(sQ)$ :
    - static energy, “same” renormalon as  $m_b$ , Wilson line.

# Higher-power Power Corrections

# Notation & Setup

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- Consider (dimensionless)

$$\mathcal{V}(Q) = v_{-1} + V(Q) + C_p \frac{\Lambda^p}{Q^p}, \quad V(Q) = \sum_{l=0} v_l(\mu/Q) \alpha_s(\mu)^{l+1},$$

- RGE: coefficients'  $\mu$  dependence must cancel that of  $\alpha_s$ .
- $\therefore$  RGE sets  $Q$  dependence of  $V(Q)$ .

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tree level

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tree level      “perturbative” part

The diagram illustrates the decomposition of the physical quantity  $\mathcal{V}(Q)$ . It is shown as a sum of three terms: a constant  $v_{-1}$ , a function  $V(Q)$ , and a term involving the coupling constant  $C_p$  and the scale  $\Lambda$ . The term  $V(Q)$  is further expanded as a series of perturbative corrections  $\sum_{l=0} \nu_l(\mu/Q) \alpha_s(\mu)^{l+1}$ . A green speech bubble labeled "physical quantity" encloses the entire expression. A black bracket labeled "tree level" points to the  $v_{-1}$  term. A blue bracket labeled "\"perturbative\" part" points to the  $V(Q)$  term.

- RGE: coefficients'  $\mu$  dependence must cancel that of  $\alpha_s$ .
- ∴ RGE sets  $Q$  dependence of  $V(Q)$ .

# Notation & Setup

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$$\mathcal{V}(Q) = v_{-1} + V(Q) + C_p \frac{\Lambda^p}{Q^p}, \quad V(Q) = \sum_{l=0} v_l(\mu/Q) \alpha_s(\mu)^{l+1},$$

tree level      “perturbative” part      perturbative series

- RGE: coefficients'  $\mu$  dependence must cancel that of  $\alpha_s$ .
- ∴ RGE sets  $Q$  dependence of  $V(Q)$ .

# Notation & Setup

- Consider (dimensionless)

$$\mathcal{V}(Q) = v_{-1} + V(Q) + C_p \frac{\Lambda^p}{Q^p}, \quad V(Q) = \sum_{l=0} v_l(\mu/Q) \alpha_s(\mu)^{l+1},$$

**physical quantity**

**power correction**

**tree level**

**“perturbative” part**

**perturbative series**

The diagram illustrates the decomposition of a physical quantity  $\mathcal{V}(Q)$  into its components. A large orange bracket labeled "physical quantity" covers the entire expression. A black bracket labeled "tree level" covers the term  $v_{-1}$ . A blue bracket labeled "‘perturbative’ part" covers the term  $V(Q)$ . A red bracket labeled "power correction" covers the term  $C_p \frac{\Lambda^p}{Q^p}$ . A yellow bracket labeled "perturbative series" covers the term  $\sum_{l=0} v_l(\mu/Q) \alpha_s(\mu)^{l+1}$ .

- RGE: coefficients'  $\mu$  dependence must cancel that of  $\alpha_s$ .
- ∴ RGE sets  $Q$  dependence of  $V(Q)$ .

# Notation & Setup

- Consider (dimensionless)

$$\mathcal{V}(Q) = v_{-1} + V(Q) + C_p \frac{\Lambda^p}{Q^p}, \quad V(Q) = \sum_{l=0} \nu_l(\mu/Q) \alpha_s(\mu)^{l+1},$$

physical quantity

power correction

tree level

“perturbative” part

perturbative series

power  $p \leftrightarrow$  factorial growth

The diagram illustrates the decomposition of a physical quantity  $\mathcal{V}(Q)$  into its components. It shows  $\mathcal{V}(Q) = v_{-1} + V(Q) + C_p \frac{\Lambda^p}{Q^p}$ , where  $V(Q) = \sum_{l=0} \nu_l(\mu/Q) \alpha_s(\mu)^{l+1}$ . The term  $C_p \frac{\Lambda^p}{Q^p}$  is labeled as a "power correction". The term  $\sum_{l=0} \nu_l(\mu/Q) \alpha_s(\mu)^{l+1}$  is labeled as a "perturbative series". A pink arrow points from the "power correction" term to the  $\Lambda^p$  term, indicating they are equivalent. A yellow arrow points from the "perturbative series" term to the  $l=0$  term, indicating they are equivalent. A green box labeled "physical quantity" encloses the entire expression  $\mathcal{V}(Q)$ . A blue box labeled "tree level" encloses the term  $v_{-1}$ . A red box labeled "‘perturbative’ part" encloses the term  $V(Q)$ . A purple box labeled "power correction" encloses the term  $C_p \frac{\Lambda^p}{Q^p}$ . A pink box labeled "power  $p \leftrightarrow$  factorial growth" encloses the  $\Lambda^p$  term. A yellow box labeled "perturbative series" encloses the term  $\sum_{l=0} \nu_l(\mu/Q) \alpha_s(\mu)^{l+1}$ .

- RGE: coefficients'  $\mu$  dependence must cancel that of  $\alpha_s$ .
- $\therefore$  RGE sets  $Q$  dependence of  $V(Q)$ .

# Renormalon Removal

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- Define  $v_{-1} + F(Q) \equiv \frac{1}{pQ^{p-1}} \frac{dQ^p \gamma}{dQ} \equiv \hat{Q}_p \gamma$
- This operation eliminates  $\Lambda^p / Q^p$ .
- As a series  $F(Q) = \sum_{k=0}^{\infty} f_k \alpha_s^{k+1}$ .  
$$f_k = v_k - \frac{2}{p} \sum_{j=0}^{k-1} (j+1) \beta_{k-1-j} v_j$$
- Differential equation  $V(\alpha) + \frac{2}{p} \beta(\alpha) V'(\alpha) = F(\alpha)$ , which Komijani solves via developed asymptotic expansions.
- NB: solution of homogeneous equation is  $C_p \Lambda^p$ .

# My Solution

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- The relation between the coefficients is a matrix equation

$$f_k = v_k - \frac{2}{p} \sum_{j=0}^{k-1} (j+1) \beta_{k-1-j} v_j$$

$$\mathbf{f} = \left[ 1 - \frac{2}{p} \mathbf{D} \right] \cdot \mathbf{v} \equiv \mathbf{Q}_p \cdot \mathbf{v}$$

and  $\mathbf{D}$  is on the lower triangle.

- Although the “matrix” is infinite, the triangular form makes an iterative solution straightforward.

- Let  $\mathbf{P}_p = \text{diag}[1, p/2\beta_0, (p/2\beta_0)^2, (p/2\beta_0)^3, \dots]$ .
- Then  $\mathbf{Q}_p = \mathbf{P}_p^{-1} \cdot \mathbf{B}_p \cdot \mathbf{P}_p$

$$\mathbf{B}_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -pb & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ -(pb)^2 & -2pb & -3 & 1 & 0 & 0 & 0 & \cdots \\ -(pb)^3 & -2(pb)^2 & -3pb & -4 & 1 & 0 & 0 & \cdots \\ -(pb)^4 & -2(pb)^3 & -3(pb)^2 & -4pb & -5 & 1 & 0 & \cdots \\ -(pb)^5 & -2(pb)^4 & -3(pb)^3 & -4(pb)^2 & -5pb & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$b = \beta_1 / 2\beta_0^2 \stackrel{n_f=3}{=} 32/81 \approx 0.4$$

- (a certain scheme for  $\alpha_s$  is chosen to simplify algebra)

- Inverse reveals that factorial growth begins at low orders:

$$\mathbf{B}_p^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{\Gamma(3+pb)}{\Gamma(2+pb)} & 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{\Gamma(4+pb)}{\Gamma(2+pb)} & 2\frac{\Gamma(4+pb)}{\Gamma(3+pb)} & 3 & 1 & 0 & 0 & 0 & \dots \\ \frac{\Gamma(5+pb)}{\Gamma(2+pb)} & 2\frac{\Gamma(5+pb)}{\Gamma(3+pb)} & 3\frac{\Gamma(5+pb)}{\Gamma(4+pb)} & 4 & 1 & 0 & 0 & \dots \\ \frac{\Gamma(6+pb)}{\Gamma(2+pb)} & 2\frac{\Gamma(6+pb)}{\Gamma(3+pb)} & 3\frac{\Gamma(6+pb)}{\Gamma(4+pb)} & 4\frac{\Gamma(6+pb)}{\Gamma(5+pb)} & 5 & 1 & 0 & \dots \\ \frac{\Gamma(7+pb)}{\Gamma(2+pb)} & 2\frac{\Gamma(7+pb)}{\Gamma(3+pb)} & 3\frac{\Gamma(7+pb)}{\Gamma(4+pb)} & 4\frac{\Gamma(7+pb)}{\Gamma(5+pb)} & 5\frac{\Gamma(7+pb)}{\Gamma(6+pb)} & 6 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

- As a formula:

$$\nu_l = \left( \frac{2\beta_0}{p} \right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left( \frac{p}{2\beta_0} \right)^k f_k + f_l$$

larger  $p \Rightarrow$  slower growth

namely, the well-known factorial growth, a sum similar to Komijani's renormalon strength factor, and an extra term.

- Suppose we have  $L$  orders. For  $l < L$ , this equation returns the  $\nu_l$  we started with.
- For  $l \geq L$ , it suggests truncating on the  $f_k$  not the  $\nu_l$ .

- As a formula:

$$v_l = \left( \frac{2\beta_0}{p} \right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \sum_{k=0}^{l-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left( \frac{p}{2\beta_0} \right)^k f_k$$

+  $f_l$

larger  $p \Rightarrow$  slower growth

$$v_l \approx V_l \equiv V_0 \left( \frac{2\beta_0}{p} \right)^l \frac{\Gamma(l+1+pb)}{\Gamma(1+pb)} \quad \cdot \quad l \geq L$$

$$V_0 \equiv \sum_{k=0}^{L-1} (k+1) \frac{\Gamma(1+pb)}{\Gamma(k+2+pb)} \left( \frac{p}{2\beta_0} \right)^k f_k$$

- For  $l \geq L$ , it suggests truncating on the  $f_k$  not the  $v_l$ .

# Comparing Truncations

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- Standard

$$\sum_{l=0}^{\infty} v_l \alpha_s^{l+1} \rightarrow \sum_{l=0}^{L-1} v_l \alpha_s^{l+1}$$

- arXiv:1701.00347 + arXiv:1712.04983:

$$\sum_{l=0}^{\infty} v_l \alpha_s^{l+1} \rightarrow \sum_{l=0}^{\infty} [v_l - V_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} V_l \alpha_s^{l+1} \rightarrow \sum_{l=0}^{L-1} [v_l - V_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} V_l \alpha_s^{l+1}$$

- This analysis:

$$\sum_{l=0}^{\infty} v_l \alpha_s^{l+1} \rightarrow \sum_{l=0}^{L-1} v_l \alpha_s^{l+1} + \sum_{l=L}^{\infty} V_l \alpha_s^{l+1} \rightarrow \sum_{l=0}^{L-1} [v_l - V_l] \alpha_s^{l+1} + \sum_{l=0}^{\infty} V_l \alpha_s^{l+1}$$

# Sequence of Power Corrections

# Next Approximation

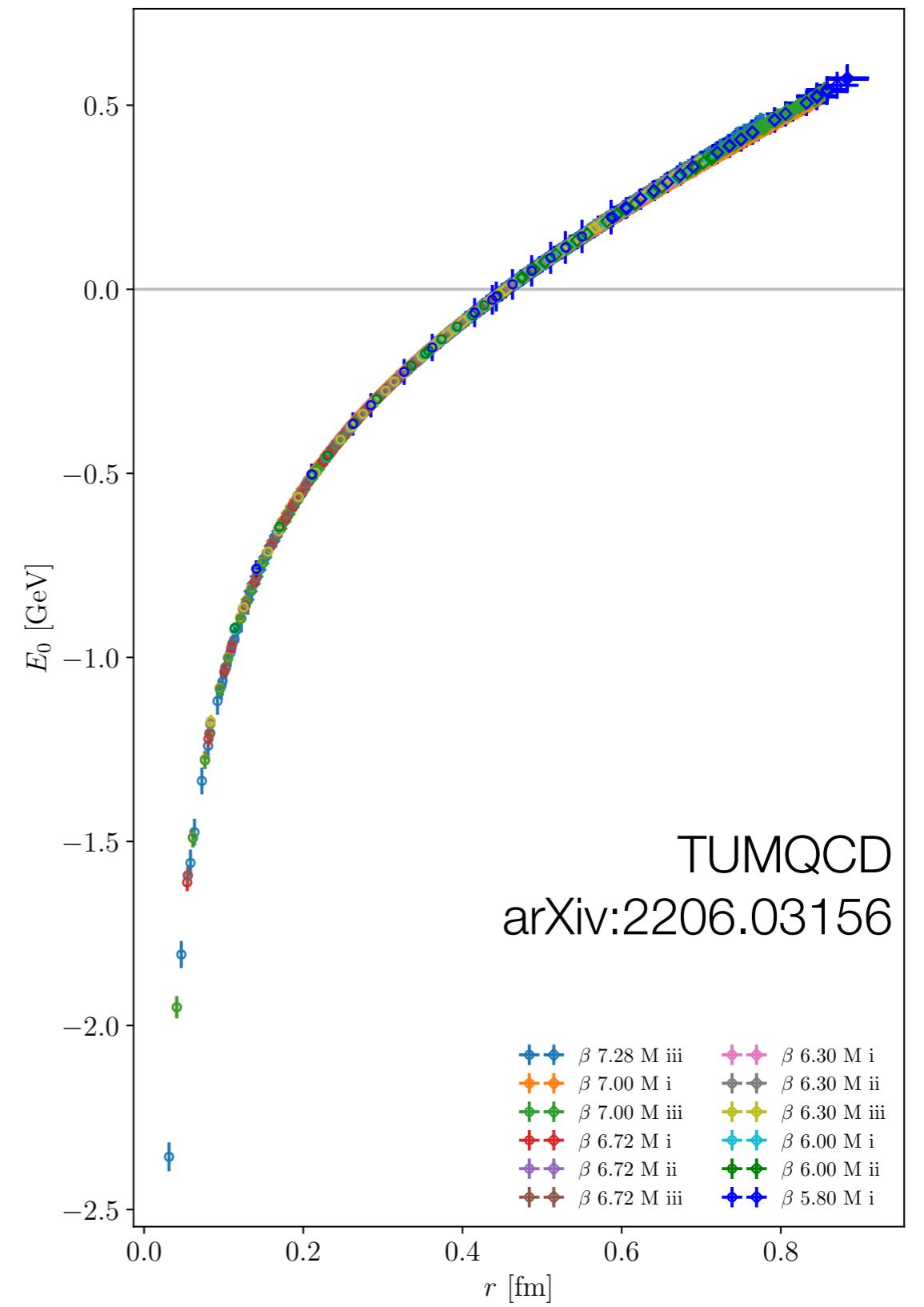
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- If there is another power correction with  $p_2 > p_1 = p$ , then  $f_k$  will grow in a **similar** but **slower** fashion.
- Apply previous procedure with  $p_1$ ; then repeat with  $p_2$ :

$$\begin{aligned} f^{\{p_1, p_2\}} &\equiv \mathbf{Q}_{p_2} \cdot \mathbf{Q}_{p_1} \cdot \nu \\ \Rightarrow \nu &= \mathbf{Q}_{p_1}^{-1} \cdot \mathbf{Q}_{p_2}^{-1} \cdot f^{\{p_1, p_2\}} \\ &= \left[ \frac{p_2}{p_2 - p_1} \mathbf{Q}_{p_1}^{-1} + \frac{p_1}{p_1 - p_2} \mathbf{Q}_{p_2}^{-1} \right] \cdot f^{\{p_1, p_2\}} \end{aligned}$$

- Extension to any sequence of higher powers by induction.

# Scale Dependence



# Static Energy

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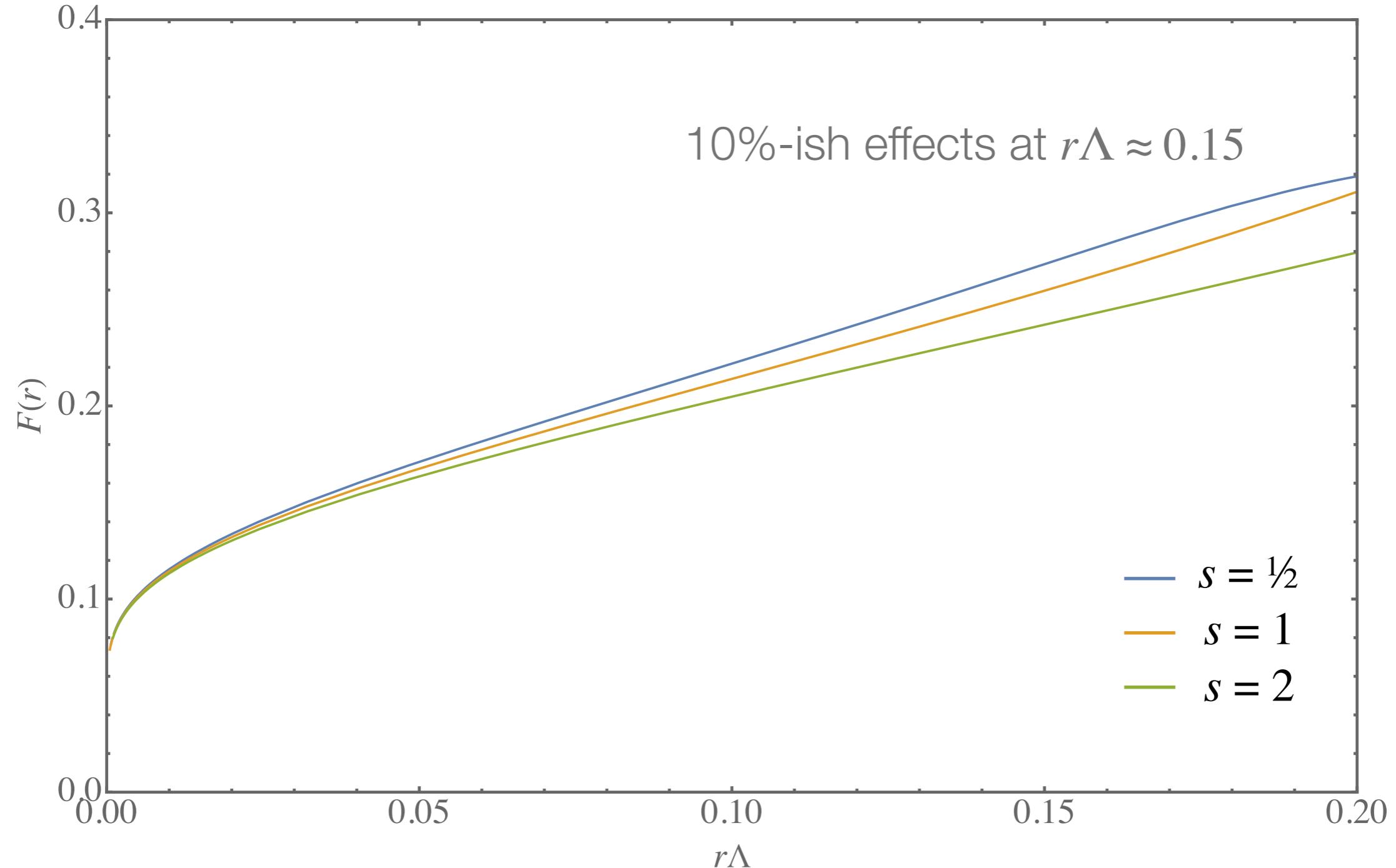
- The quantity we extract from oblong Wilson loops, aka “static potential” or “heavy-quark potential”:
  - perturbative potential has IR divergences starting at 3 loops, compensated by multipole (retardation) term.

$$E_0(r) = -\frac{C_R}{r} \sum_{l=0} \nu_l(\mu r) \alpha_s(\mu)^{l+1} + \Lambda_0$$

- In notation used above,  $Q = 1/r$ ,  $\mathcal{V}(1/r) = rE_0(r)$ ,  $p = 1$ .
- Plots set  $\mu = s/r$ , and study  $r$  dependence for various  $s$ , using an algebra-simplifying scheme for .

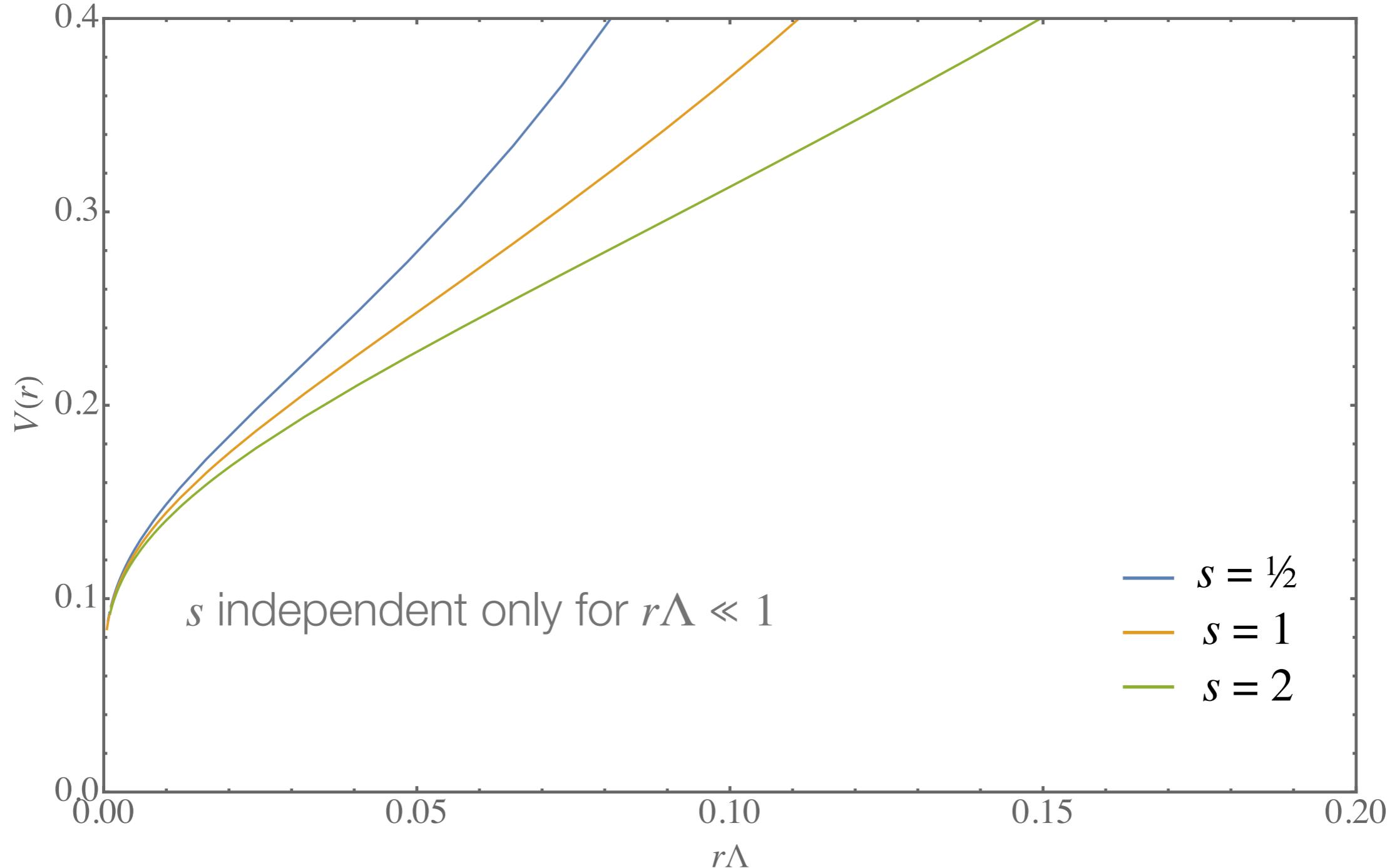
# A good series (at most subleading growth)

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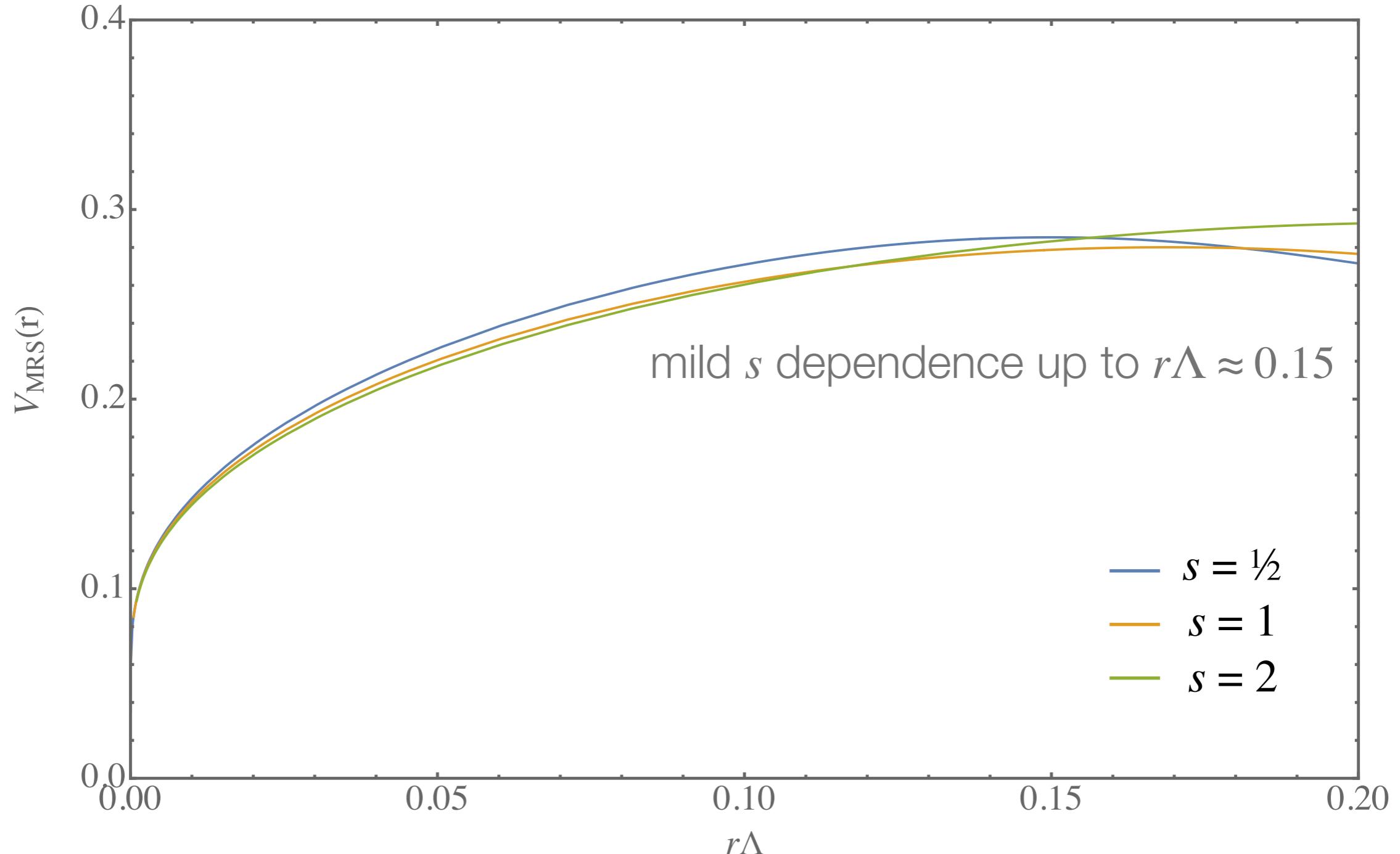
# The bad series, $rE_0(r)$

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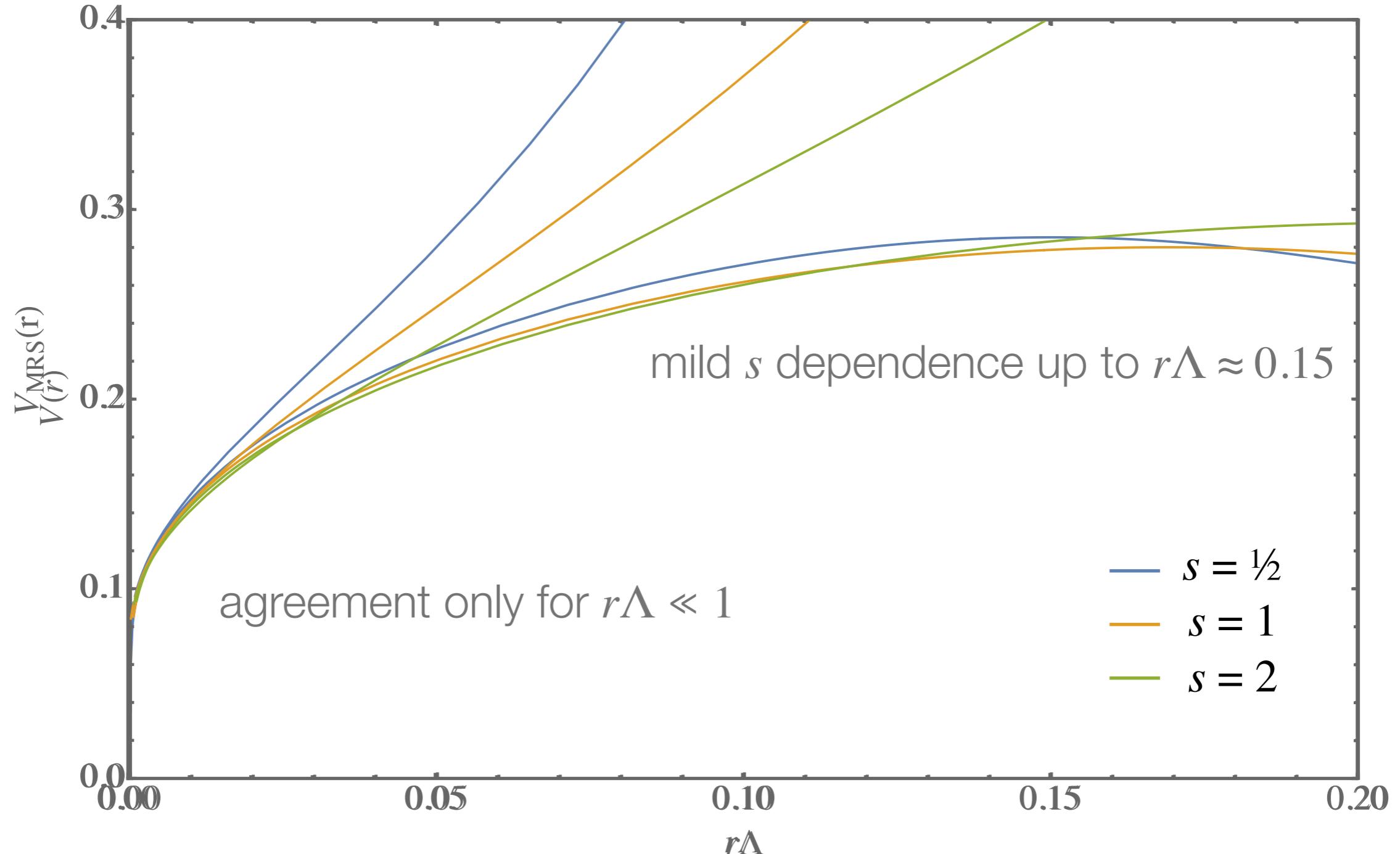
# MRS applied to $rE_0(r)$

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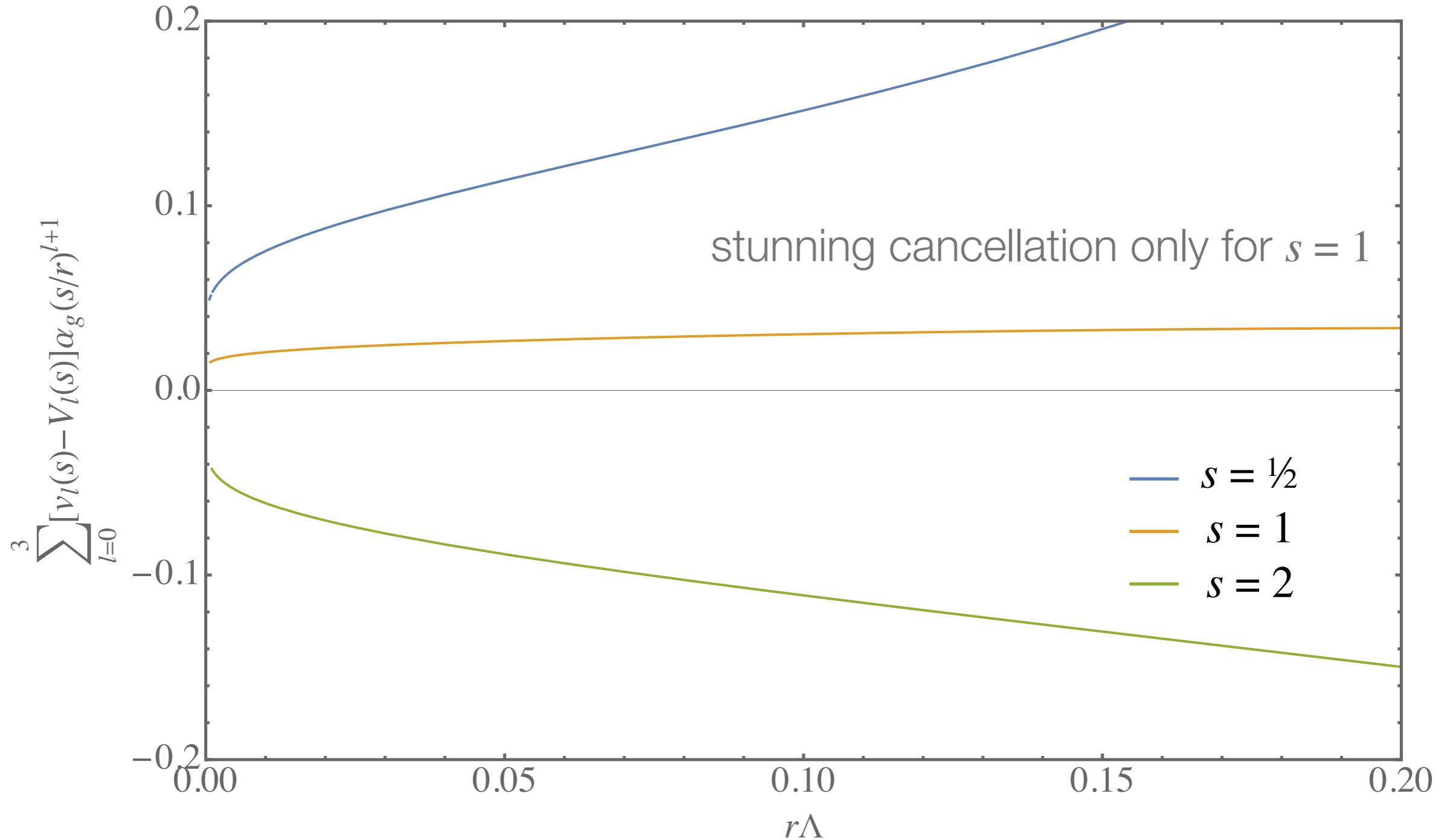
# MRS applied to $rE_0(r)$

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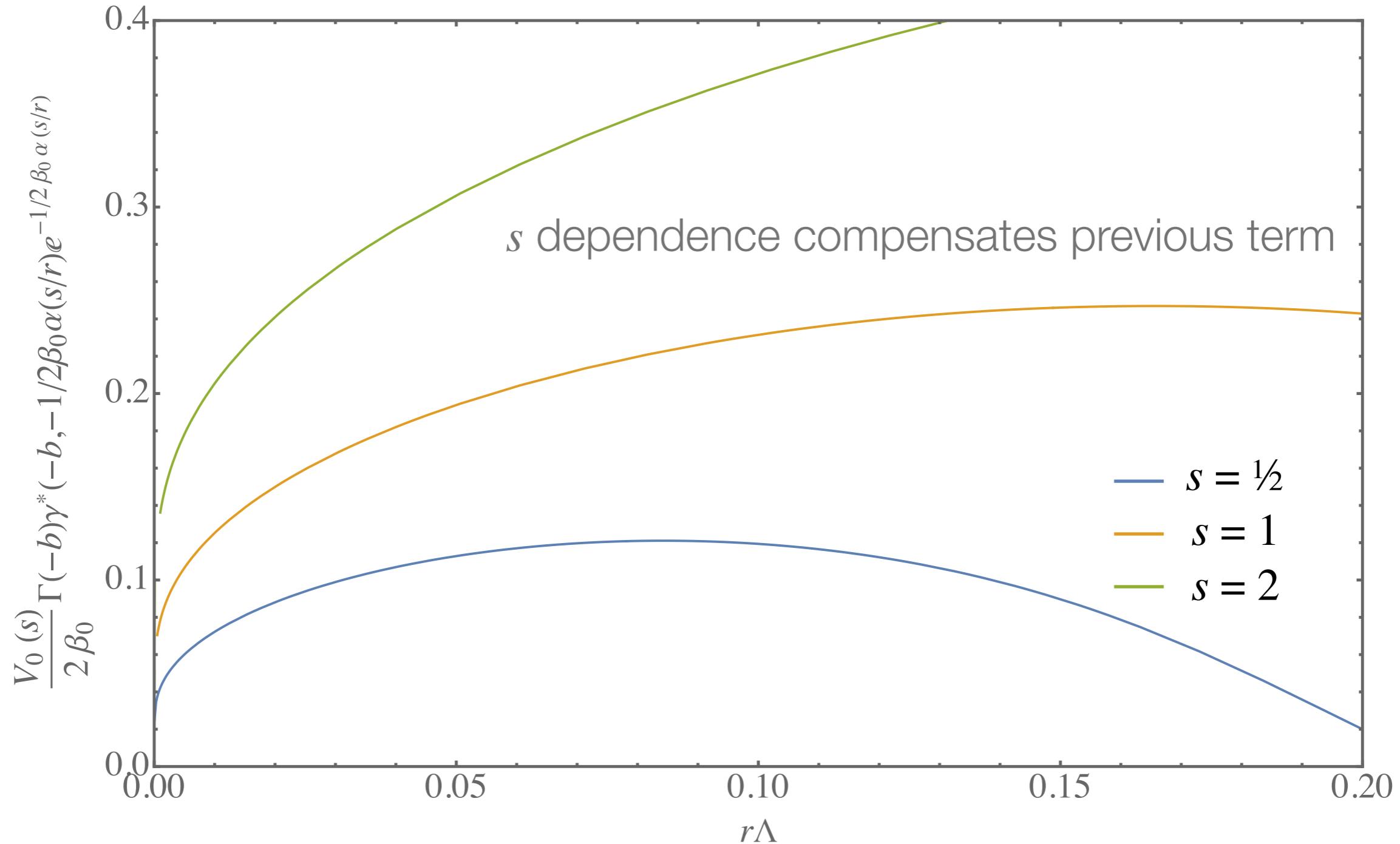
# The subtracted series

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The part that is a convergent series in  $1/\alpha_s$

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# Summary

# Summary

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- Minimal renormalon subtraction revisited:
  - formalism for any sequence of power corrections  $\leftrightarrow$  dominant, subdominant, sub-subdominant, ... growth.
- Formulas for growth and normalization both follow from RGE and hold exactly at low orders.
- Cancellation scale dependent, but total is not.
- Scale dependence is mild.

Thank you for your attention

**Questions?**