Prospect for the stout smearing as an equivalent approach to the Wilson flow

M. Nagatsuka, K. Sakai, S. Sasaki
Tohoku University

Motivation

- Spatial stout smearing and spatial gradient flow almost shows same behavior in 2pt fuctions of glueballs. K. Sakai, S. Sasaki, PRD107, 034510 (2022)

- Same feature was previously observed using plaquette values. S. D. Thomas et al., PRD92, 094515 (2015)

- We want to know the parameter dependence of difference between two method.
Yang-Mills Gradient flow

A time evolution equation toward 5th direction $t$ (flow time)

$$\frac{\partial B_\mu}{\partial t}(x, t) = D_\nu G_{\nu\mu}(x, t)$$

$$B_\mu(x, t) \mid_{t=0} = A_\mu(x) : \text{Initial condition}$$

Correlation functions made of $B_\mu(x, t)$ are always UV finite without the wave function renormalization.

Gradient flow is regarded as 4+1 dimension field theory in continuous space time.

→ one can use perturbation theory.
Wilson flow (Yang-Mills Gradient flow)

M. Lüscher JHEP 08 (2010) 071

A time evolution equation toward 5th direction $t$ (flow time)

$$\frac{\partial V_\mu}{\partial t}(x, t)V_\mu^{-1}(x, t) = - g_0^2 \partial_{x, \mu} S_W(V_\mu(x, t))$$

$$V_\mu(x, t) \bigg|_{t=0} = U_\mu(x)$$

: Initial condition

Correlation functions made of $V_\mu(x, t)$ are always UV finite without the wave function renormalization.

M. Lüscher, P. Weisz, JHEP 02 (2011) 051

Applications

- Construction of Energy Momentum tensor on the lattice
  H. Suzuki, PTEP 2013, 083B03 (2013)

- Non-perturbative Renormalization
  A. Ramos, JHEP 11, 101 (2014)
Stout-link smearing

\[ U^{(n+1)}_{\mu}(x) = e^{i\rho Q^{(n)}_{\mu}(x)} U^{(n)}_{\mu}(x) \]

It is used to reduce a statistical error in numerical simulations. \( U^{(n)}_{\mu}(x) \in SU(3) \) is always true.

\[
C_\mu(x) = \sum_{\nu \neq \mu} \left( U_{\nu}(x) U_{\mu}(x + \hat{\nu}) \Omega_\nu^\dag(x + \hat{\mu}) + \Omega_\nu^\dag(x - \hat{\nu}) U_{\mu}(x - \hat{\nu}) U_\nu(x - \hat{\nu} + \hat{\mu}) \right)
\]

\[\Omega_\mu(x) = C_\mu(x) U_\mu^\dag(x)\]

\[Q_\mu(x) = \frac{i}{2} \left( \Omega_\mu^\dag(x) - \Omega_\mu(x) \right) - \frac{i}{6} \text{Tr} \left( \Omega_\mu^\dag(x) - \Omega_\mu(x) \right) \]

(Traceless Hermitian matrix)
Equivalence of two methods

Step 1.

\[ U^{(n+1)}_{\mu}(x) = e^{i\rho Q^{(n)}_{\mu}(x)} U^{(n)}_{\mu}(x) \]

\[ U_{\mu}(x, t + \rho) = U_{\mu}(x, t) + i\rho Q_{\mu}(x, t) U_{\mu}(x, t) + \frac{1}{2} \left( i\rho Q_{\mu}(x, t) \right)^2 U_{\mu}(x, t) + \cdots \]

\[ \rho \to 0 \quad \frac{\partial U_{\mu}(x, t)}{\partial t} = i Q^{(n)}_{\mu}(x) U_{\mu}(x, t) \]

\[ t = \rho n \]

Step 2.

\[ g_0^2 \partial_{x,\mu} S_W = -i Q_{\mu}(x, t) \]

\[ \frac{\partial U_{\mu}}{\partial t}(x, t) U_{\mu}^{-1}(x, t) = -g_0^2 \partial_{x,\mu} S_W \]

- Stout-link smearing = Wilson flow at \( \rho \to 0 \) (independent of \( a \)).
- LO correction \( O(\rho) \) induce the \( O(a^2) \) correction.
Setups

- Simulation at four lattice spacing \( a \). 100 statistics for each \( a \).

- Comparison of the energy density \( \langle E(t) \rangle \) (flow time) = \( \rho \times (\text{smearing step}) \) : analytical relation

These are scaled by \( a^2 \).

- We smear the gauge configurations with \( \rho = 0.1, 0.025, 0.01 \) for each couplings (12 combinations of \( \rho \) and \( a \)).

- Wilson flow is calculated using 4th order Runge-Kutta method with \( \epsilon = 0.025 \) (fixed, independent of choice).

<table>
<thead>
<tr>
<th>( \beta = 6/g_0^2 )</th>
<th>( L^3 \times T )</th>
<th>( a ) [fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.76</td>
<td>16^3 \times 32</td>
<td>0.1486(7)</td>
</tr>
<tr>
<td>5.96</td>
<td>24^3 \times 48</td>
<td>0.1000(5)</td>
</tr>
<tr>
<td>6.17</td>
<td>32^3 \times 64</td>
<td>0.0708(3)</td>
</tr>
<tr>
<td>6.42</td>
<td>48^3 \times 96</td>
<td>0.0500(2)</td>
</tr>
</tbody>
</table>
Behavior of $X(t) = t^2 \langle E(t) \rangle$

- $X(t)$ is dimensionless.
- The gray line and the yellow band is the perturbative result.
- Similar behavior in two methods.
Evaluation of the difference

\[ R(t) = \frac{X_{\text{stout}}(t) - X_{\text{flow}}(t)}{X_{\text{flow}}(t)} \]

The gray band is noise/signal ratio of \( X_{\text{flow}}(t) \): \( \Delta X_{\text{flow}} / X_{\text{flow}} \)

\( R(t) \to 0 \) as \( \rho, a \to 0 \).

\( \beta = 5.96 \) fixed

\( \rho = 0.1 \) fixed

(We scale \( t \) with \( t_{0.3} \) satisfying \( X|_{t=t_{0.3}} = 0.3 \))
**Dependence of $R(t = t_{0.3})$**

Evaluation of $R(t = t_{0.3})$ \( t_{0.3} \text{ satisfies } t^2 \langle E \rangle |_{t=t_{0.3}} = 0.3 \)

Dashed line: $\Delta X_{\text{flow}}/X_{\text{flow}}$

\[ R(t_{0.3}) \]

\[ a^2/t_0 \]
(a) dependence of \( R(t = t_{0.3}) \)

Evaluation of \( R(t = t_{0.3})/\rho \) \( \left( t_{0.3} \text{satisfies } t^2 \langle E \rangle |_{t=t_{0.3}} = 0.3 \right) \)}
Summary and future works

- The Wilson flow and the stout-link smearing is equivalent at $\rho \to 0$.
- We can regard the two methods are same within some numerical precision if we choose proper combinations.

- Calculation with Stout-link smearing is a factor of $O(10)$ faster than Wilson flow → efficient simulations
- Perturbation theory in the gradient flow formalism can be use for the calculation of one loop quantities in lattice perturbation theory for smeared-link fermion action.
Backup
Energy density $\langle E \rangle$

Traceless

$$G_{\mu\nu}(x, t) = \text{Anti-Hermitian part of}$$

$$= \frac{1}{4} \left( V_{\mu}(x, t)V_{\nu}(x + \hat{\mu}, t)^\dagger V_{\mu}(x + \hat{\nu}, t)V_{\nu}^\dagger(x, t) ight.$$

$$+ V_{\nu}^\dagger(x - \hat{\nu}, t)V_{\mu}(x - \hat{\nu}, t)V_{\nu}(x + \hat{\mu} - \hat{\nu}, t)V_{\mu}^\dagger(x, t)$$

$$+ V_{\mu}^\dagger(x - \hat{\mu}, t)V_{\nu}^\dagger(x - \hat{\mu} - \hat{\nu}, t)V_{\mu}(x - \hat{\mu} - \hat{\nu}, t)V_{\nu}(x - \hat{\nu}, t)$$

$$+ V_{\nu}^\dagger(x, t)V_{\mu}(x - \hat{\mu} + \hat{\nu}, t)V_{\nu}(x - \hat{\mu}, t)V_{\mu}^\dagger(x - \hat{\mu}, t) \right)_{AH}$$

$$E = \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a : \text{Energy density}$$
Gauge conf.

TABLE I. Summary of the gauge ensembles: gauge coupling, lattice size ($L^3 \times T$), plaquette value, lattice spacing ($a$), spatial extent ($La$), the Sommer scale ($r_0$), the number of the gauge field configurations ($N_{\text{conf}}$), the number of thermalization sweeps ($n_{\text{therm}}$) and the number of update sweeps ($n_{\text{update}}$). All lattice spacings are set by the Sommer scale ($r_0 = 0.5$ fm) [20, 21].

<table>
<thead>
<tr>
<th>$\beta = 6/g_0^2$</th>
<th>$L^3 \times T$</th>
<th>plaquette</th>
<th>$a$ [fm]</th>
<th>$La$ [fm]</th>
<th>$r_0/a$ (Ref. [21])</th>
<th>$N_{\text{conf}}$</th>
<th>$n_{\text{therm}}$</th>
<th>$n_{\text{update}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.76</td>
<td>$16^3 \times 32$</td>
<td>0.560938(9)</td>
<td>0.1486(7)</td>
<td>2.38</td>
<td>3.364(17)</td>
<td>100</td>
<td>5000</td>
<td>200</td>
</tr>
<tr>
<td>5.96</td>
<td>$24^3 \times 48$</td>
<td>0.589159(3)</td>
<td>0.1000(5)</td>
<td>2.40</td>
<td>5.002(25)</td>
<td>100</td>
<td>2000</td>
<td>200</td>
</tr>
<tr>
<td>6.17</td>
<td>$32^3 \times 64$</td>
<td>0.610867(1)</td>
<td>0.0708(3)</td>
<td>2.27</td>
<td>7.061(35)</td>
<td>100</td>
<td>2000</td>
<td>200</td>
</tr>
<tr>
<td>6.42</td>
<td>$48^3 \times 96$</td>
<td>0.632217(1)</td>
<td>0.0500(2)</td>
<td>2.40</td>
<td>10.00(5)</td>
<td>100</td>
<td>2000</td>
<td>200</td>
</tr>
</tbody>
</table>
$t_0, \ w_0$

TABLE III. Results of $t_0/a^2$ obtained from the stout smearing with three smearing parameters $\rho = 0.1, 0.025, 0.01$ and the Wilson flow ($\epsilon = 0.025$).

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$t_0/a^2$ (stout-link smearing)</th>
<th>$t_0/a^2$ (Wilson flow)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho = 0.1$</td>
<td>$\rho = 0.025$</td>
</tr>
<tr>
<td>5.76</td>
<td>1.2502(30)</td>
<td>1.2690(30)</td>
</tr>
<tr>
<td>5.96</td>
<td>2.7714(62)</td>
<td>2.7919(62)</td>
</tr>
<tr>
<td>6.17</td>
<td>5.476(13)</td>
<td>5.494(13)</td>
</tr>
<tr>
<td>6.42</td>
<td>11.218(22)</td>
<td>11.236(23)</td>
</tr>
</tbody>
</table>

TABLE IV. Results of $w_0/a$ obtained from the stout-link smearing with three smearing parameters $\rho = 0.1, 0.025, 0.01$ and the Wilson flow ($\epsilon = 0.025$).

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$w_0/a$ (stout-link smearing)</th>
<th>$w_0/a$ (Wilson flow)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\rho = 0.1$</td>
<td>$\rho = 0.025$</td>
</tr>
<tr>
<td>5.76</td>
<td>1.1098(18)</td>
<td>1.1199(18)</td>
</tr>
<tr>
<td>5.96</td>
<td>1.6755(24)</td>
<td>1.6819(24)</td>
</tr>
<tr>
<td>6.17</td>
<td>2.3684(41)</td>
<td>2.3729(41)</td>
</tr>
<tr>
<td>6.42</td>
<td>3.4042(48)</td>
<td>3.4075(48)</td>
</tr>
</tbody>
</table>
Computational speed

Computational time for 100 step (single core, 2.5GHz)

<table>
<thead>
<tr>
<th>directions</th>
<th>stout-link smearing ($\rho = 0.025$)</th>
<th>Wilson flow ($\epsilon = 0.025$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2nd-order RK</td>
<td>3rd-order RK</td>
</tr>
<tr>
<td>space-time</td>
<td>136.80 [sec]</td>
<td>985.55 [sec]</td>
</tr>
<tr>
<td>space</td>
<td>90.39 [sec]</td>
<td>511.84 [sec]</td>
</tr>
</tbody>
</table>