# Structure-dependent form factors in radiative leptonic decays of the $D_s$ meson with Domain Wall fermions


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2023 LATTICE

The 40th International Symposium on Lattice Field Theory

Fermilab

31st Jul – 4th Aug 2023

## OUTLINE

- Motivations
- Leptonic decays of pseudoscalar mesons

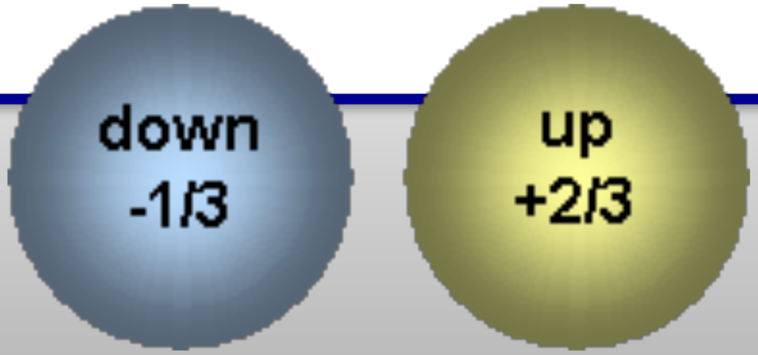
  $H \to \ell \nu_\ell \gamma$
- Outlook


In collaboration with

C. F. Kane, C. Lehner, S. Meinel and A. Soni

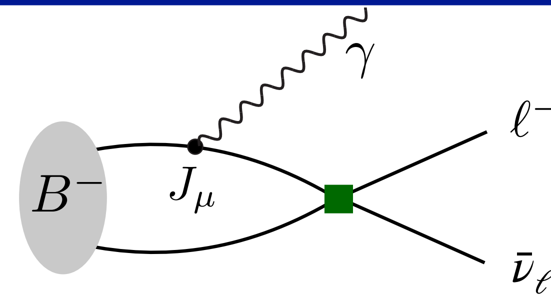

(mainly based on: arXiv:2302.01298 published this year in PRD)

# Phenomenological motivations



# Radiative corrections to leptonic B-meson decays

## $B^- \to \ell^- \bar{\nu}_\ell \gamma$




- The emission of a real hard photon removes the $(m_\ell / M_B)^2$ helicity suppression
- This is the simplest process that probes (for large $E_\gamma$) the first inverse moment of the B-meson LCDA

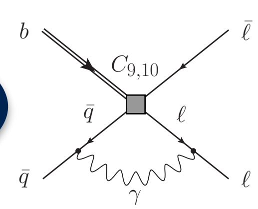
$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega}{\omega} \Phi_{B+}(\omega, \mu)$$

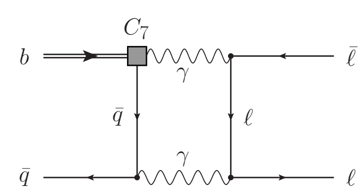
$\lambda_B$ is an important input in QCD-factorization predictions for non-leptonic B decays but is poorly known

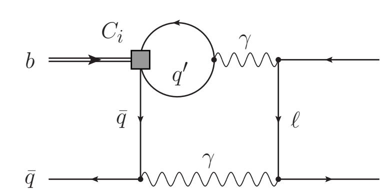
M. Beneke, V. M. Braun, Y. Ji, Y.-B. Wei, 2018

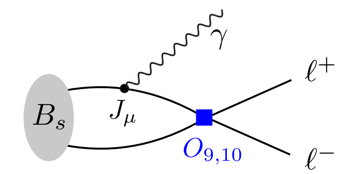
- Belle 2018: $\mathscr{B}(B^- \to \ell^- \bar{\nu}_\ell \gamma, E_\gamma > 1 \text{ GeV}) < 3.0 \cdot 10^{-6}$ $\longrightarrow$ $\lambda_B > 0.24$ GeV
- QCD sum rules in HQET: $\lambda_B(1 \text{ GeV}) = 0.46\,(11)$ GeV

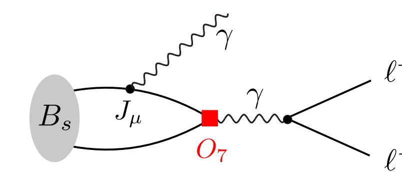
## $B_q \to \ell^+ \ell^- (\gamma)$




- Enhancement of the virtual corrections by a factor $M_B / \Lambda_{QCD}$ and by large logarithms

M. Beneke, C. Bobeth, R. Szafron, 2019

- The real photon emission process is a clean probe of NP: sensitiveness to $C_9$, $C_{10}$, $C_7$

# Lattice calculation of

# $H \to \ell \nu_\ell \gamma$

PHYSICAL REVIEW D **107,** 074507 (2023)

arXiv:2302.01298

**Methods for high-precision determinations of radiative-leptonic decay form factors using lattice QCD**

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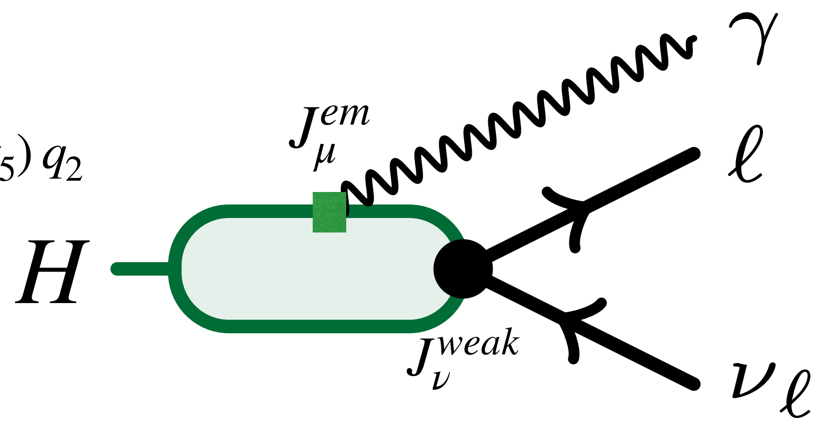
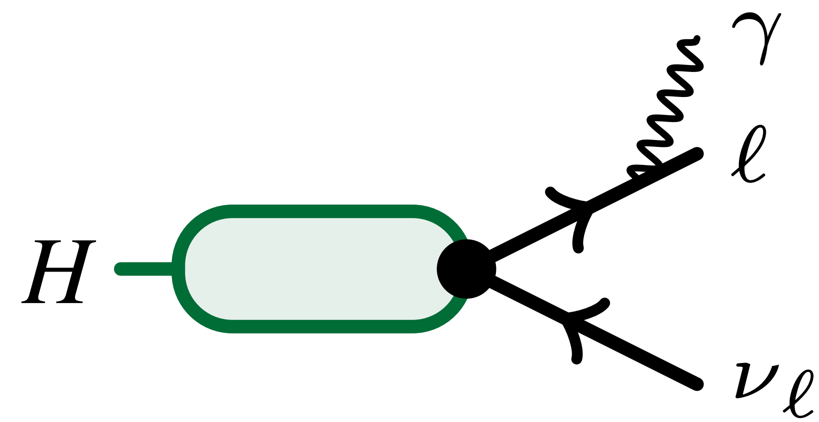
(Received 9 February 2023; accepted 21 March 2023; published 19 April 2023)

# Hadronic tensor and form factors

$$J_\mu^{em} = \sum_q Q_q \bar{q}\gamma_\mu q$$

$$J_\nu^{weak} = \bar{q}_1\gamma_\nu(1-\gamma_5)\,q_2$$




$$T_{\mu\nu} = -i\int d^4x\; e^{ip_\gamma\cdot x}\,\langle 0|\,\mathbf{T}\left(J_\mu^{em}(x)\,J_\nu^{weak}(0)\right)|H(\vec{p}_H)\rangle \qquad (p_H = m_H v)$$

$$= \varepsilon_{\mu\nu\tau\rho}p_\gamma^\tau v^\rho F_V + i\left[-g_{\mu\nu}(p_\gamma\cdot v) + v_\mu(p_\gamma)_\nu\right]F_A - i\frac{v_\mu v_\nu}{p_\gamma\cdot v}m_H f_H + (p_\gamma)_\mu - \text{terms}$$

$$F_A = F_{A,SD} + (-Q_\ell f_H/E_\gamma^{(0)}), \qquad E_\gamma^{(0)} = p_\gamma\cdot v$$

Goal: Calculate $F_V, F_{A,SD}$ as a function of $E_\gamma^{(0)}$

$$\phi_H^\dagger = -\bar{q}_2\gamma_5 q_1$$

$$C_{3,\mu\nu}(t_{em}, t_H) = \int d^3x \int d^3y\; e^{-i\vec{\mathbf{p}}_\gamma\cdot\vec{\mathbf{x}}}e^{i\vec{\mathbf{p}}_H\cdot\vec{\mathbf{y}}}\langle J_\mu^{em}(t_{em},\vec{\mathbf{x}})J_\nu^{weak}(0)\phi_H^\dagger(t_H,\vec{\mathbf{y}})\rangle$$

- safe analytic continuation from Minkowsky to Euclidean spacetime, because of the absence of intermediate states lighter than the pseudoscalar meson

C. F. Kane *et al.*, arXiv:1907.00279, RM123 & Soton Coll., arXiv:2006.05358

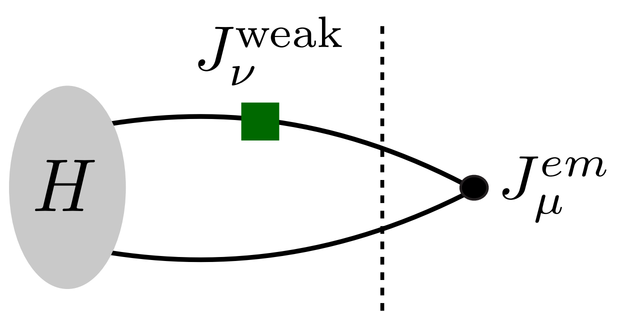
# Euclidean correlation function

$$C_{3,\mu\nu}(t_{em}, t_H) = \int d^3x \int d^3y \; e^{-i\vec{\mathbf{p}}_\gamma\cdot\vec{\mathbf{x}}} e^{i\vec{\mathbf{p}}_H\cdot\vec{\mathbf{y}}} \langle J_\mu^{\rm em}(t_{em},\vec{\mathbf{x}}) J_\nu^{\rm weak}(0) \phi_H^\dagger(t_H,\vec{\mathbf{y}})\rangle$$

$$I_{\mu\nu}^{<}(T, t_H) = \int_{-T}^{0} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

$$I_{\mu\nu}^{>}(T, t_H) = \int_{0}^{T} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Time ordering: $t_{em} > 0$




$$T_{\mu\nu}^{>} = -\sum_n \frac{\langle 0|\, J_\mu^{em}(0)\, |n(\vec{\mathbf{p}}_\gamma)\rangle \langle n(\vec{\mathbf{p}}_\gamma)|\, J_\nu^{weak}(0)\, |H(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})}$$

$$I_{\mu\nu}^{>}(t_H, T) = \int_0^T dt_{em}\; e^{E_\gamma t_{em}} C_{\mu\nu}(t_{em}, t_H)$$

$$= -\sum_m e^{E_m t_H} \frac{\langle m(\vec{\mathbf{p}}_H)|\, \phi_H^\dagger(0)\, |0\rangle}{2E_{m,\vec{\mathbf{p}}_H}}$$

$$\times \sum_n \frac{\langle 0|\, J_\mu^{em}(0)\, |n(\vec{\mathbf{p}}_\gamma)\rangle \langle n(\vec{\mathbf{p}}_\gamma)|\, J_\nu^{weak}(0)\, |m(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})} \left[1 - e^{(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})T}\right]$$

$t_H \to -\infty$ to achieve ground state saturation

$T \to \infty$ to remove unwanted exponentials that come with intermediate states

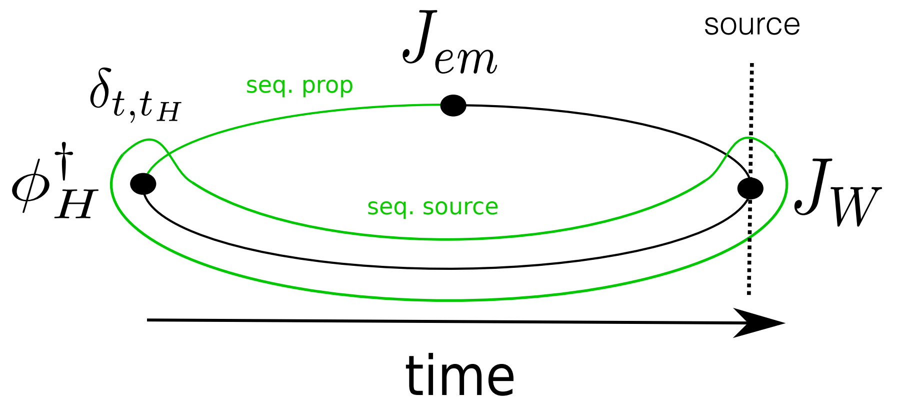
# Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T\to\infty} \lim_{t_H\to-\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{\mathbf{p}}_H)|\, \phi_H^\dagger \,|0\rangle} \underbrace{\int_{-T}^{T} dt_{em}\; e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)}_{I_{\mu\nu}(T,t_H)}$$

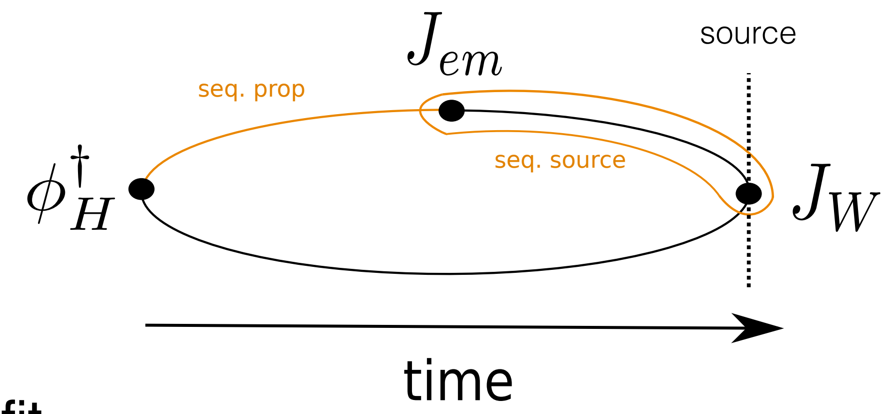
Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

arXiv:1907.00279 & arXiv:2110.13196




2: 4d sequential propagator through $J_\mu^{em}$ → calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed $T$ get all $t_H$ for free




RM123 & Soton Coll., arXiv:2006.05358: Set $T = N_T/2$ and fit to constant in $t_H$ where data has plateaued

Number of propagator solves:

| Source | 3d | 4d |
|---|---|---|
| point | $2(1 + N_{t_H} N_{p_H})$ | $2(1 + 4N_T N_{p_\gamma})$ |
| $\mathbb{Z}_2$ wall | $2(1 + N_{t_H} N_{p_H} + N_{p_H} N_{p_\gamma})$ | $2(1 + 4N_T N_{p_\gamma} + N_{p_\gamma} N_{p_H})$ |

# Simulation details

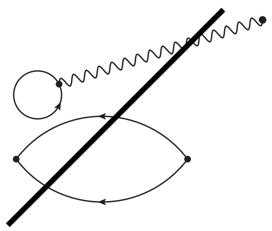
- $N_f = 2+1$ DWF, 3 RBC/UKQCD gauge ensembles

| ensemble | $(L/a)^3 \times (T/a)$ | $L_5/a$ | $\approx a^{-1}$(GeV) | $am_l$ | $am_s$ | $\approx M_\pi$(MeV) | $N_{conf}$ |
|---|---|---|---|---|---|---|---|
| 24I | $24^3 \times 64$ | 16 | 1.785 | 0.005 | 0.04 | 340 | 25 |
| 32I | $32^3 \times 64$ | 16 | 2.383 | 0.004 | 0.03 | 304 | 26 |
| 48I | $48^3 \times 96$ | 24 | 1.730 | 0.00078 | 0.0362 | 139 | 7 |

- charm valence quarks $\longrightarrow$ Möbius DW with "stout" smearing
- $\mathbb{Z}_2$ random wall sources & randomly placed point sources
- Two datasets: $J^{weak}(0)$ or $J^{em}(0)$

| Method | Source | Meson Momentum | Photon Momentum |
|---|---|---|---|
| 3d | $\mathbb{Z}_2$-wall | $\vec{p}_{D_s} = (0,0,0)$ | $\lvert\vec{p}_\gamma\rvert^2 \in (2\pi/L)^2\{1,2,3,4\}$ |
| 3d | point | $p_{D_s,z} \in 2\pi/L\{0,1,2\}$ | all |
| 4d | $\mathbb{Z}_2$-wall | $p_{D_s,z} \in 2\pi/L\{-1,0,1,2\}$ | $p_{\gamma,z} = 2\pi/L$ |
| $4d^{>,<}$ | $\mathbb{Z}_2$-wall | $p_{D_s,z} \in 2\pi/L\{-1,0,1,2\}$ | $p_{\gamma,z} = 2\pi/L$ |

- Local electromagnetic current + mostly non-perturbative RCs
- Disconnected diagrams are neglected



- For point sources use translational invariance to fix em/weak operator at $\mathbf{0}$

use an "infinite-volume approximation" to generate data for arbitrary photon momenta (only exp. small FVEs are introduced)

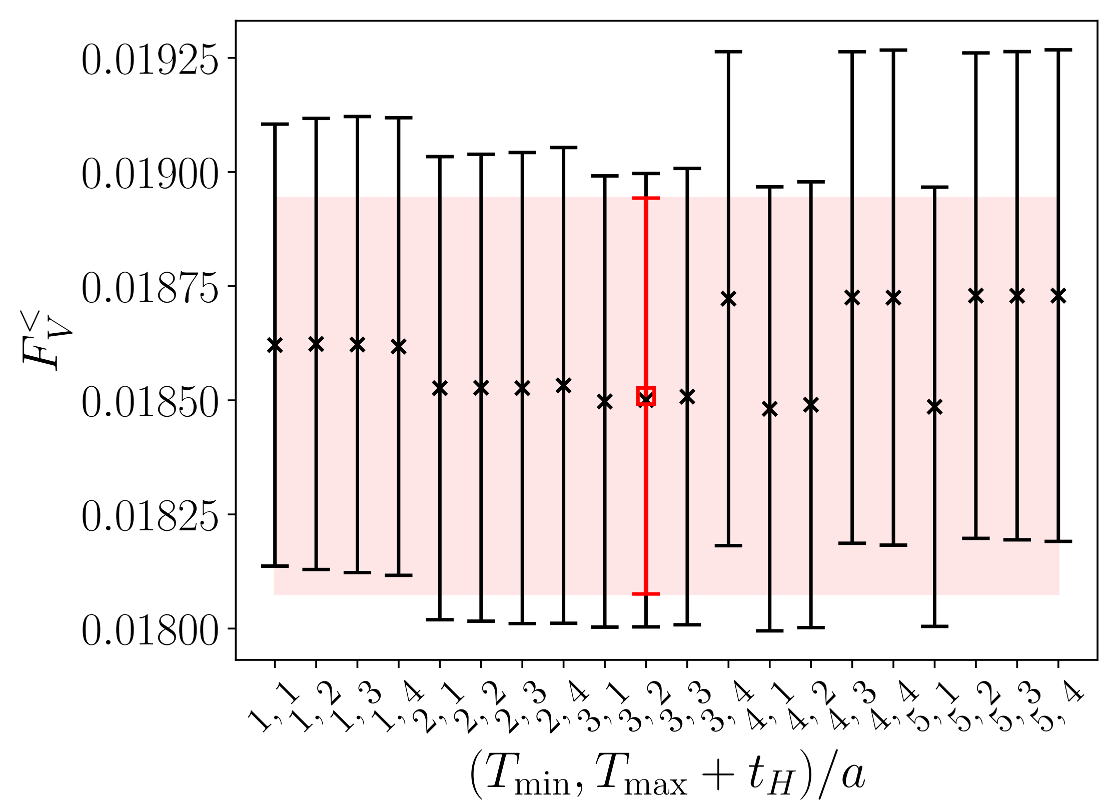
$$C_{3,\mu\nu} = \int d^3x \int d^3y\, e^{-i\vec{p}_\gamma\cdot\vec{x}} \langle J^{em}_\mu(t_{em},\vec{x}) J^{weak}_\nu(0)\phi^\dagger_H(t_H,\vec{y})\rangle \qquad \vec{p}_H = 0, \text{ several } \vec{p}_\gamma$$

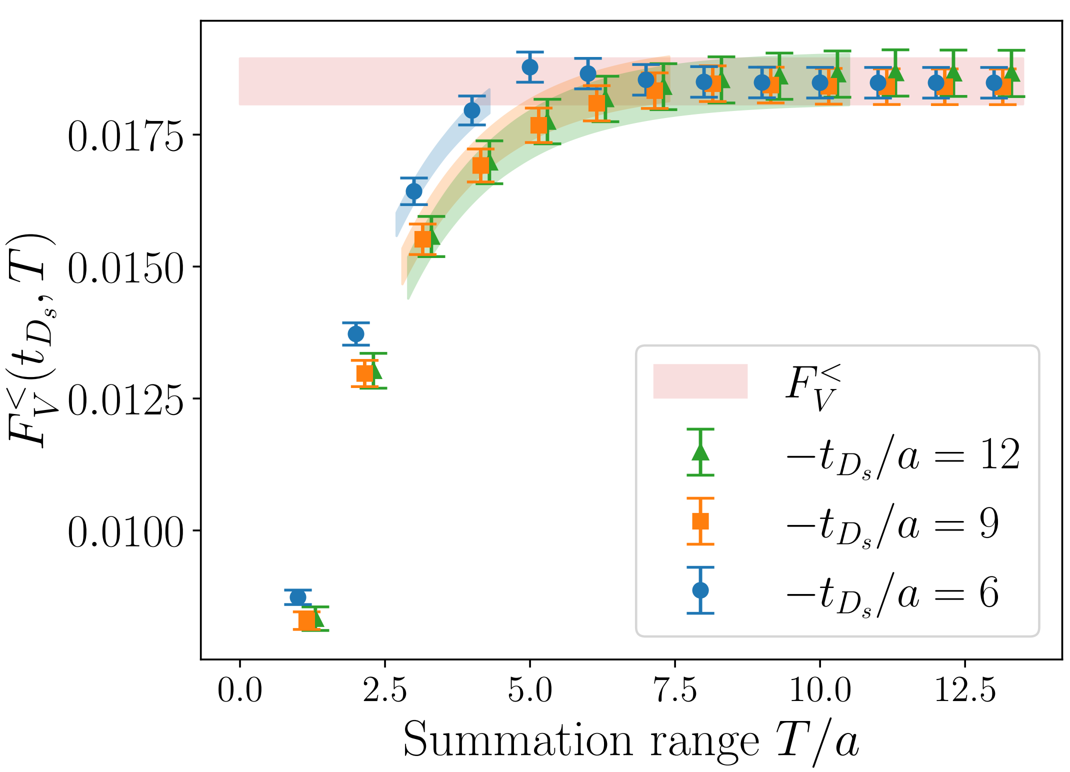
Include terms to fit
(1) unwanted exponential from first intermediate state
(2) first excited state

Fit form factors $F_V$ and $F_{A,SD}$ directly instead of $I_{\mu\nu}$

Time ordering $t_{em} < 0$:

$$F^<(t_H, T) = F^< + B_F^<\left(1 + B_{F,\text{exc}}^< \overbrace{e^{\Delta E(T+t_H)}}\right)\overbrace{e^{-(E_\gamma - E_H + E^<)T}} + C_F^< \overbrace{e^{\Delta E t_H}}$$



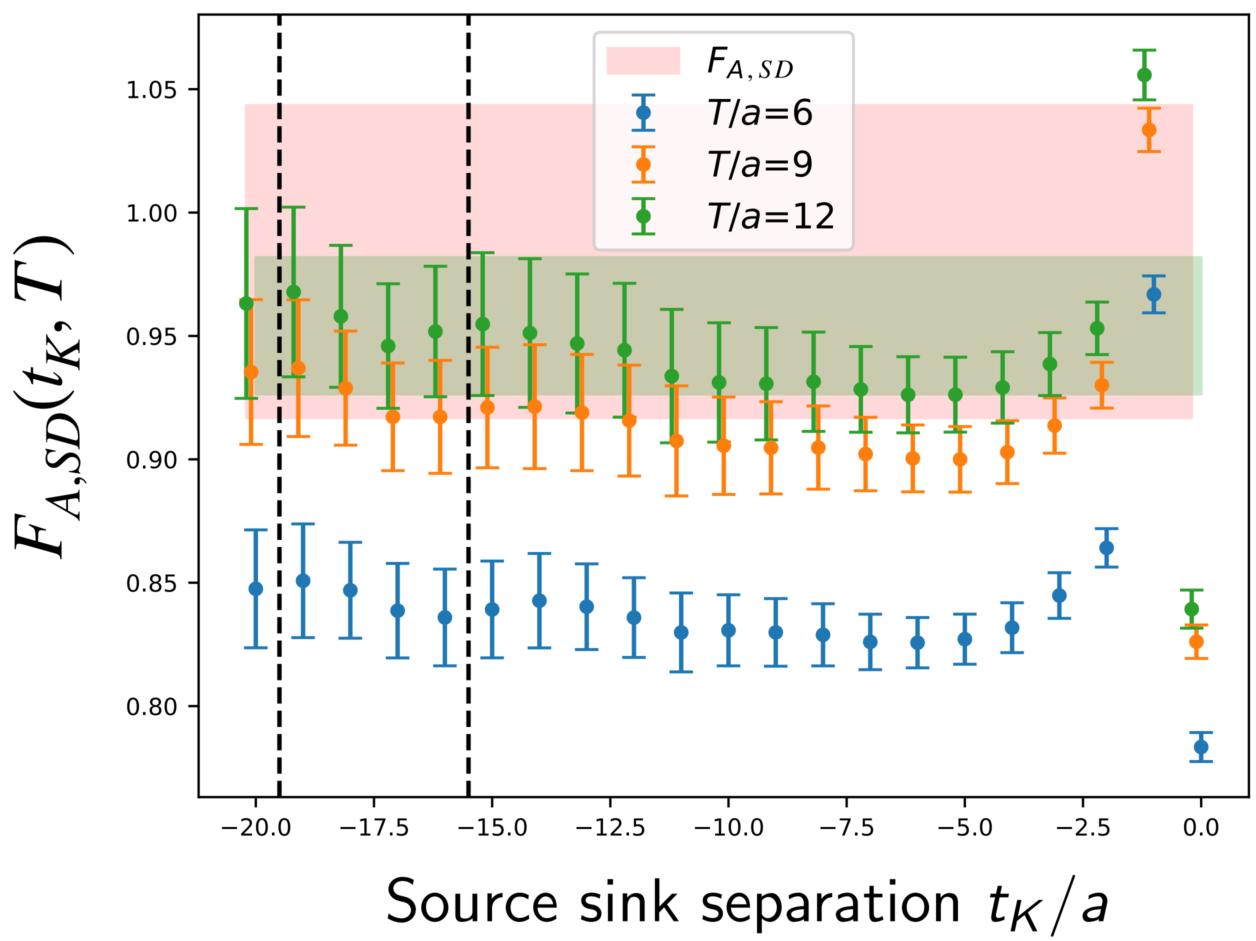
# Fit form: 4d method

Use fit ranges where data has plateaued in $t_H$, i.e. $t_H \to -\infty$

Include terms to fit
(1) unwanted exponential from first intermediate state

Sum of both time orderings $I_{\mu\nu}(T, t_H) = I^{<}_{\mu\nu}(T, t_H) + I^{>}_{\mu\nu}(T, t_H)$

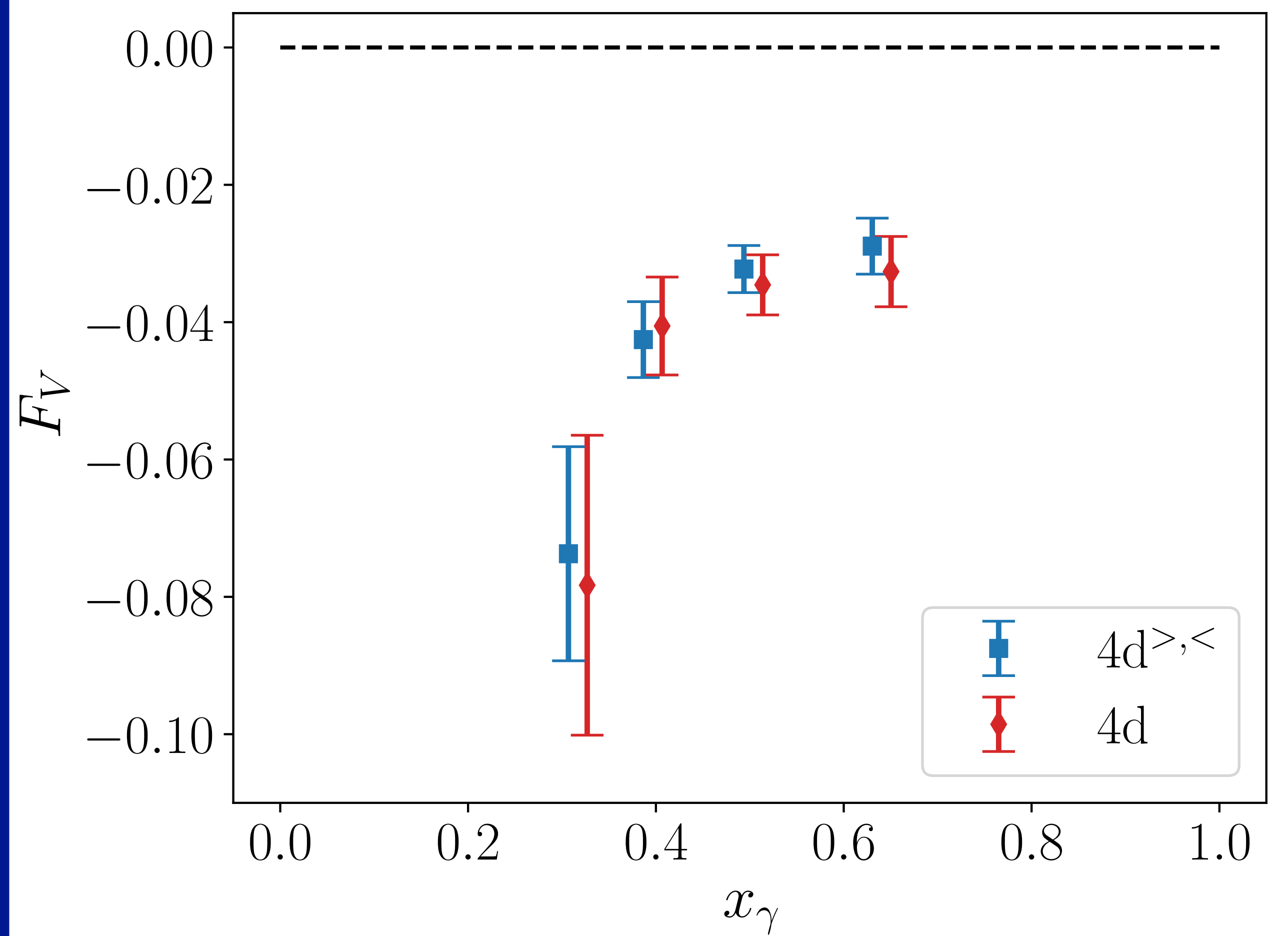
$$F(t_H, T) = F + B_F^{<} \underbrace{e^{-(E_\gamma - E_H + E^{<})T}}_{t_{em}<0} + B_F^{>} \underbrace{e^{(E_\gamma - E^{>})T}}_{t_{em}>0}$$

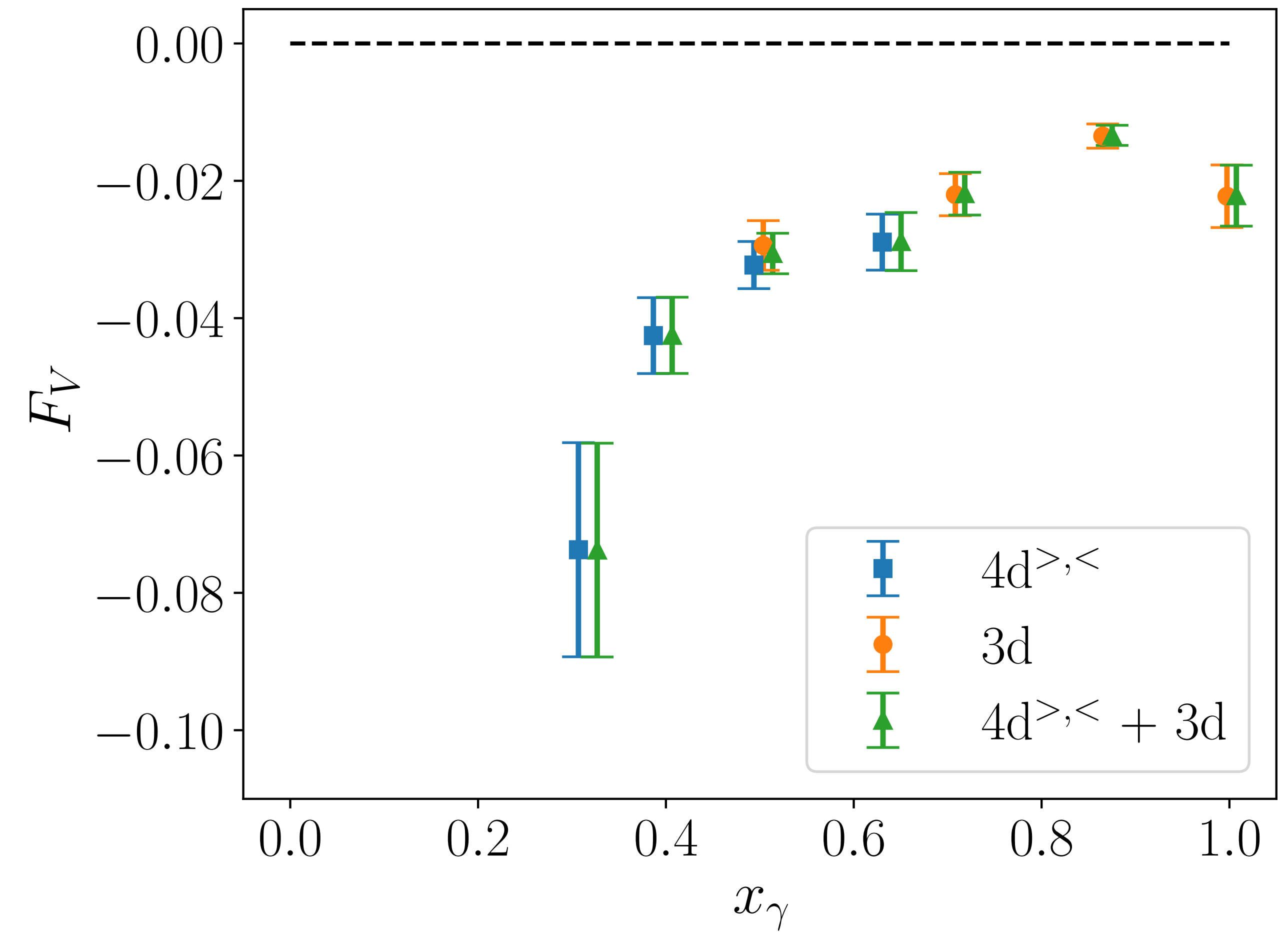



Limitation of 4d method:
the two different time orderings of $I_{\mu\nu}(t_H, T)$ cannot be resolved

$4d^{>,<}$ method
performing two sequential solves through the em current

# $D_s \to \ell\nu_\ell\gamma$: 3d vs 4d analysis results



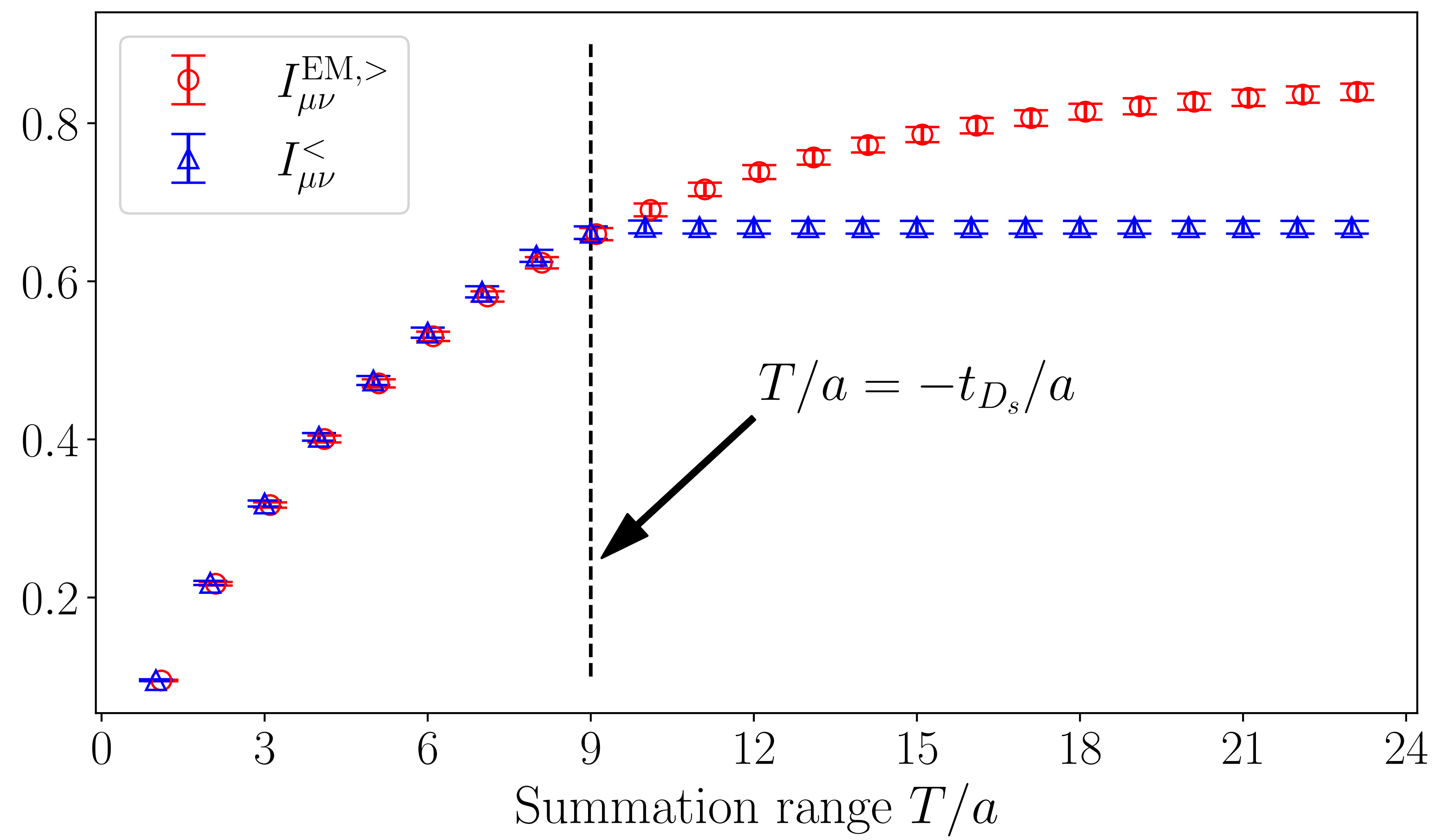




$$x_\gamma = \frac{2p_H \cdot p_\gamma}{m_H^2} \xrightarrow{\vec{p}_H = \mathbf{0}} x_\gamma = \frac{2E_\gamma^{(0)}}{m_H} \qquad 0 \le x_\gamma \le 1 - \frac{m_\ell^2}{m_H^2}$$

- 4d method cannot resolve the sum of the unwanted exponentials of the separate time orderings
- 3d method offers good control over the unwanted exponentials for a significantly cheaper computational cost

# 3pt function with e.m. current at origin

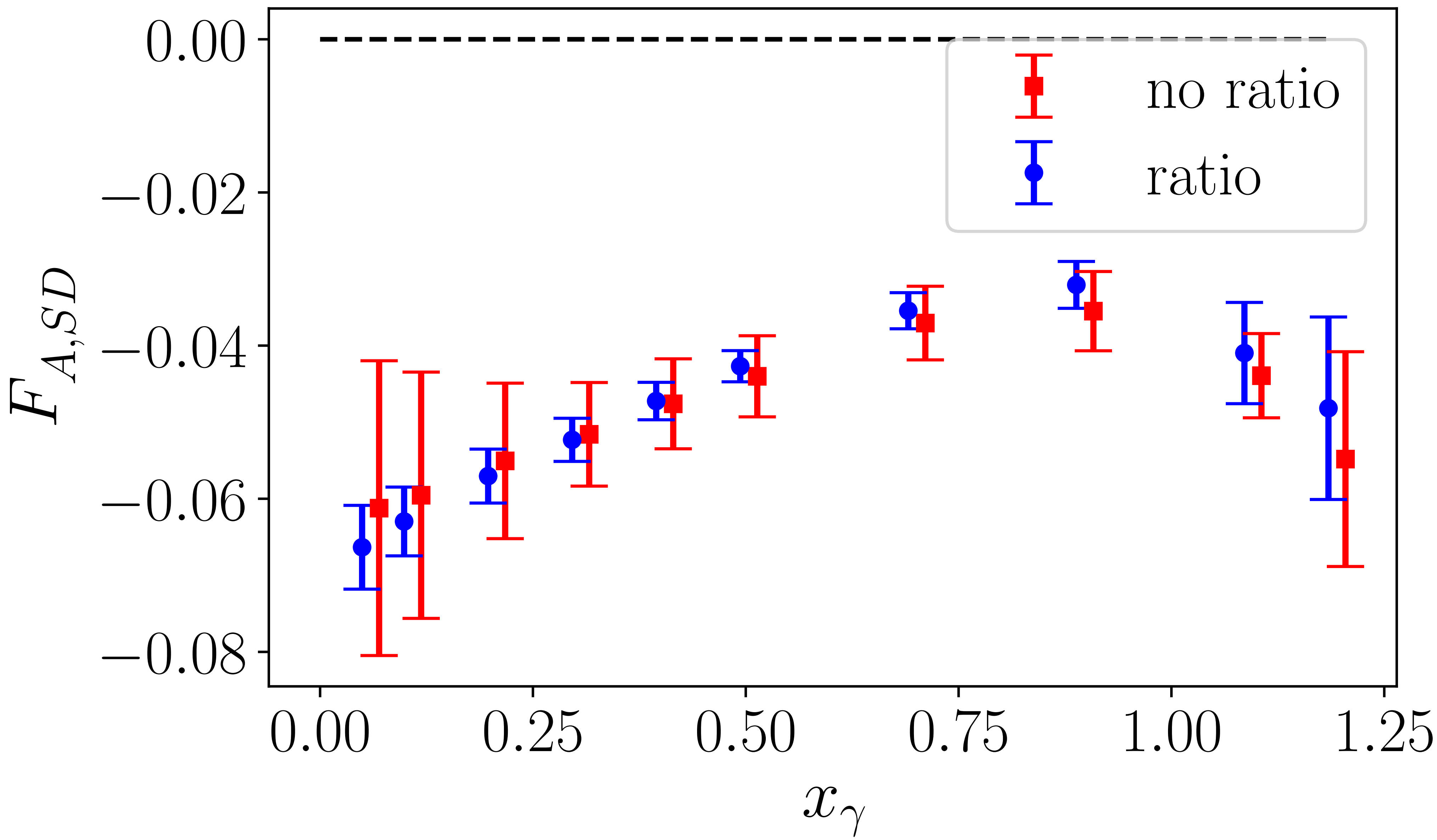
$$C^{\mathrm{EM}}_{3,\mu\nu}(t_W, t_H) = e^{E_H t_W} \int d^3x \int d^3y \, e^{i(\vec{p}_\gamma - \vec{p}_H)\cdot\vec{x}} e^{i\vec{p}_H\cdot\vec{y}} \langle J^{\mathrm{em}}_\mu(0) J^{\mathrm{weak}}_\nu(t_W, \vec{x}) \phi^\dagger_H(t_H, \vec{y})\rangle$$




The spectral decomposition of the $t_W > 0$ time ordering of $I^{\mathrm{EM}}_{\mu\nu}$ and the $t_{em} < 0$ time ordering of $I_{\mu\nu}$ are equal up to excited state effects

# Improved form factors estimators

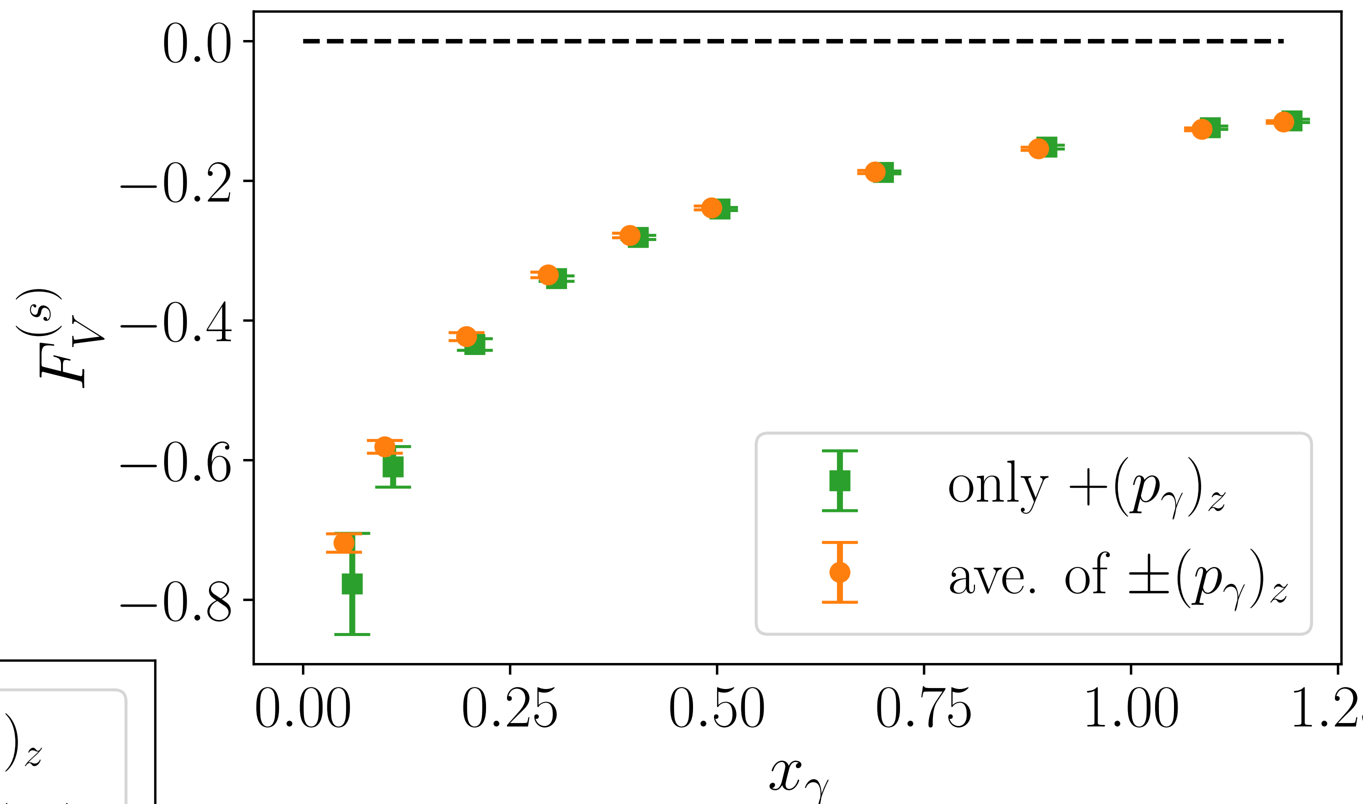
$$C_i(\overrightarrow{p}_\gamma, t) = \frac{C_p(\overrightarrow{p}_\gamma, t)}{C_p(\overrightarrow{p}_\gamma = \overrightarrow{p}^\star_\gamma, t)} C_z(\overrightarrow{p}_\gamma = \overrightarrow{p}^\star_\gamma, t)$$

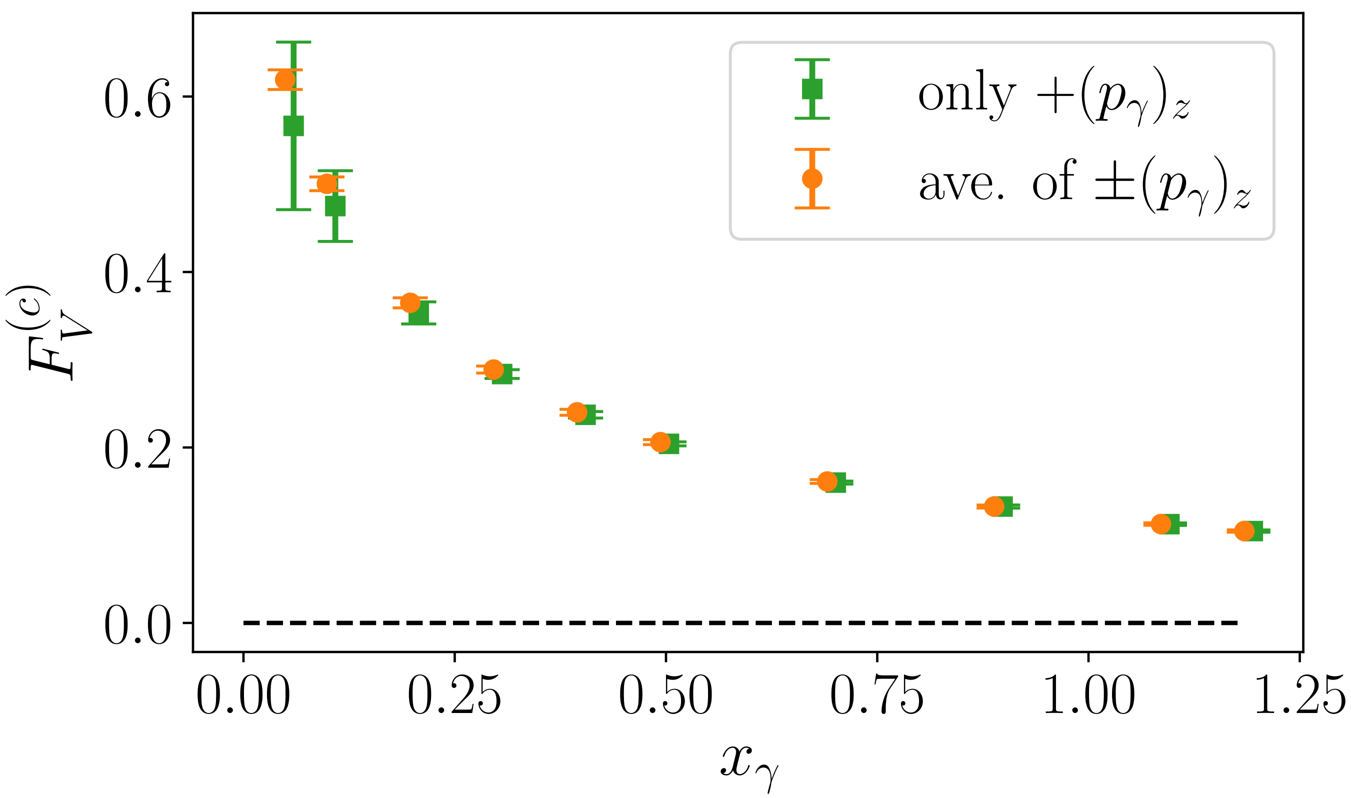


# Improved form factors estimators [2]

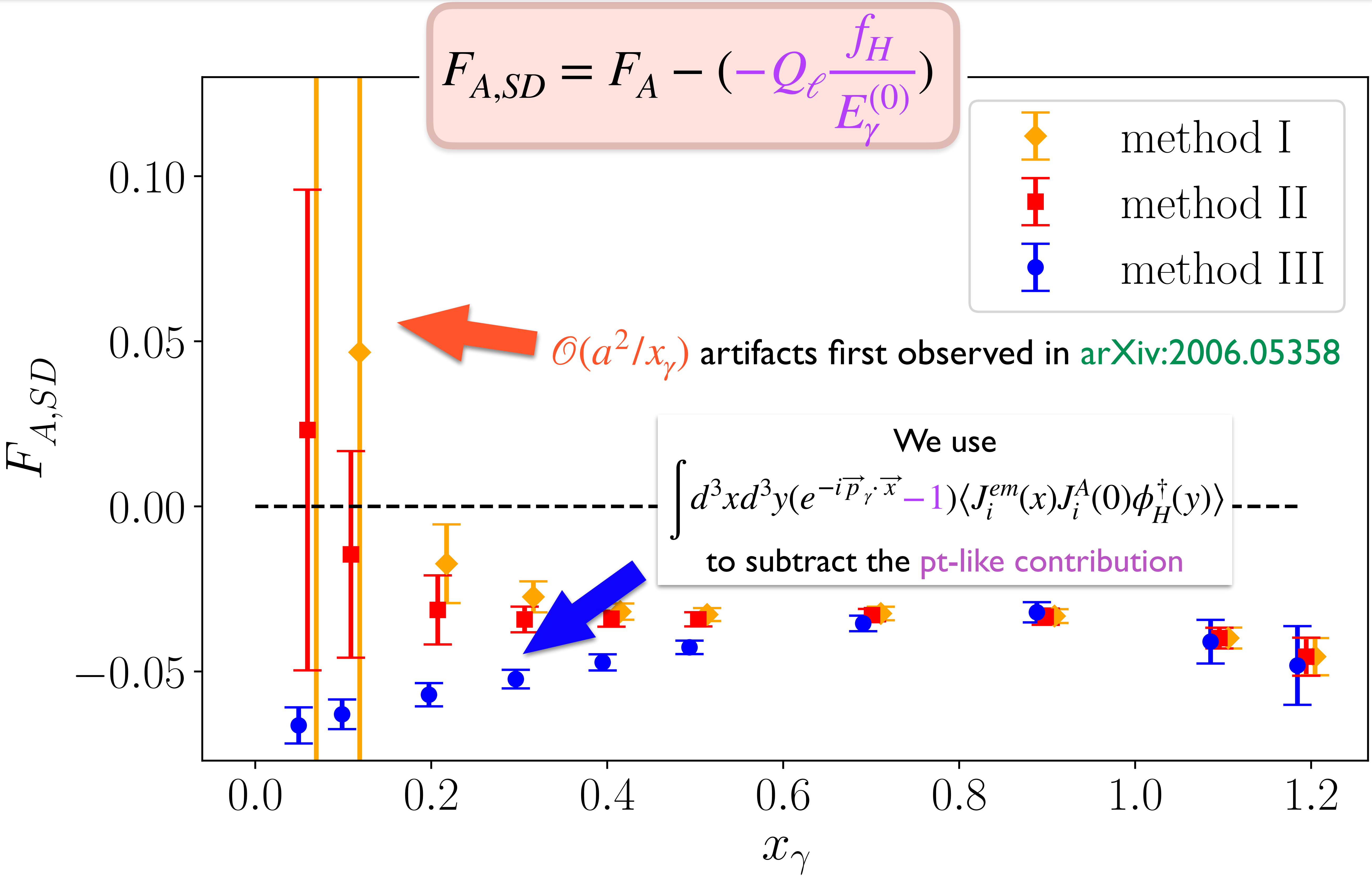
$$C_{3,\mu\nu}(t_{em}, t_H) = \int d^3x \int d^3y \; e^{-i\vec{\mathbf{p}}_\gamma\cdot\vec{\mathbf{x}}} e^{i\vec{\mathbf{p}}_H\cdot\vec{\mathbf{y}}} \langle J^{\mathrm{em}}_\mu(t_{em}, \vec{\mathbf{x}}) J^{\mathrm{weak}}_\nu(0) \phi^\dagger_H(t_H, \vec{\mathbf{y}})\rangle$$

$\pm \overrightarrow{p}_\gamma$ average



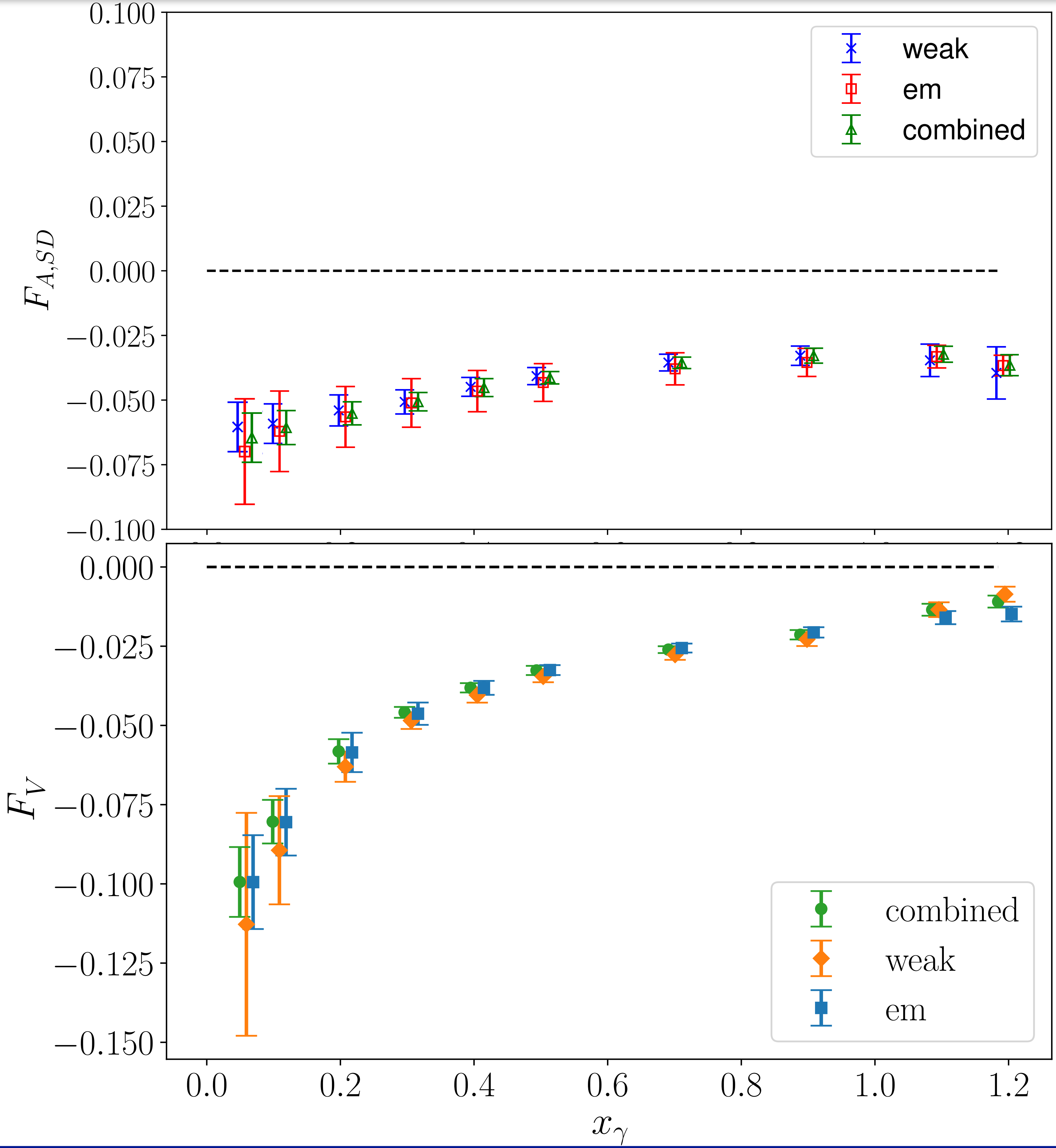



# NP subtraction of IR-divergent discretization effects

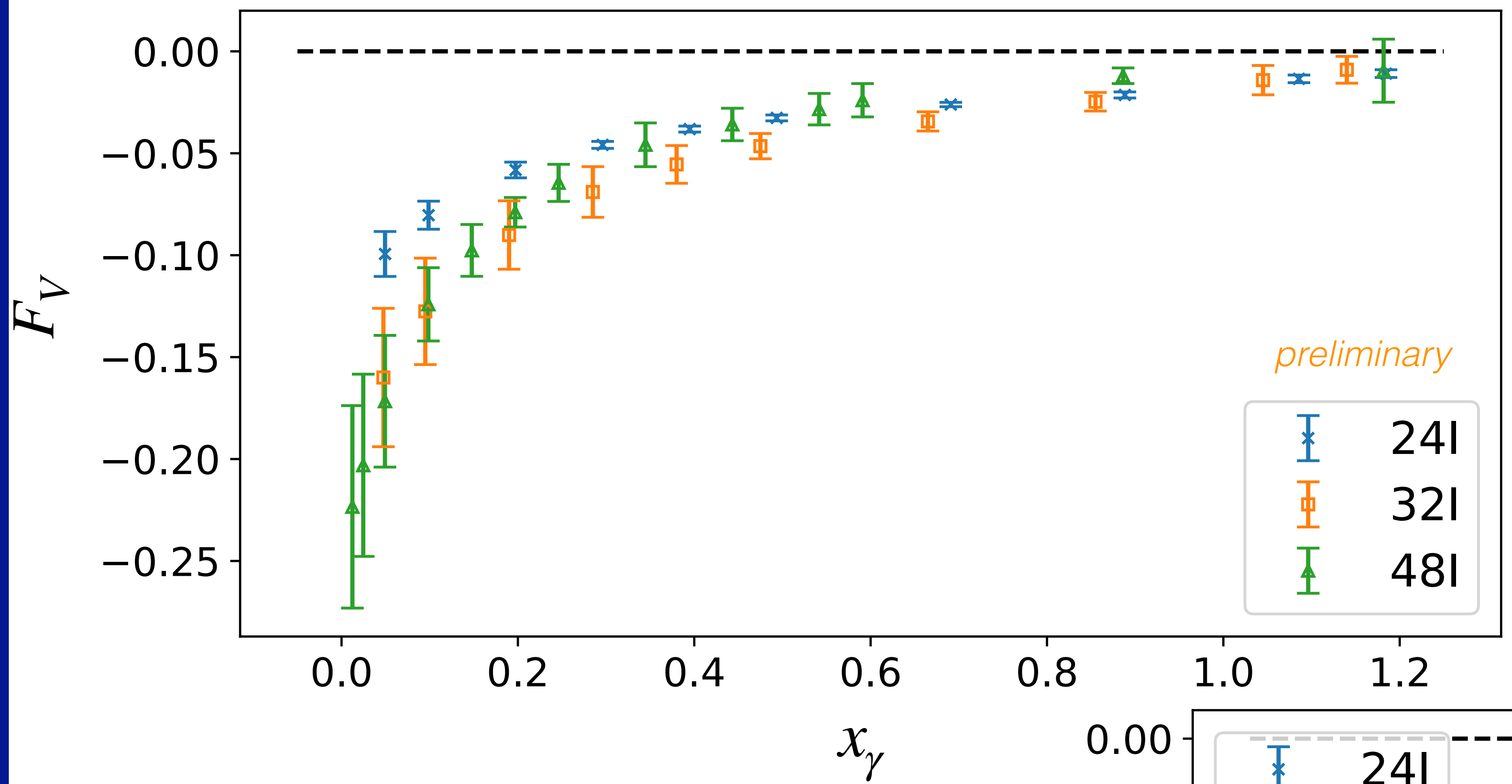


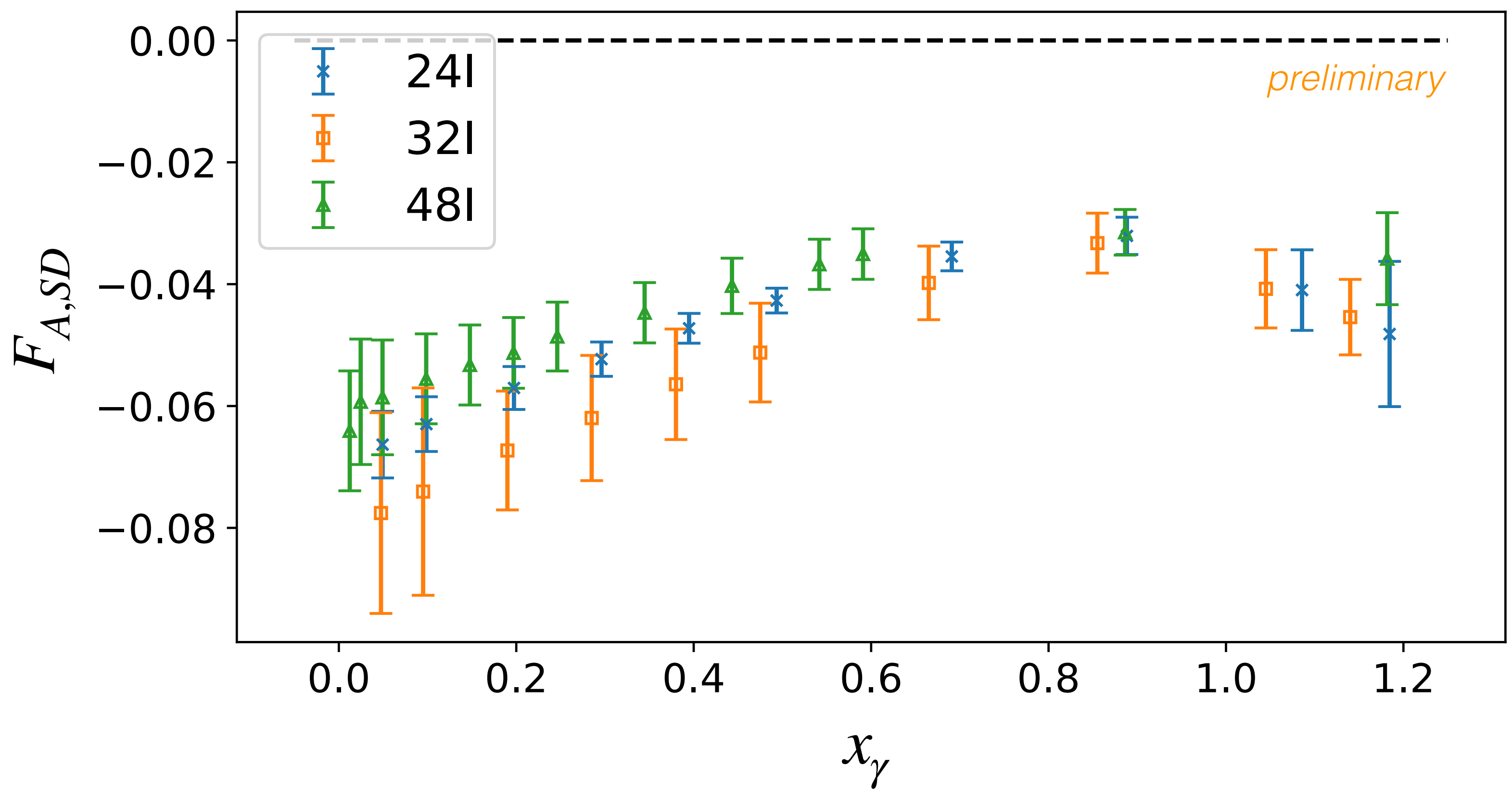

Blue data: improved subtraction of pt-like contribution

# $D_s \to \ell \nu_\ell \gamma$: weak and em datasets



# $D_s \to \ell\nu_\ell\gamma$: preliminary results



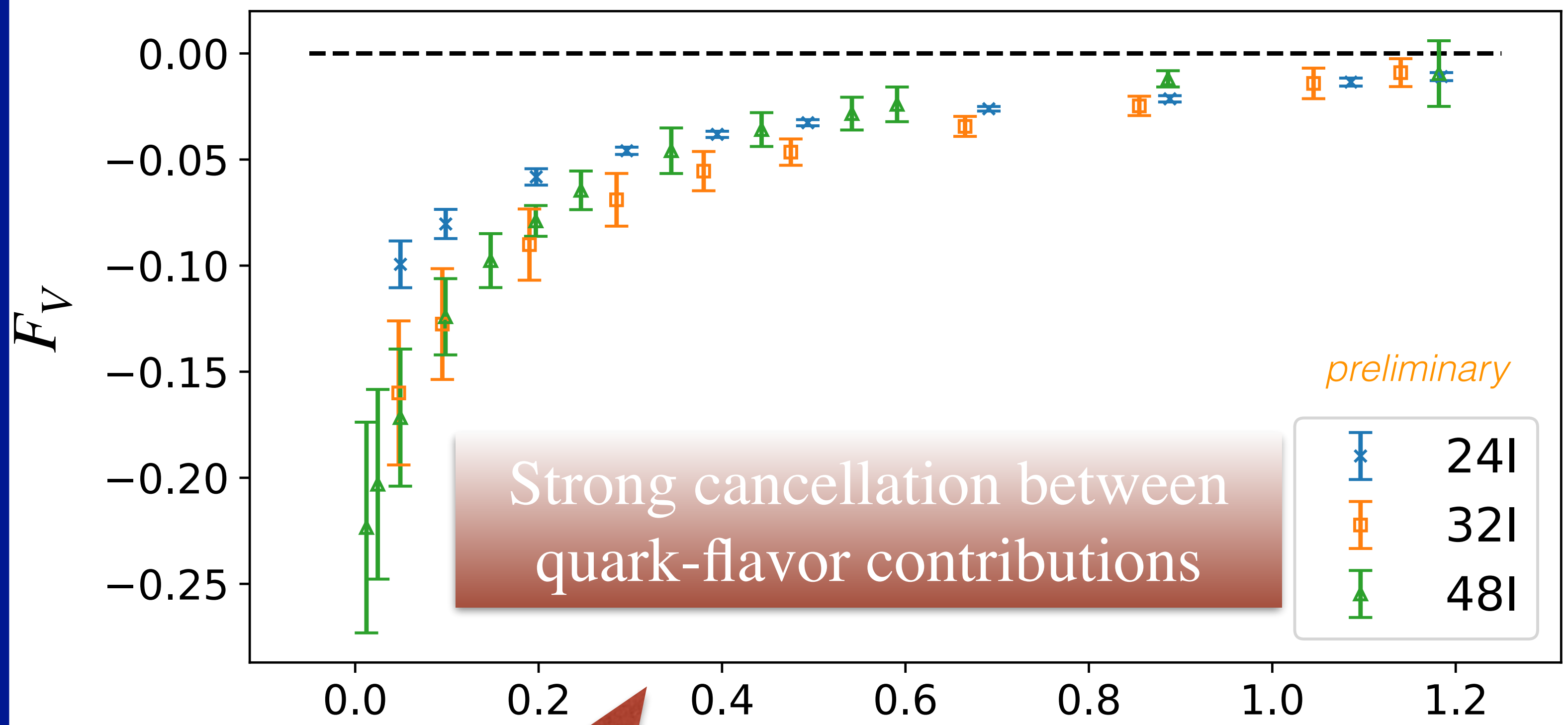



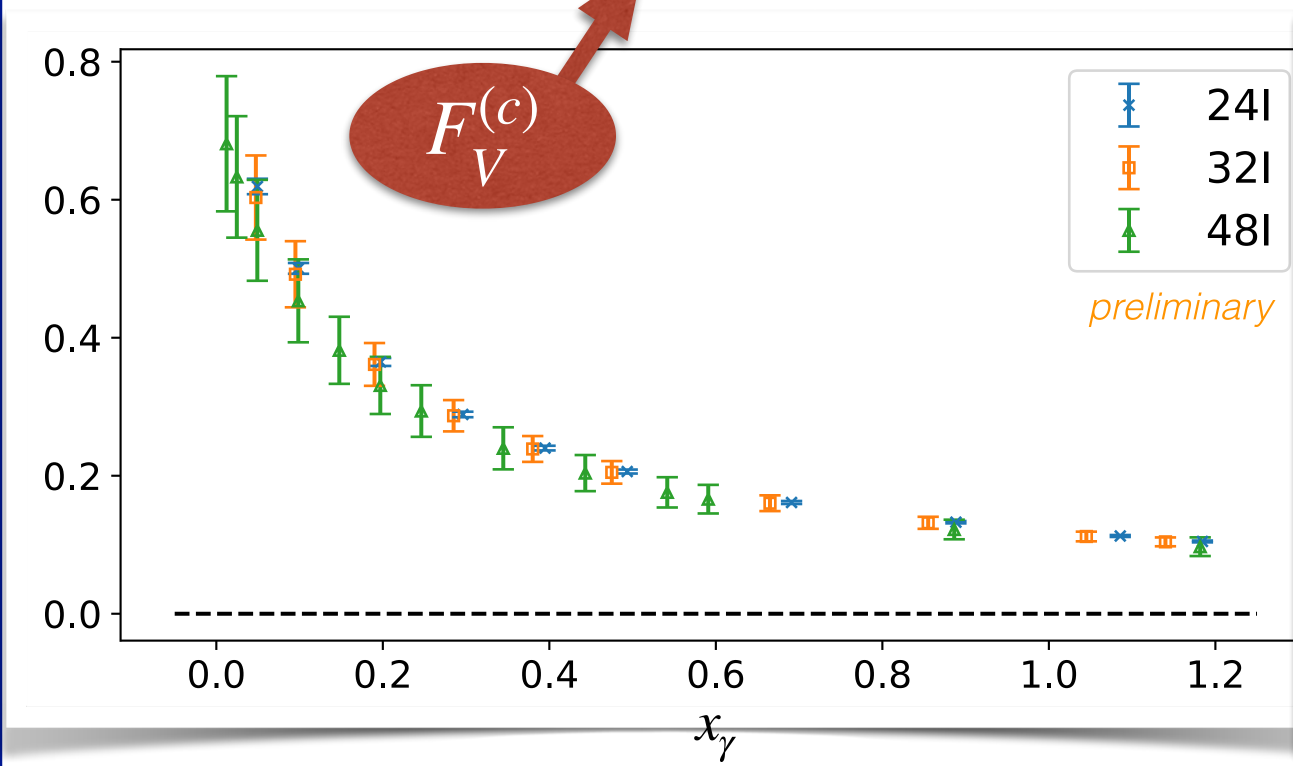

We are testing various fit functions to provide a parameterization of the form-factor lattice data

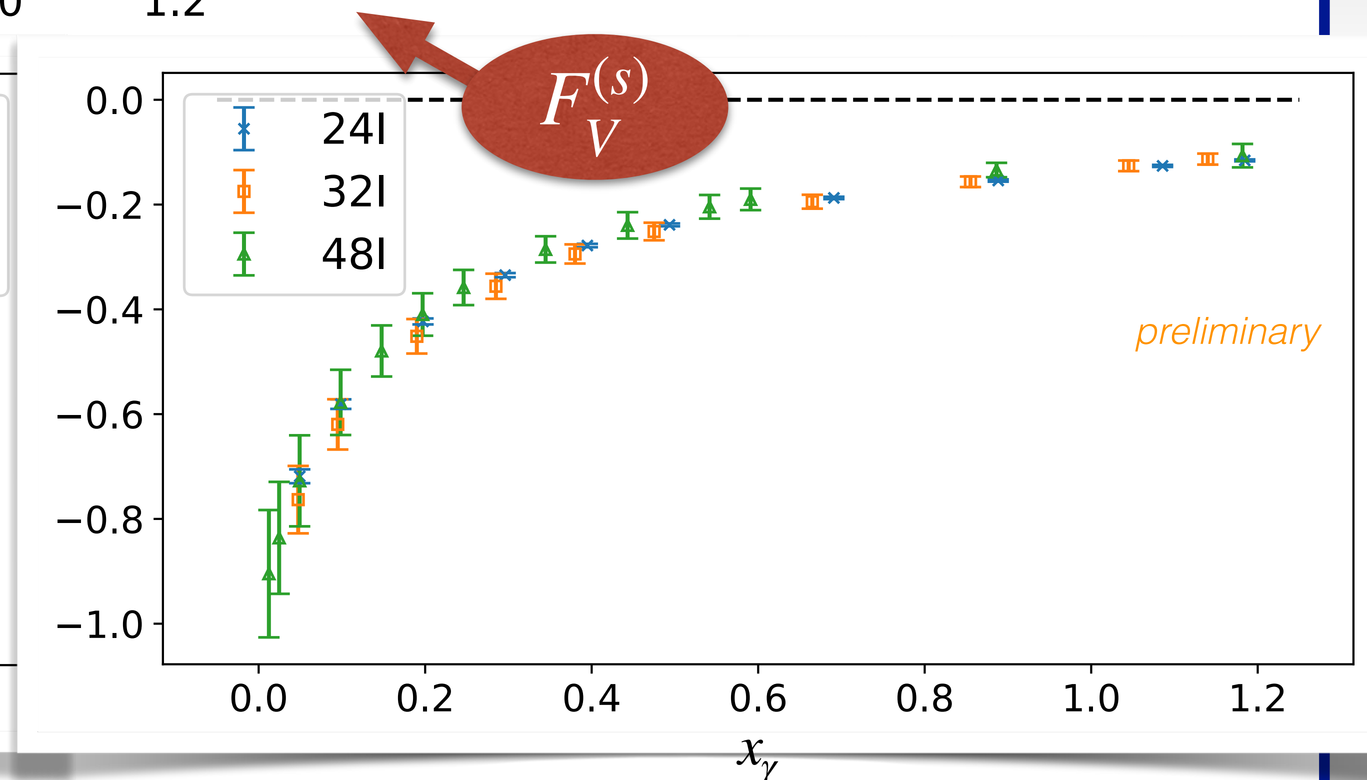
$D_s^+ \to e^+\nu\gamma$: $\mathcal{B}(E_\gamma > 10\text{ MeV}) < 1.3 \times 10^{-4}$ SM: $\mathcal{O}(10^{-4})$

[BESIII Collaboration, arXiv:1902.03351]

# $D_s \to \ell \nu_\ell \gamma$: preliminary results

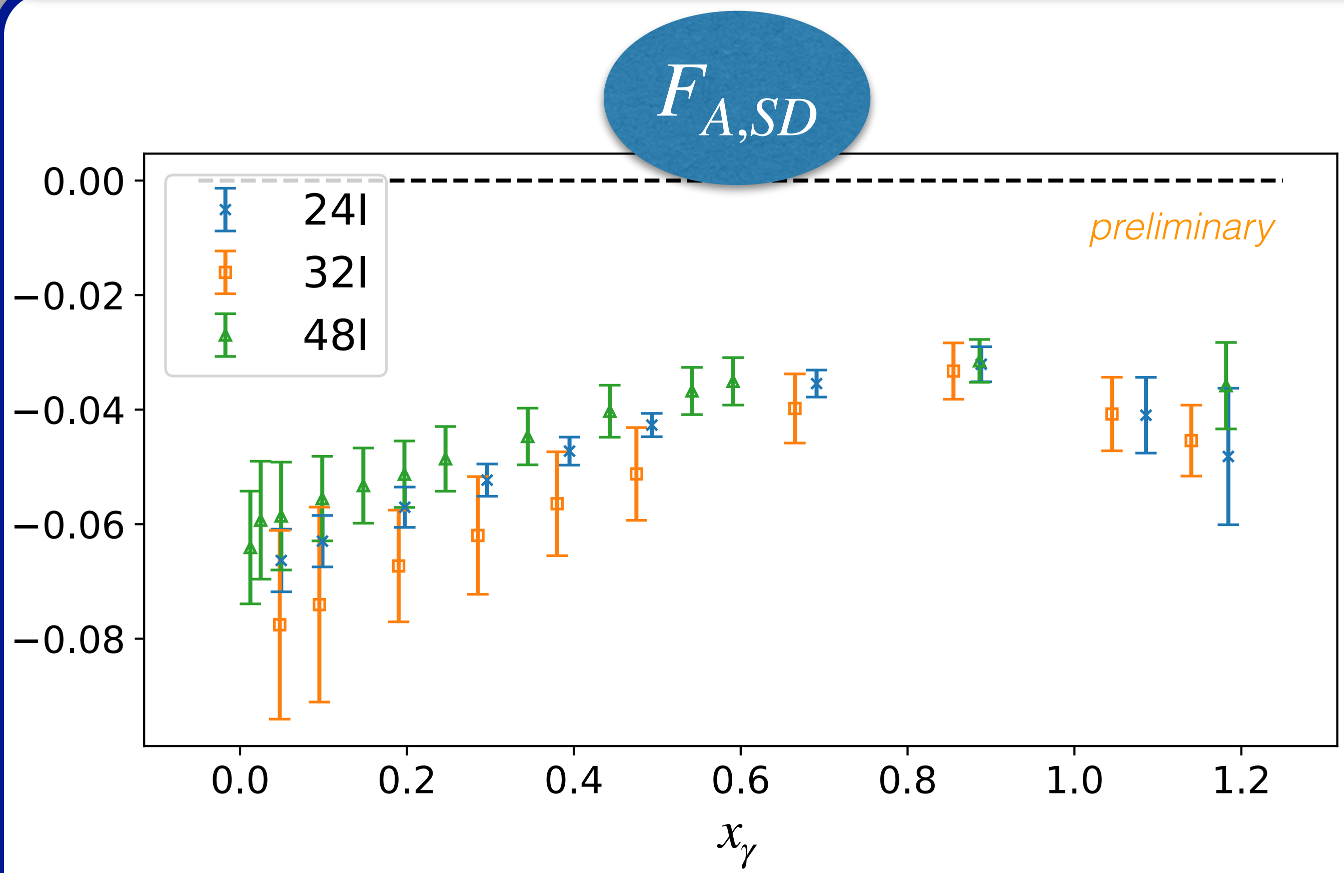


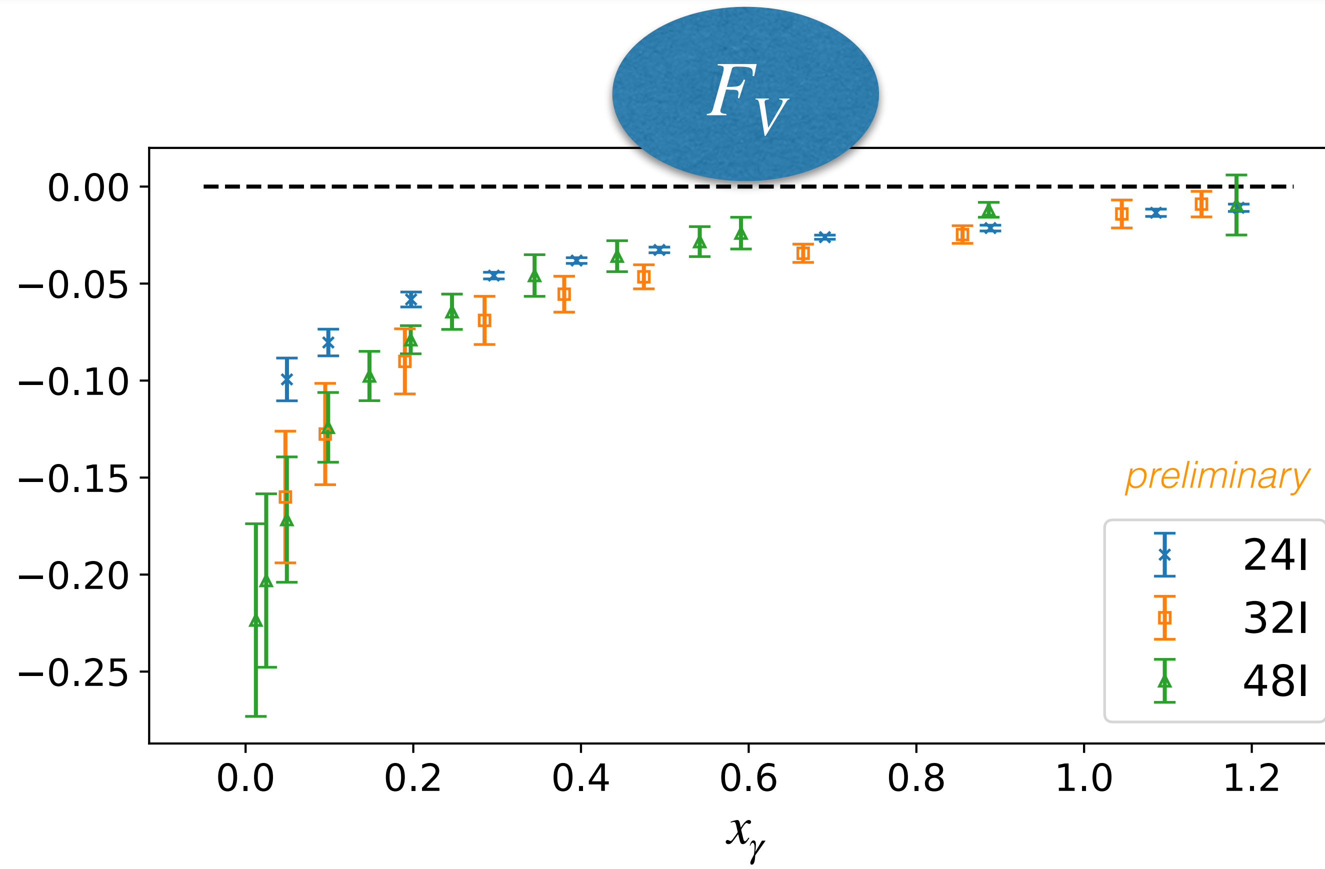



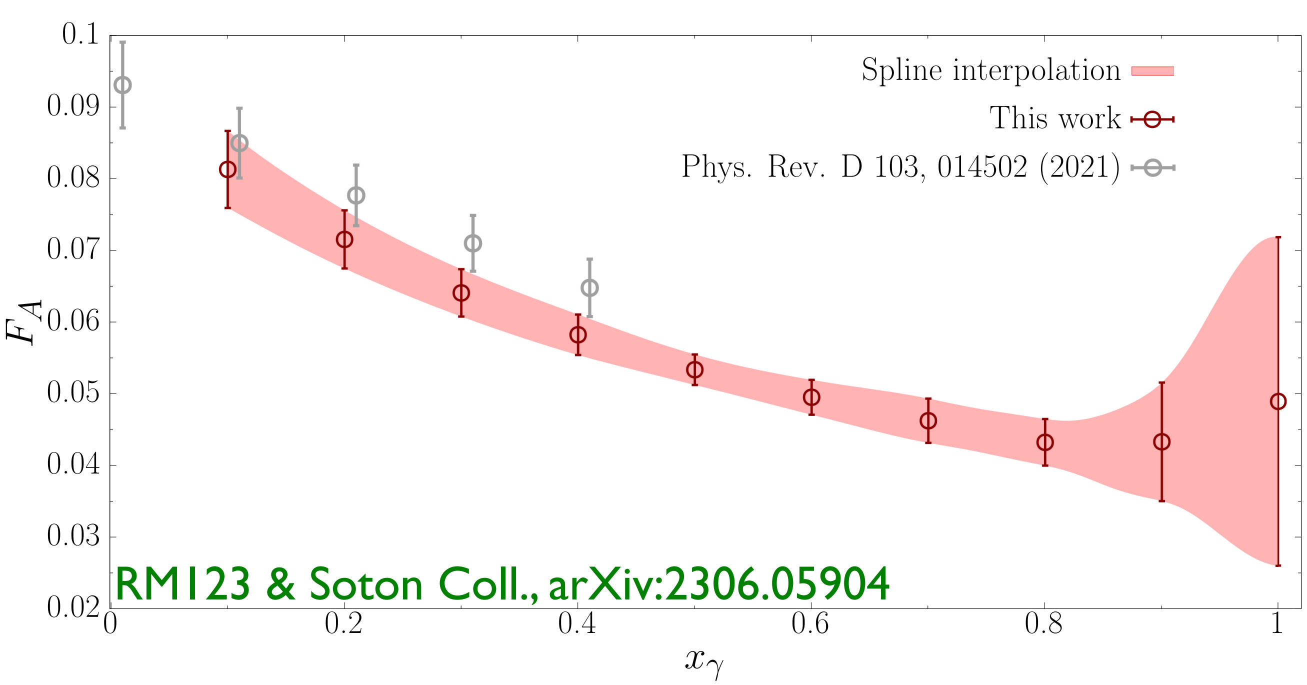

- Similar cancellations observed in $D_s D_s^* \gamma$ couplings, corresponding to pole residues in $D_s \to \ell \nu_\ell \gamma$ form factors G. C. Donald *et al.*, 2014 & B. Pullin and R. Zwicky, 2021

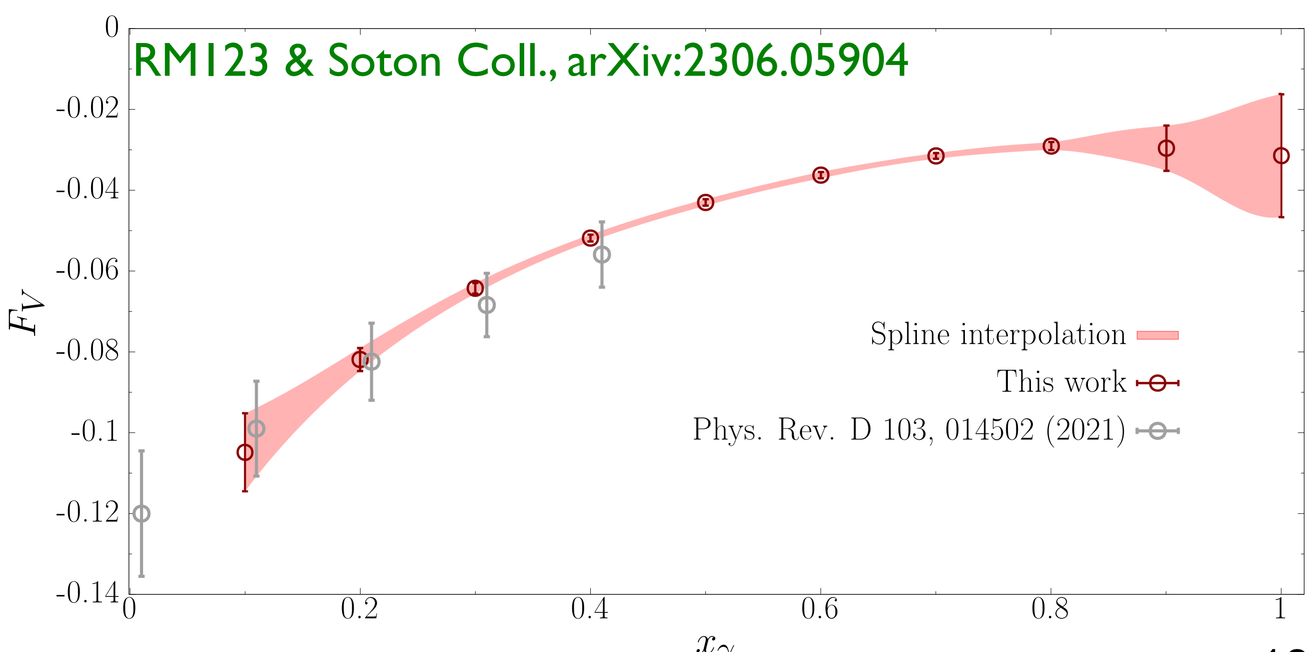
# $D_s \to \ell \nu_\ell \gamma$: comparison



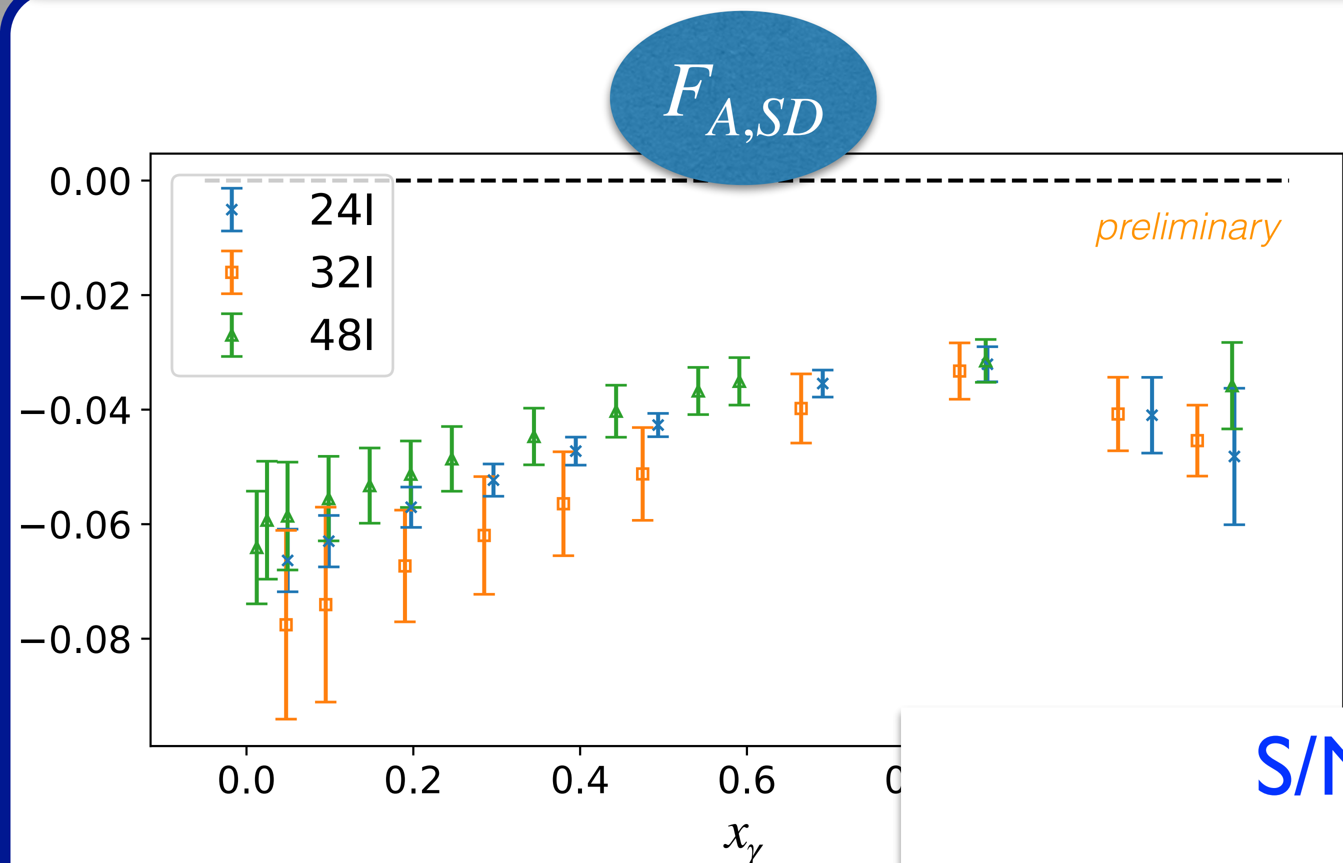



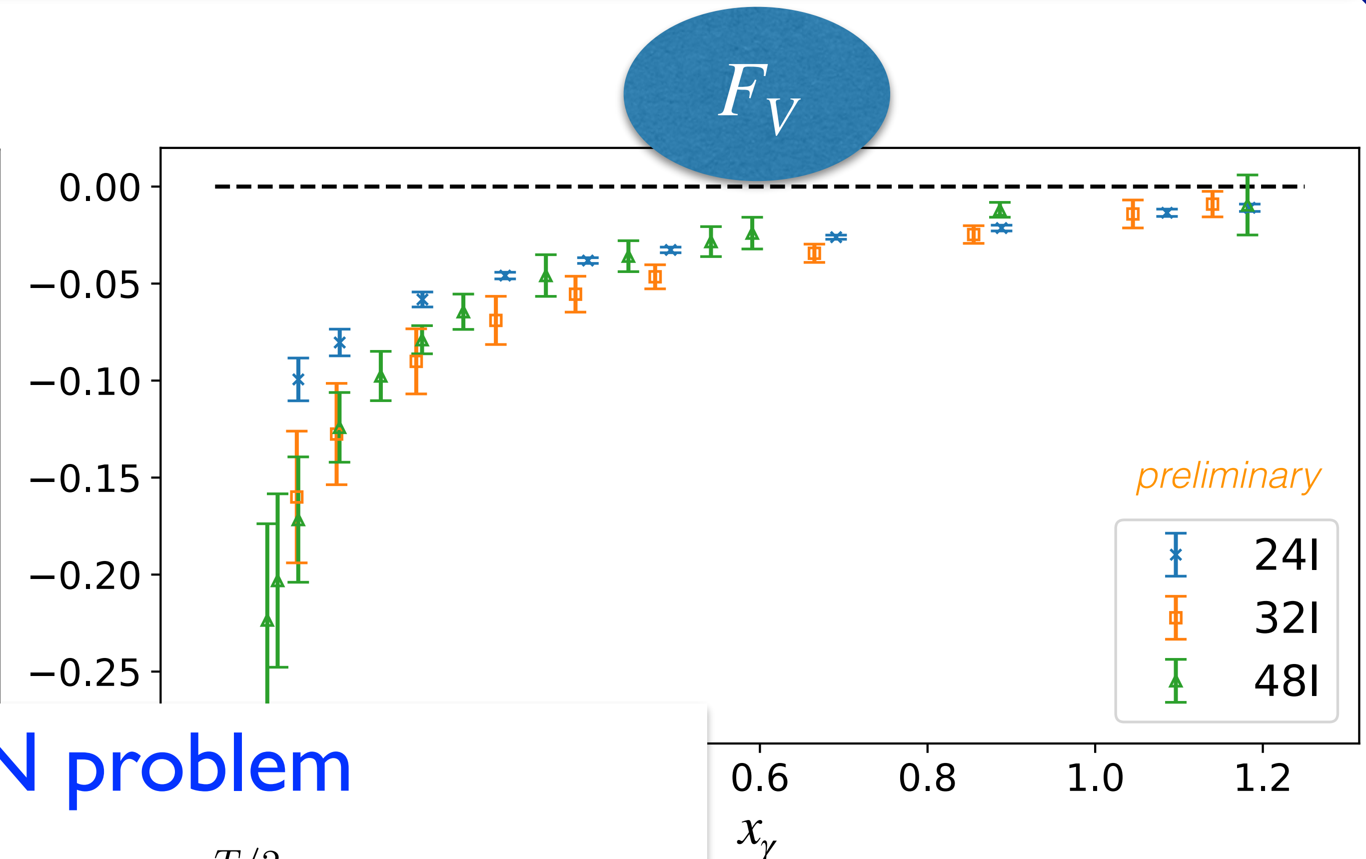

<u>sign</u>: different FFs parameterization



# $D_s \to \ell \nu_\ell \gamma$: comparison

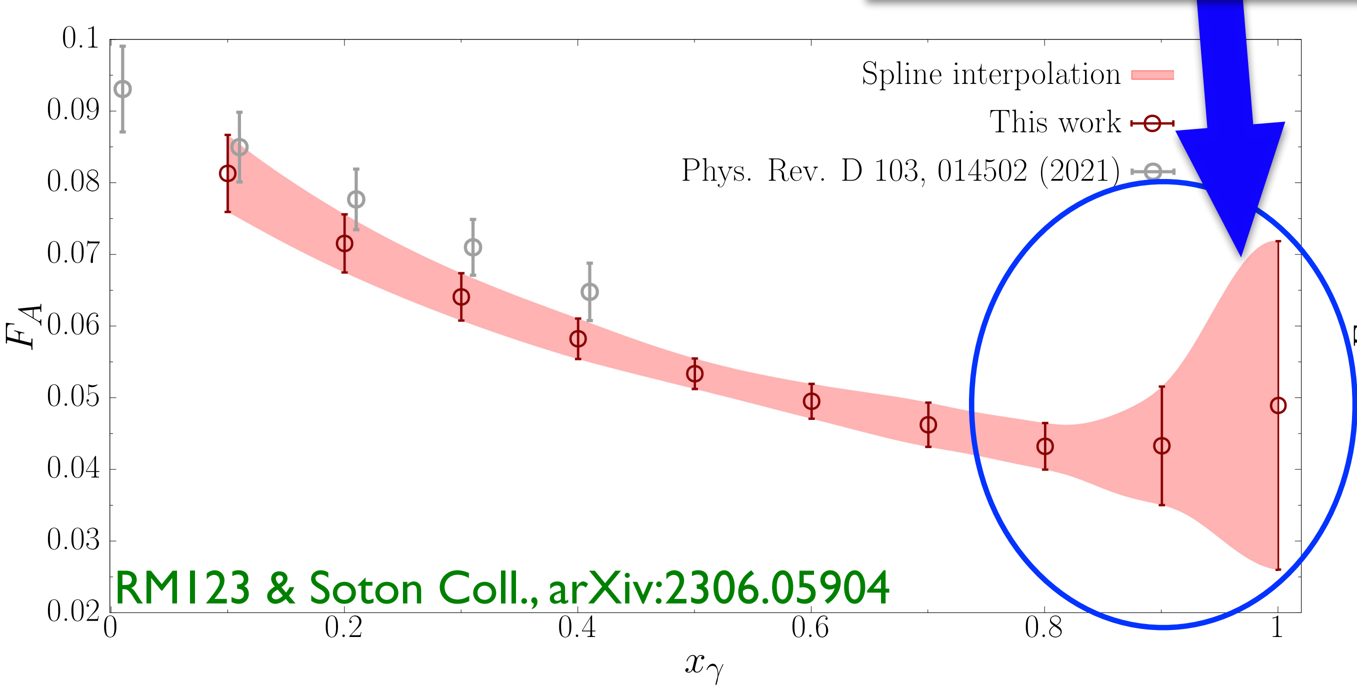


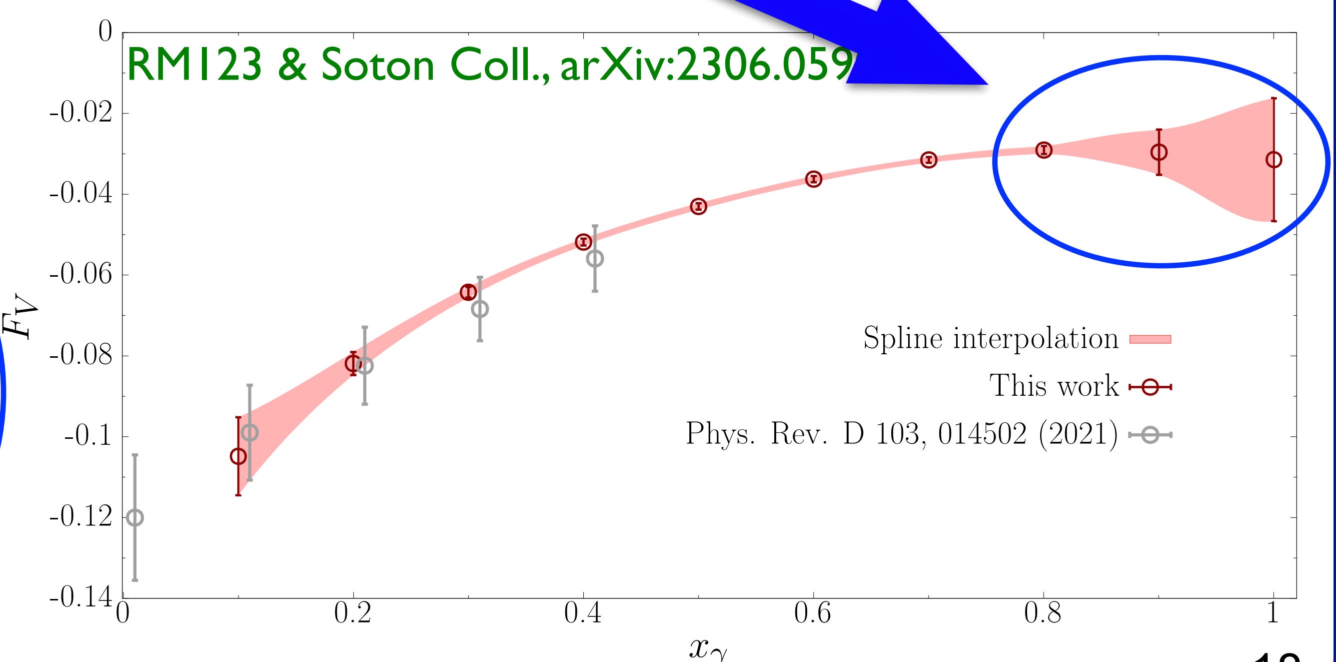




## S/N problem

$$\sigma_{C^{\mu\nu;2}_{f,W}}(t,\boldsymbol{k}) \simeq B_{f,0}\, e^{M^{\mathrm{PS}}_{\bar{f}f} t} \int_t^{T/2} dt_y\, e^{-(M^{\mathrm{PS}}_{\bar{f}f} - E_\gamma)t_y}$$

sign: different FFs parameterization



# Conclusions and future perspectives

- The form factors for real emissions are accessible from Euclidean correlators

- We compared analysis methods using 3d and 4d data. 3d method results in smallest statistical uncertainties and allows to tame S/N problems at large photon energies. Those findings have been illustrated in a method paper we published this year

- With moderate statistics we are able to provide rather precise, first-principles results for the form factors in the full kinematical (photon-energy) range

- Lattice calculations of radiative leptonic heavy-meson decays at high photon energy could provide useful information to better understand the internal structure of hadrons

- Statistical improvement for all ensembles used is in progress thanks to dedicated ACCESS computing resources. A new paper with continuum-limit results will appear soon. To extend the study to B-meson decays we will take advantage of new RBC/UKQCD ensembles at $a^{-1} \approx (3.5, 4.5)$ GeV

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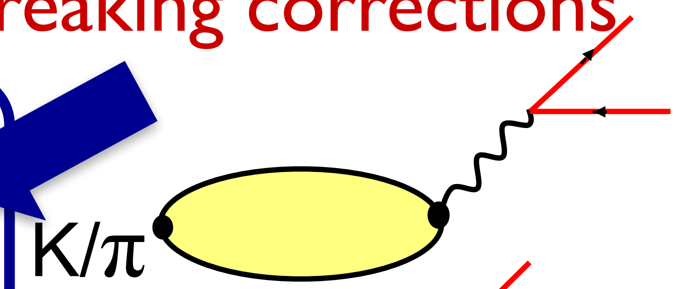
# Supplementary slides

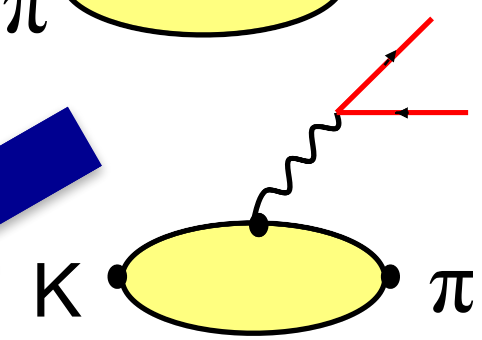
# Electromagnetic and isospin-breaking effects

Given the present exper. and theor. (LQCD) accuracy, an important source of uncertainty are long distance electromagnetic and SU(2)-breaking corrections

$$\frac{\Gamma\left(K^+\to\ell^+\nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+\to\ell^+\nu_\ell(\gamma)\right)}=\left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2\frac{M_{K^+}\left(1-m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1-m_\ell^2/M_{\pi^+}^2\right)^2}\boxed{\left(1+\delta_{EM}+\delta_{SU(2)}\right)}$$




$$\Gamma\left(K^{+,0}\to\pi^{0,-}\ell^+\nu_\ell(\gamma)\right)=\frac{G_F^2M_{K^{+,0}}^5}{192\pi^3}C_{K^{+,0}}^2\left|V_{us}f_+^{K^0\pi^-}(0)\right|^2I_{K\ell}^{(0)}S_{EW}\boxed{\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)}$$




For $\Gamma_{Kl2}/\Gamma_{\pi l2}$ At leading order in ChPT both $\delta_{EM}$ and $\delta_{SU(2)}$ can be expressed in terms of physical quantities (e.m. pion mass splitting, $f_K/f_\pi$, …)

- $\boxed{\delta_{EM}=-0.0069(17)}$ 25% of error due to higher orders ➡ 0.2% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  M.Knecht *et al.*, 2000; V.Cirigliano and H.Neufeld, 2011

- $\boxed{\delta_{SU(2)}=\left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi}\right)^2-1=-0.0044(12)}$ 25% of error due to higher orders ➡ 0.1% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  J.Gasser and H.Leutwyler, 1985; V.Cirigliano and H.Neufeld, 2011

**ChPT** is not applicable to D and B decays

# Real photon emission amplitude

By setting $p_\gamma^2 = 0$, at fixed meson mass, the form factors depend on $p_H \cdot p_\gamma$ only.
Moreover, by choosing a *physical* basis for the polarization vectors, *i.e.* $\epsilon_r(\mathbf{p}_\gamma) \cdot p_\gamma = 0$, one has

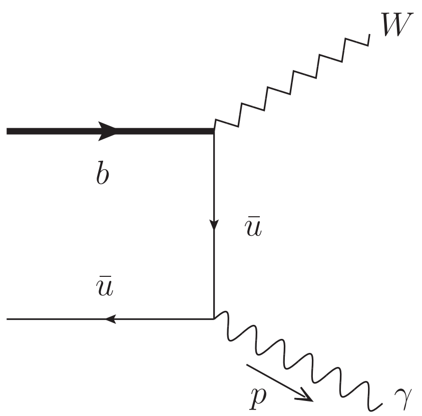
$$\epsilon_\mu^r(\mathbf{p}_\gamma)\, T^{\mu\nu}(p_\gamma, p_H) = \epsilon_\mu^r(\mathbf{p}_\gamma) \left\{ \varepsilon^{\mu\nu\tau\rho}(p_\gamma)_\tau v_\rho F_V + i\left[-g^{\mu\nu}(p_\gamma \cdot v) + v^\mu p_\gamma^\nu\right] F_A - i\frac{v^\mu v^\nu}{p_\gamma \cdot v} m_H f_H \right\}$$

In the case of off-shell photons $(p_\gamma^2 \neq 0) \longrightarrow \Gamma[H \to \ell \nu_\ell \ell^+ \ell^-]$ expressed in terms of 4 form factors

For large photon energies and in the B-meson rest frame the form factors can be written as

$$F_V(E_\gamma) = \frac{e_u M_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \xi(E_\gamma) + \Delta\xi(E_\gamma)$$

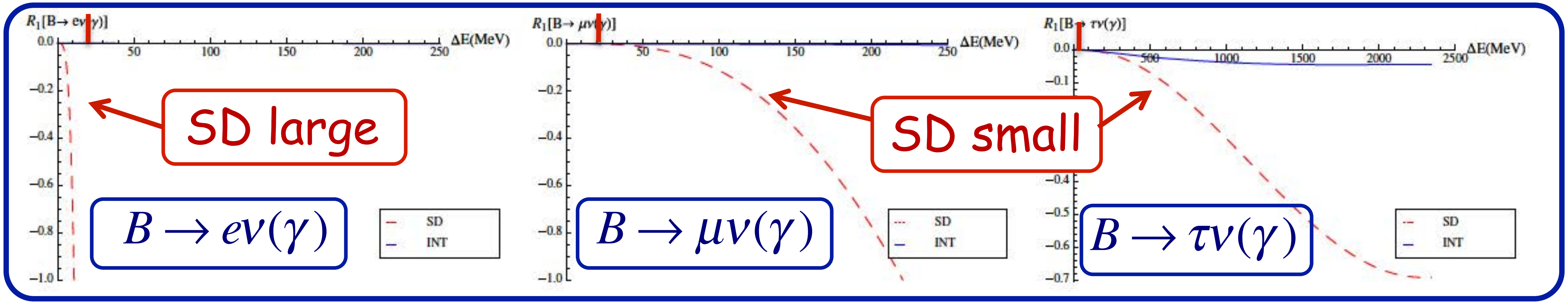
$$F_A(E_\gamma) = \frac{e_u M_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \xi(E_\gamma) - \Delta\xi(E_\gamma)$$




M. Beneke and J. Rohrwild, 2011

# Structure dependent contributions to decays of D and B mesons

- For the studies of D and B mesons decays we cannot apply ChPT
- For B mesons in particular we have another small scale, $m_{B^*} - m_B \simeq 45 \text{ MeV}$
  ➡ the radiation of a soft photon may still induce sizeable SD effects
- A phenomenological analysis based on a simple pole model for $F_V$ and $F_A$ confirms this picture
  D. Becirevic *et al.*, PLB 681 (2009) 257




$$F_V \simeq \frac{\tilde{C}_V}{1-(p_B-k)^2/m_{B^*}^2}$$

$$F_A \simeq \frac{\tilde{C}_A}{1-(p_B-k)^2/m_{B_1}^2}$$

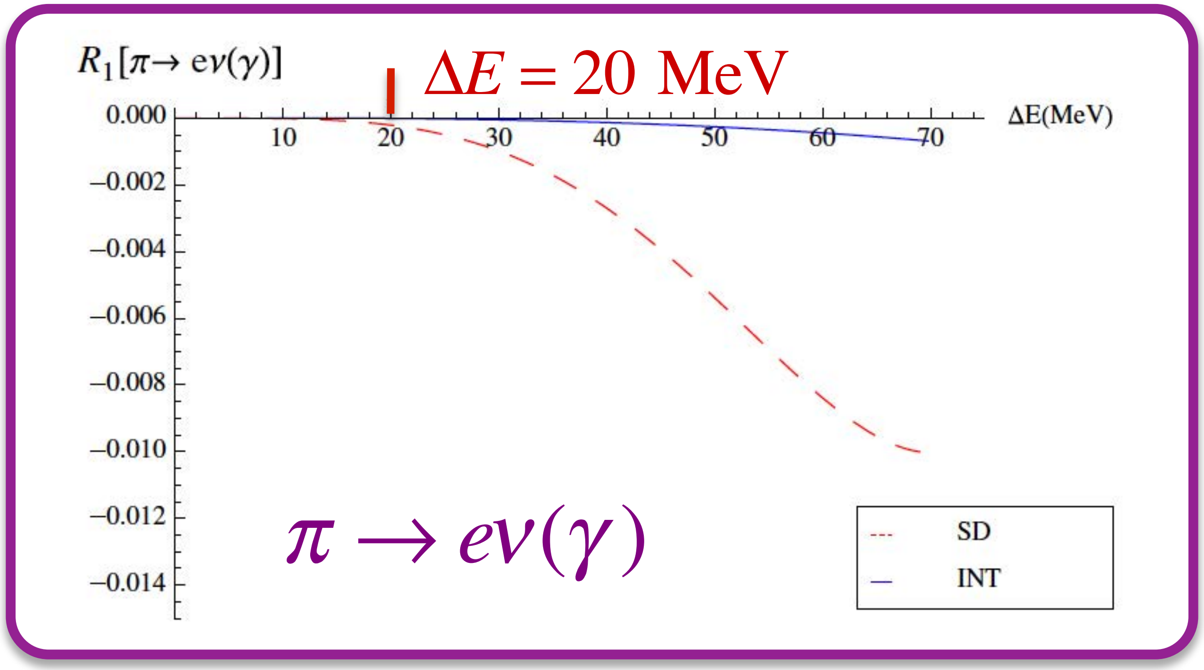
Under this assumption the SD contributions to $B \to e\nu(\gamma)$ for $E_\gamma \simeq 20$ MeV can be very large, but are small for $B \to \mu\nu(\gamma)$ and $B \to \tau\nu(\gamma)$

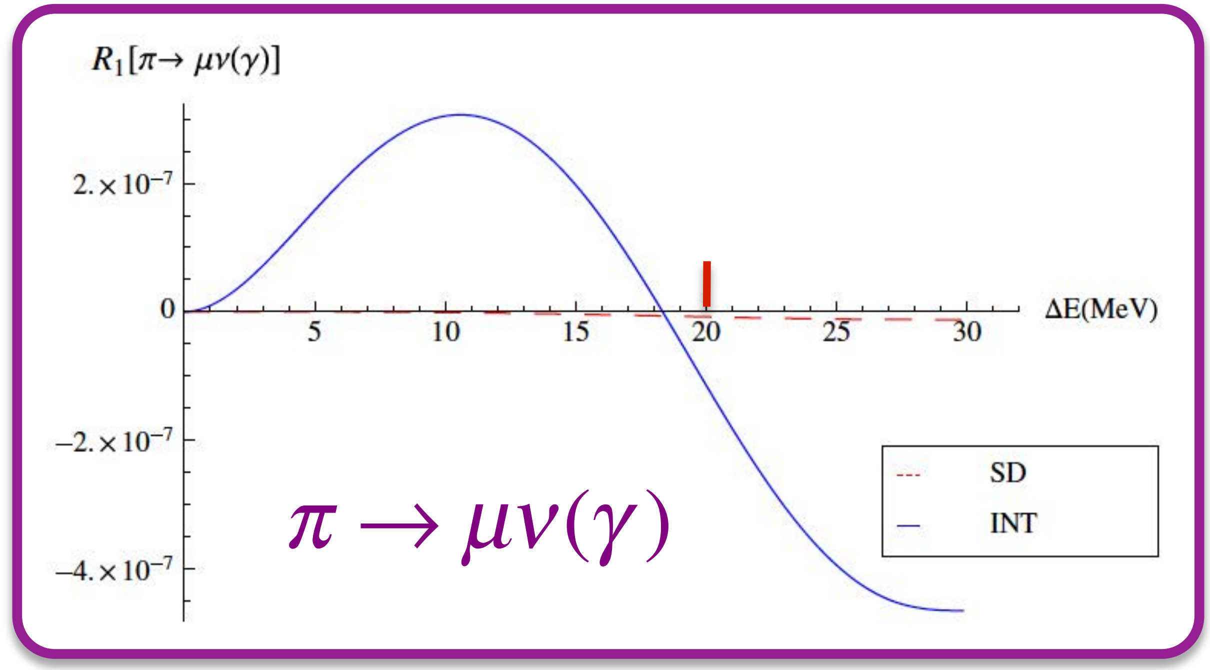
A lattice calculation of $F_V$ and $F_A$ would be very useful

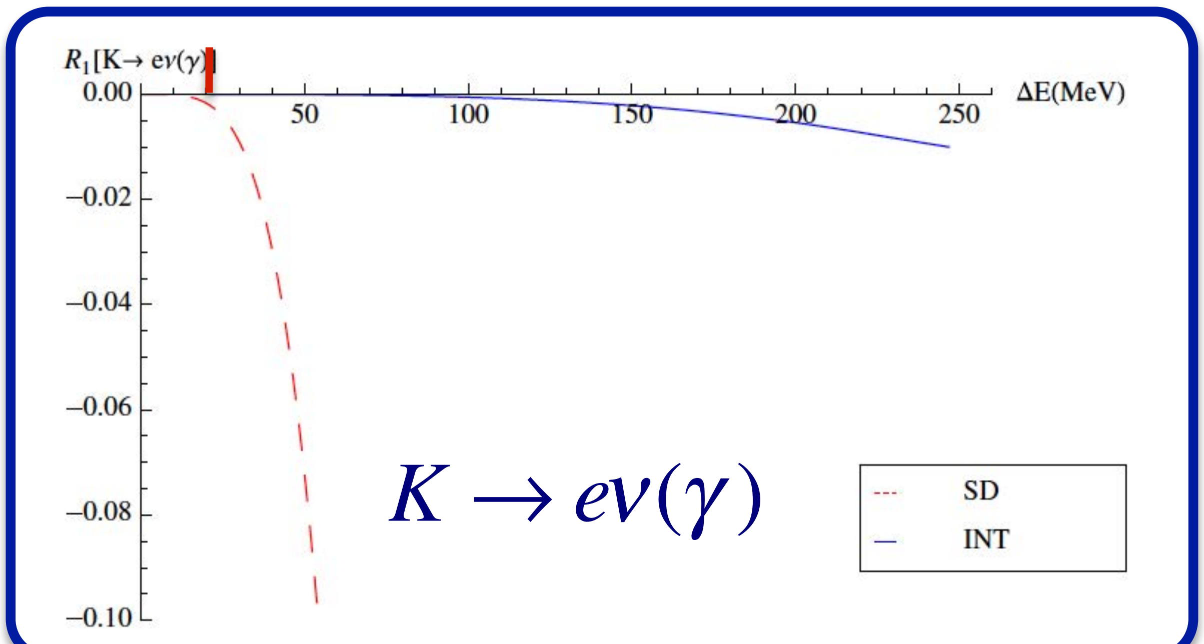
$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)} \quad , \quad A = \{\mathrm{SD}, \mathrm{INT}\}$$

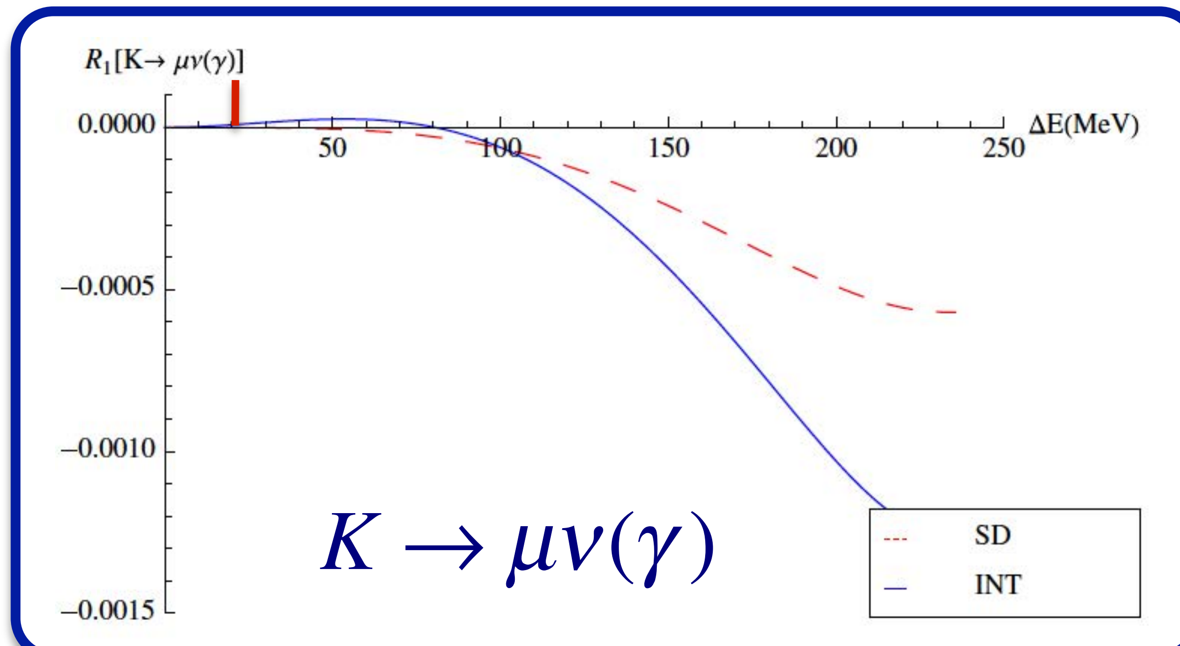
SD = structure dependent

INT = interference



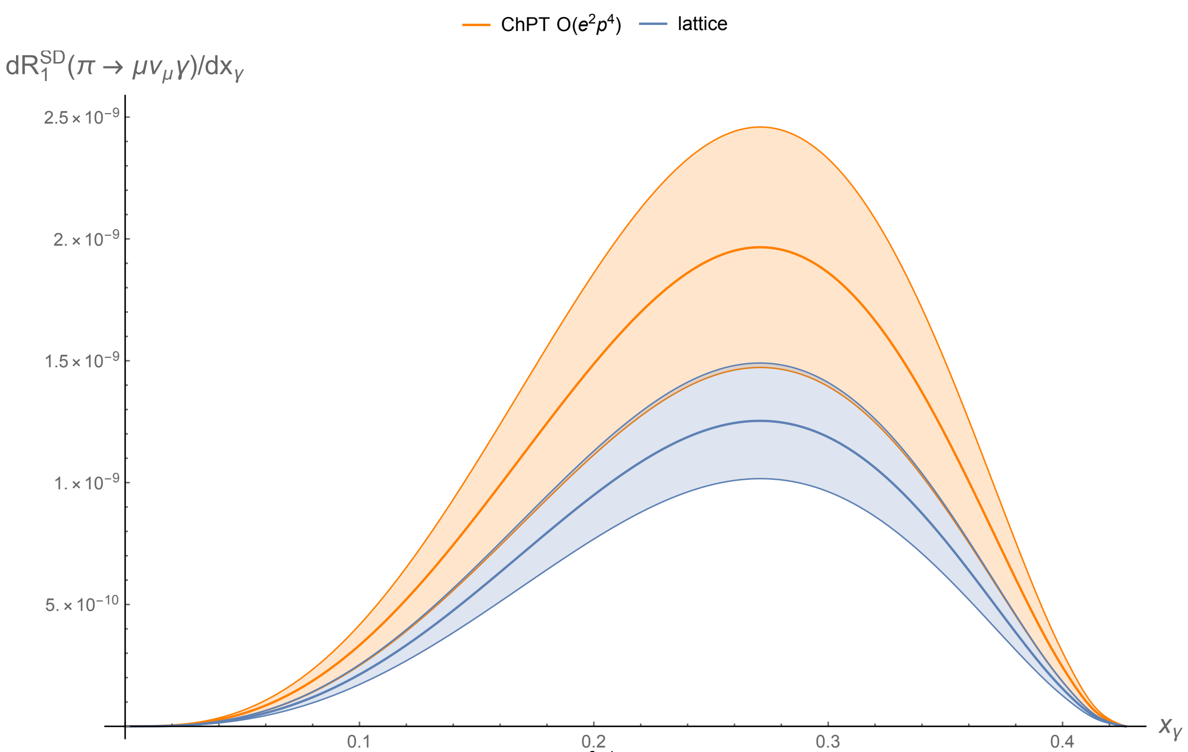



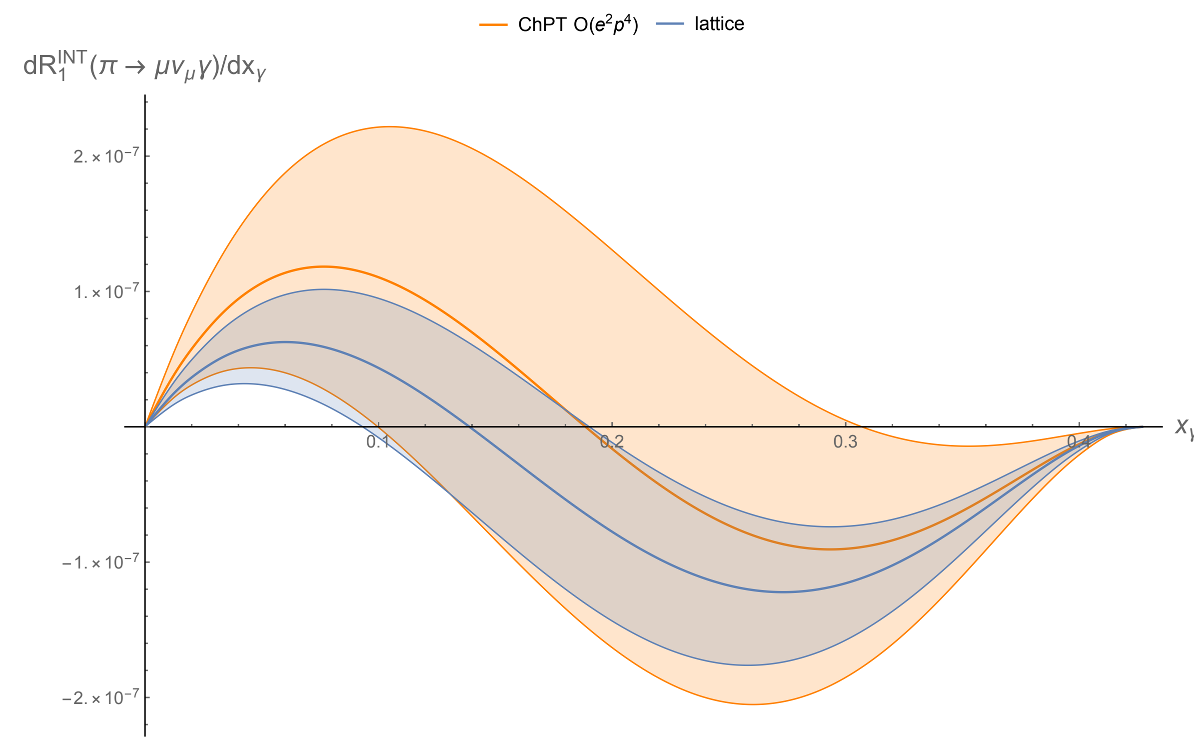



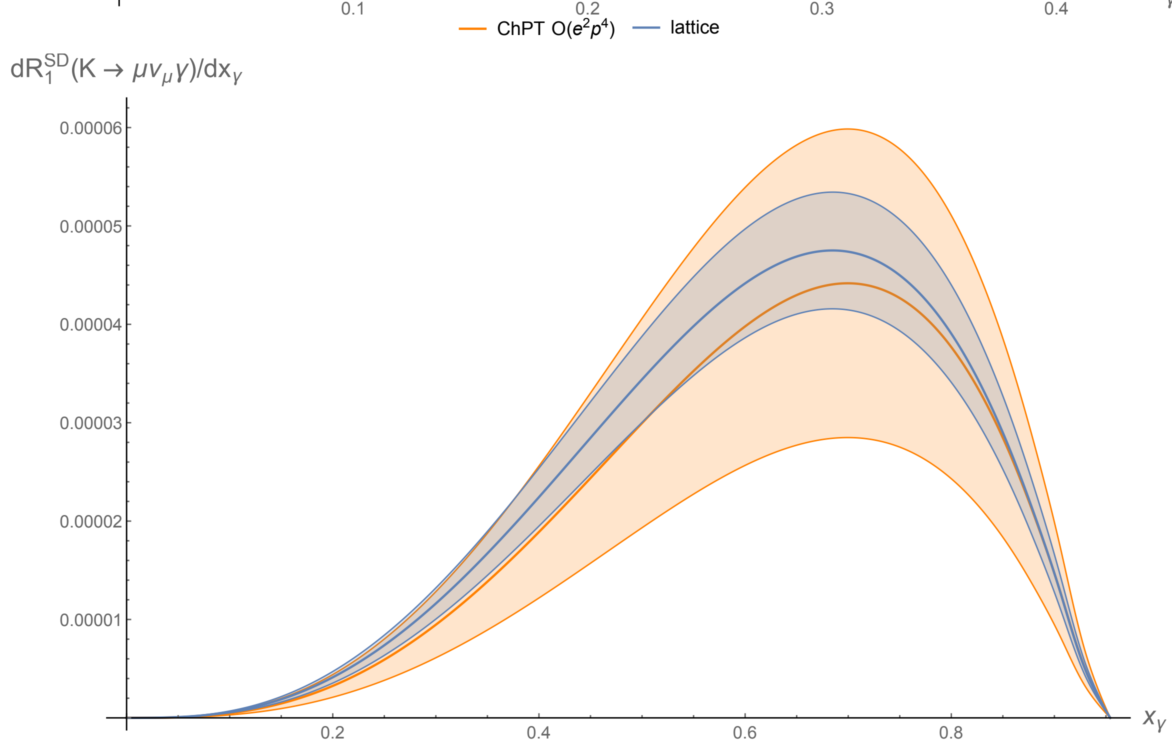

- Interference contributions are negligible in all the decays
- Structure-dependent contributions can be sizable for $K \to e\nu(\gamma)$ but they are negligible for $\Delta E < 20\ \mathrm{MeV}$ (which is experimentally accessible)

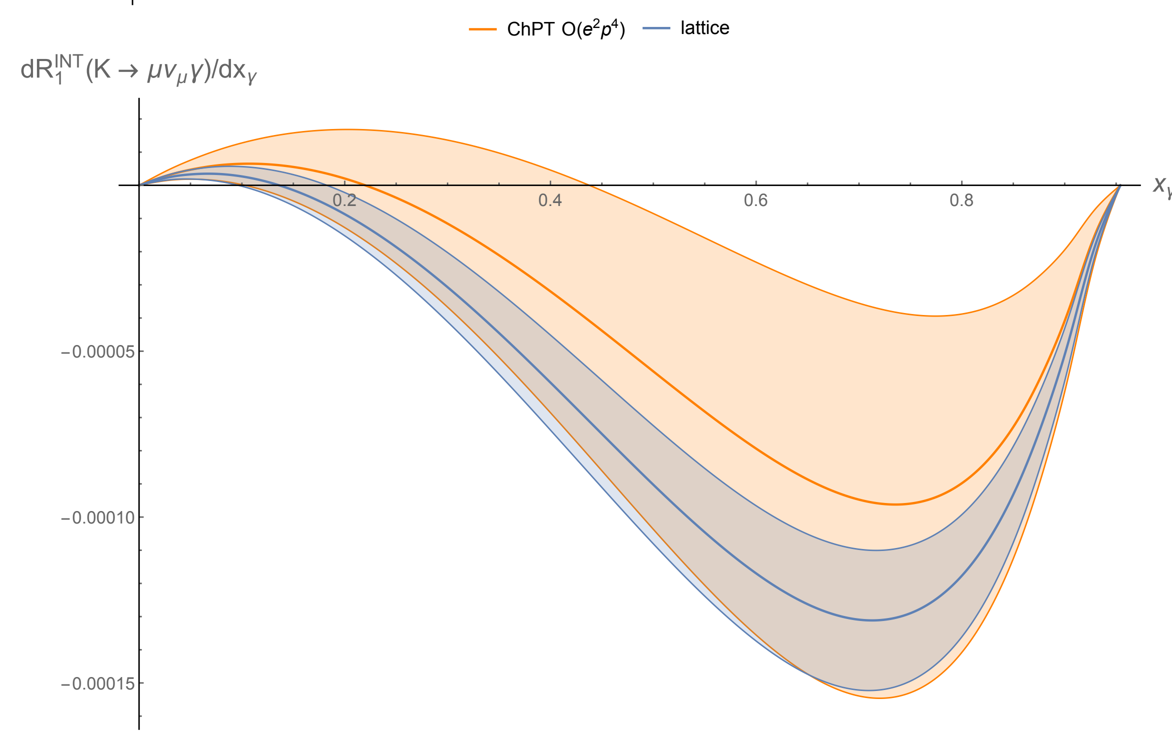
$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\,\frac{d\Gamma_1^{\rm SD}}{dx_\gamma} = \frac{m_P^2}{6 f_P^2\, r_\ell^2\, (1-r_\ell^2)^2}\left[F_V(x_\gamma)^2 + F_A(x_\gamma)^2\right] f^{\rm SD}(x_\gamma)$$

$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\,\frac{d\Gamma_1^{\rm INT}}{dx_\gamma} = -\frac{2\, m_P}{f_P\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)\, f_V^{\rm INT}(x_\gamma) + F_A(x_\gamma)\, f_A^{\rm INT}(x_\gamma)\right]$$



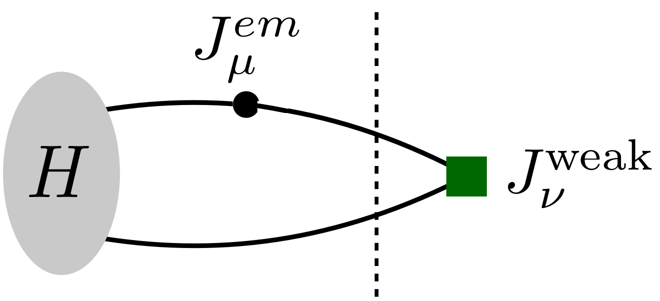



# Analytic continuation from Minkowski to Euclidean spacetime [2]

Time ordering: $t_{em} < 0$




$$T_{\mu\nu}^{<} = -\sum_n \frac{\langle 0| J_\nu^{\mathrm{weak}}(0) |n(\vec{p}_H - \vec{p}_\gamma)\rangle \langle n(\vec{p}_H - \vec{p}_\gamma)| J_\mu^{\mathrm{em}}(0) |H(\vec{p}_H)\rangle}{2E_{n,\vec{p}_H - \vec{p}_\gamma}(E_\gamma + E_{n,\vec{p}_H - \vec{p}_\gamma} - E_{H,\vec{p}_H})}$$

$$I_{\mu\nu}^{<}(t_H, T) = \int_{-T}^{0} dt_{\mathrm{em}}\, e^{E_\gamma t_{\mathrm{em}}} C_{3,\mu\nu}(t_{\mathrm{em}}, t_H)$$

$$= \sum_{l,n} \frac{\langle 0| J_\nu^{\mathrm{weak}}(0) |n(\vec{p}_H - \vec{p}_\gamma)\rangle \langle n(\vec{p}_H - \vec{p}_\gamma)| J_\mu^{\mathrm{em}}(0) |l(\vec{p}_H)\rangle \langle l(\vec{p}_H)| \phi_H^\dagger(0) |0\rangle}{2E_{n,\vec{p}_H - \vec{p}_\gamma} 2E_{l,\vec{p}_H}(E_\gamma + E_{n,\vec{p}_H - \vec{p}_\gamma} - E_{l,\vec{p}_H})}$$
$$\times\, e^{E_{l,\vec{p}_H} t_H} \left[1 - e^{-(E_\gamma - E_{l,\vec{p}_H} + E_{n,\vec{p}_H - \vec{p}_\gamma})T}\right]$$

Since the electromagnetic current operator cannot change the flavor quantum numbers of a state, the lowest-energy state appearing in the sum over $n$ is the meson $H$.

The unwanted exponential vanishes if $|\vec{p}_\gamma| + \sqrt{m_H^2 + (\vec{p}_H - \vec{p}_\gamma)^2} > \sqrt{m_H^2 + \vec{p}_H^2}$ , which is always true for $|\vec{p}_\gamma| > 0$

# Infinite-volume approximation

We assume there exist $c, d, \Lambda, \Lambda' \in \mathbb{R}^+$ and $L_0 \in \mathbb{N}$ for which

$$\tilde{C}^L(q) \equiv \sum_{x=-L/2}^{L/2-1} C^L(x) e^{iqx}$$

for all $x$ with $-L/2 \leq x \leq L/2$ and $L \geq L_0$ and

$$|C^\infty(x)| \leq d e^{-\Lambda' |x|}$$

for all $x$ with $|x| > L/2$. We now define

$$|C^\infty(x) - C^L(x)| \leq c e^{-\Lambda L} \qquad \text{and} \qquad \tilde{C}^\infty(q) \equiv \sum_{x=-\infty}^{\infty} C^\infty(x) e^{iqx} \,.$$

Under the above assumptions, it then follows that there is a $\tilde{c} \in \mathbb{R}^+$ for which

$$|\tilde{C}^\infty(q) - \tilde{C}^L(q)| \leq \tilde{c} e^{-\Lambda_0 L}$$

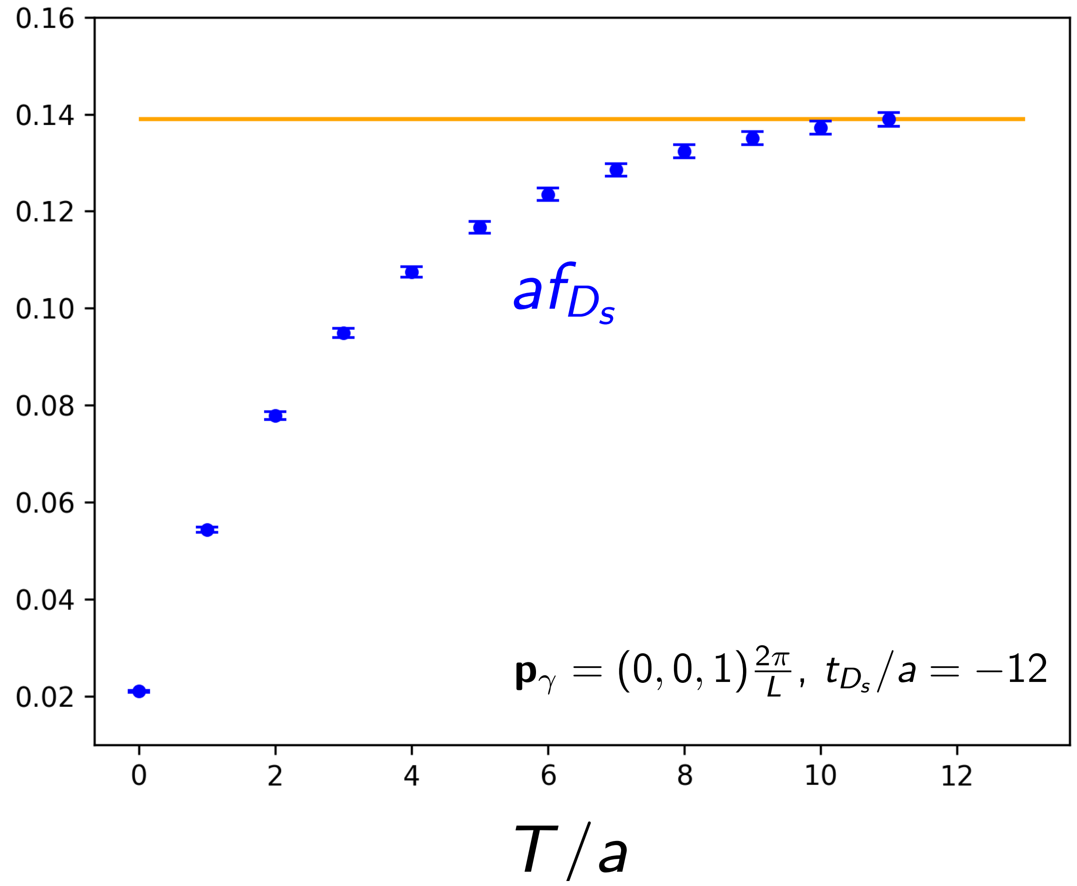
for all $q \in [-\pi, \pi]$ and all $L \geq L_0$, with $\Lambda_0 \equiv \min(\Lambda, \Lambda'/2)$.

# Cross-checks

Recall

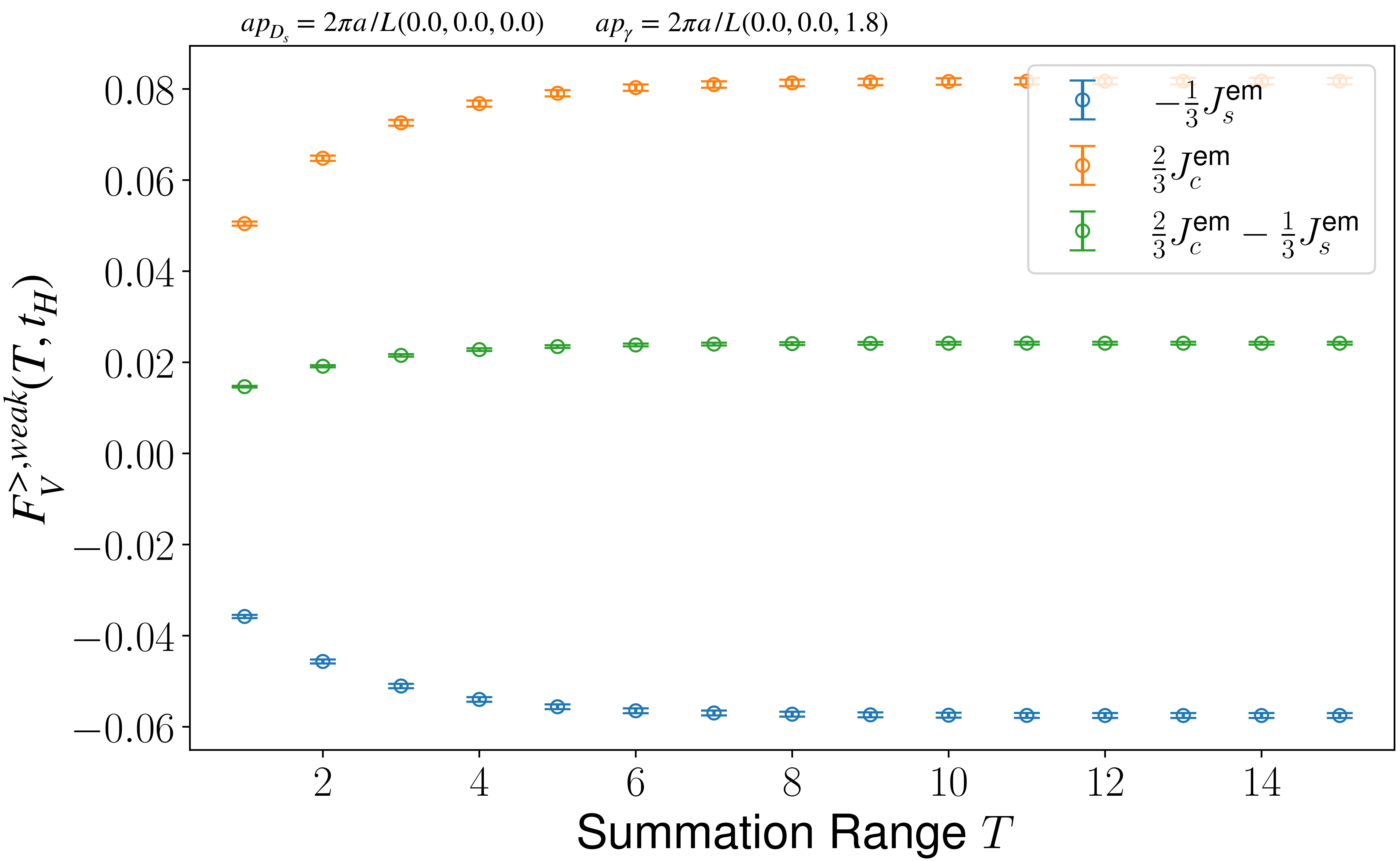
$$T_{\mu\nu} = \epsilon_{\mu\nu\tau\rho} p_\gamma^\tau v^\rho F_V + i[-g_{\mu\nu}(p_\gamma \cdot v) + v_\mu (p_\gamma)_\nu] F_A - i \frac{v_\mu v_\nu}{p_\gamma \cdot v} m_{D_s} f_{D_s}$$
$$+ (p_\gamma)_\mu\text{-terms}$$

$\longrightarrow$ also extract $f_{D_s}$ as a cross-check




Yellow line = FLAG 2021 average

# Cancellation between quark components



# Fit form: 3d method

Include terms to fit

(1) unwanted exponential from first intermediate state
(2) first excited state

Fit form factors $F_V$ and $F_{A,SD}$ directly instead of $I_{\mu\nu}$

$$\boxed{t_H < t_{em} < 0 \quad t_H < 0 < t_W}$$

$$F_<^{weak}(t_H, T) = F^< + B_F^< \left(1 + B_{F,exc}^< \overbrace{e^{\Delta E(T+t_H)}}\right) \overbrace{e^{-(E_\gamma - E_H + E^<)T}} + C_F^< \overbrace{e^{\Delta E t_H}}$$

$$F_>^{em}(t_H, T) = F^< + B_F^< \left[1 + B_{F,exc}^< \frac{E_\gamma + E^< - (\Delta E + E_H)}{E_\gamma + E^< - E_H} e^{\Delta E t_H}\right] e^{-(E_\gamma - E_H + E^<)T} + \tilde{C}_F^< e^{\Delta E t_H}$$

$$\boxed{t_H < 0 < t_{em} \quad t_H < t_W < 0}$$

$$F_>^{weak}(t_H, T) = F^> + B_F^> \left(1 + B_{F,exc}^> e^{\Delta E t_H}\right) e^{(E_\gamma - E^>)T} + C_F^> e^{\Delta E t_H}$$

$$F_<^{em}(t_H, T) = F^> + B_F^> \left[1 + B_{F,exc}^> \frac{E_\gamma - E^>}{E_\gamma - E^> + \Delta E} e^{\Delta E(T+t_H)}\right] e^{(E_\gamma - E^>)T} + \tilde{C}_F^> e^{\Delta E t_H}$$

Only have two values of $t_H$, fitting multiple exponentials not possible

$\rightarrow$ Determine $\Delta E$ from the pseudoscalar two-point correlation function

$\rightarrow$ use result as Gaussian prior in form factor fits

# $D_s \to \ell \nu_\ell \gamma$: 3d method