Prospect for the stout smearing as an equivalent approach to the Wilson flow

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M. N, K. Sakai, S. Sasaki, arXiv:2303.09938 [hep-lat]

Motivation

- Spatial stout smearing and spatial gradient flow almost shows same behavior in 2pt fuctions of glueballs.
  K. Sakai, S. Sasaki, PRD107, 034510 (2022)

- Same feature was previously observed using plaquette values.
  S. D. Thomas et al., PRD92, 094515 (2015)

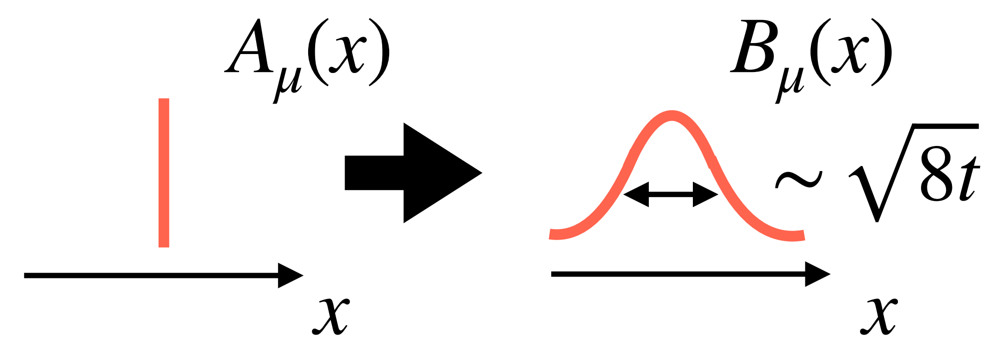
- We want to know the parameter dependence of difference between two method.

Yang-Mills Gradient flow

M. Lüscher JHEP **08** (2010) 071

A time evolution equation toward 5th direction t (flow time)

$$\frac{\partial B_\mu}{\partial t}(x,t) = D_\nu G_{\nu\mu}(x,t)$$



$$B_\mu(x,t)\,|_{t=0} = A_\mu(x)$$: Initial condition

Correlation functions made of $B_\mu(x,t)$ are always UV finite without the wave function renormalization.

M. Lüscher, P. Weisz, JHEP **02** (2011) 051

Gradient flow is regarded as 4+1 dimension field theory in continuous space time.
→one can use perturbation theory.

Wilson flow (Yang-Mills Gradient flow)

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A time evolution equation toward 5th direction t (flow time)

$$\frac{\partial V_\mu}{\partial t}(x,t)V_\mu^{-1}(x,t) = -g_0^2 \partial_{x,\mu} S_W(V_\mu(x,t))$$

$$V_\mu(x,t)|_{t=0} = U_\mu(x)$$: Initial condition

Correlation functions made of $V_\mu(x,t)$ are always UV finite without the wave function renormalization.

M. Lüscher, P. Weisz, JHEP **02** (2011) 051

Applications

- Construction of Energy Momentum tensor on the lattice

H. Suzuki, PTEP 2013, 083B03 (2013)

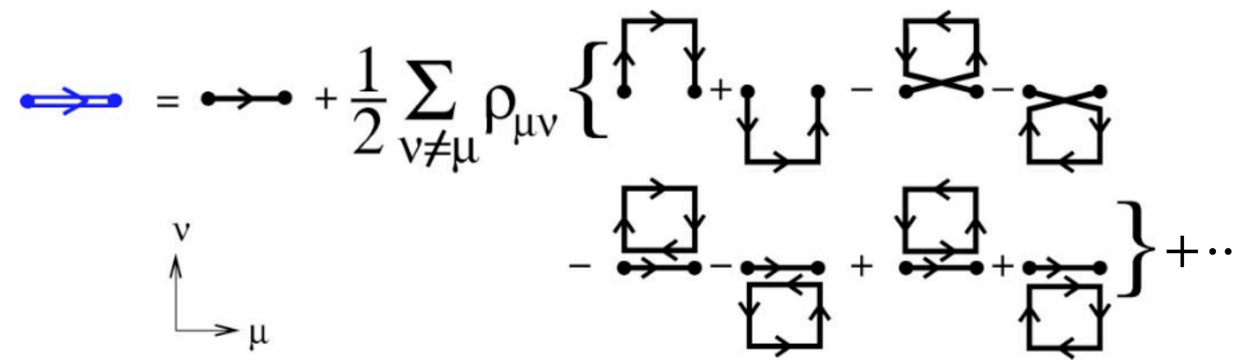
- Non-perturbative Renormalization

A. Ramos, JHEP 11, 101 (2014)

Stout-link smearing

C.Morningstar, M. Peardon (2004)

$$U_\mu^{(n+1)}(x) = e^{i\rho Q_\mu^{(n)}(x)} U_\mu^{(n)}(x)$$



It is used to reduce a statistical error in numerical simulations.

$U_\mu^{(n)}(x) \in \mathrm{SU}(3)$ is always true.

$$C_\mu(x) = \sum_{\nu\neq\mu} \left(U_\nu(x) U_\mu(x+\hat{\nu}) U_\nu^\dagger(x+\hat{\mu}) + U_\nu^\dagger(x-\hat{\nu}) U_\mu(x-\hat{\nu}) U_\nu(x-\hat{\nu}+\hat{\mu}) \right)$$

$$\Omega_\mu(x) = C_\mu(x) U_\mu^\dagger(x)$$

$$Q_\mu(x) = \frac{i}{2}\left(\Omega_\mu^\dagger(x) - \Omega_\mu(x)\right) - \frac{i}{6}\mathrm{Tr}\left(\Omega_\mu^\dagger(\mathrm{x}) - \Omega_\mu(\mathrm{x})\right)$$

(Traceless Hermitian matrix)

Equivalence of two methods

Step 1.

$$U_\mu^{(n+1)}(x) = e^{i\rho Q_\mu^{(n)}(x)} U_\mu^{(n)}(x)$$

$$U_\mu(x, t+\rho) = U_\mu(x,t) + i\rho Q_\mu(x,t) U_\mu(x,t) + \frac{1}{2}\left(i\rho Q_\mu(x,t)\right)^2 U_\mu(x,t) + \cdots$$

$$\xrightarrow{\rho\to 0} \quad \frac{\partial U_\mu(x,t)}{\partial t} = iQ_\mu^{(n)}(x) U_\mu(x,t) \qquad \begin{pmatrix} t = \rho n \\ U_\mu(x,t) = U_\mu^{(n)}(x) \end{pmatrix}$$

Step 2. $g_0^2 \partial_{x,\mu} S_W = -iQ_\mu(x,t)$

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$$\downarrow$$

$$\frac{\partial U_\mu}{\partial t}(x,t) U_\mu^{-1}(x,t) = -g_0^2 \partial_{x,\mu} S_W$$

- Stout-link smearing = Wilson flow at $\rho \to 0$ (independent of a).
- LO correction $O(\rho)$ induce the $O(a^2)$ correction.

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Setups

$\beta = 6/g_0^2$	$L^3 \times T$	a [fm]
5.76	$16^3 \times 32$	0.1486(7)
5.96	$24^3 \times 48$	0.1000(5)
6.17	$32^3 \times 64$	0.0708(3)
6.42	$48^3 \times 96$	0.0500(2)

- Simulation at four lattice spacing a. 100 statistics for each a.
- Comparison of the energy density $\langle E(t) \rangle$

  $(\text{flow time}) = \rho \times (\text{smearing step})$: analytical relation

  These are scaled by a^2.
- We smear the gauge configurations with $\rho = 0.1,\ 0.025,\ 0.01$ for each couplings (12 combinations of ρ and a).
- Wilson flow is calculated using 4th order Runge-Kutta method with $\epsilon = 0.025$ (fixed, independent of choice).

Behavior of $X(t) = t^2\langle E(t)\rangle$

t (flow time)

- $X(t)$ is dimensionless.

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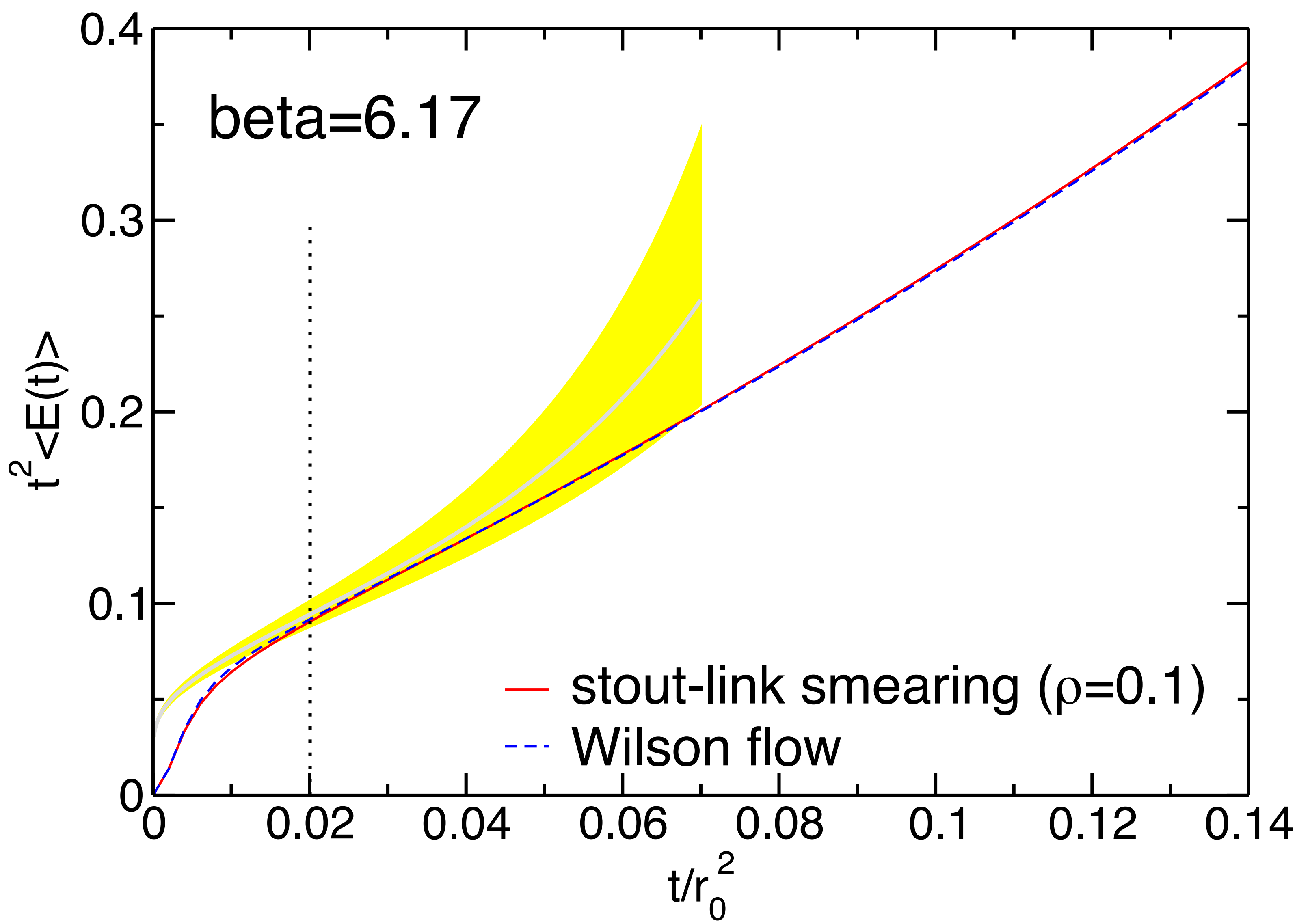
- The gray line and the yellow band is the perturbative result.
- Similar behavior in two methods.



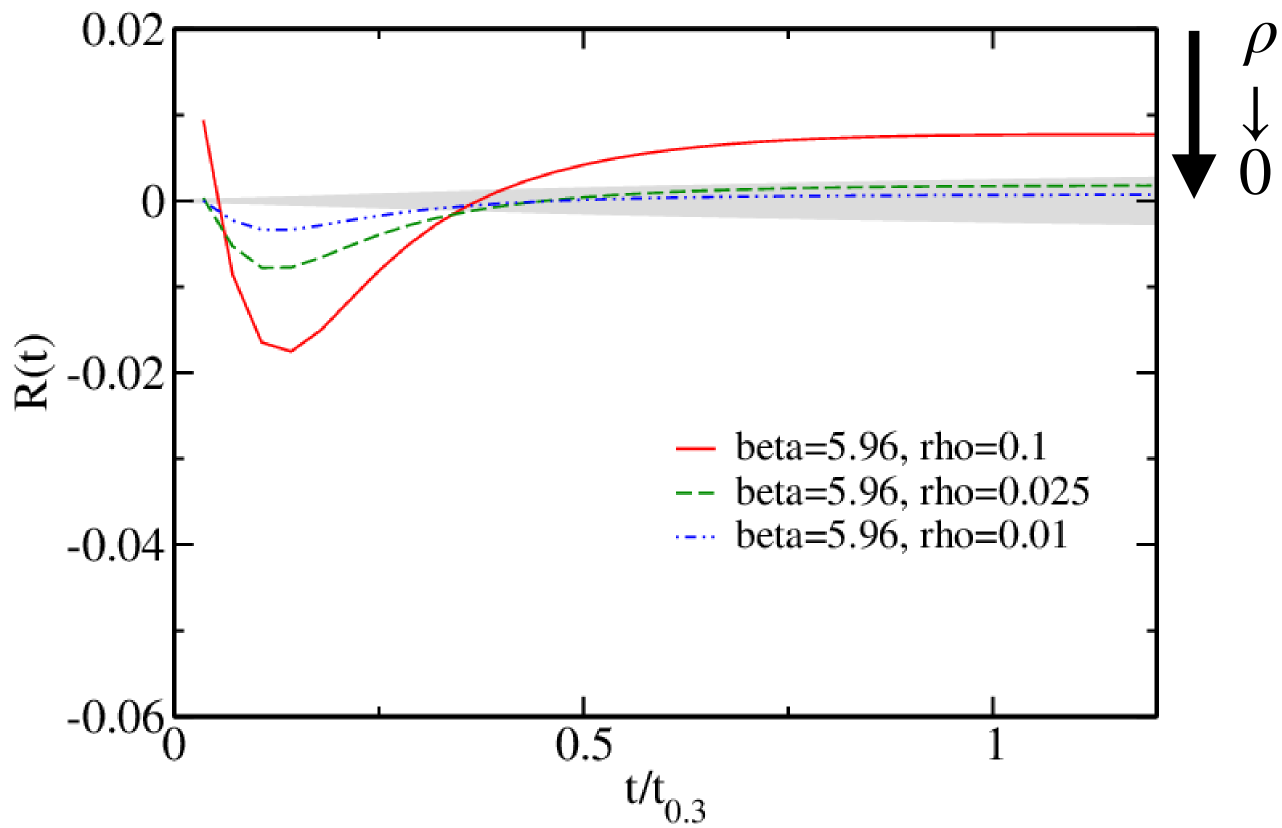
Evaluation of the difference

$$R(t) = \frac{X_{\mathrm{stout}}(t) - X_{\mathrm{flow}}(t)}{X_{\mathrm{flow}}(t)}$$

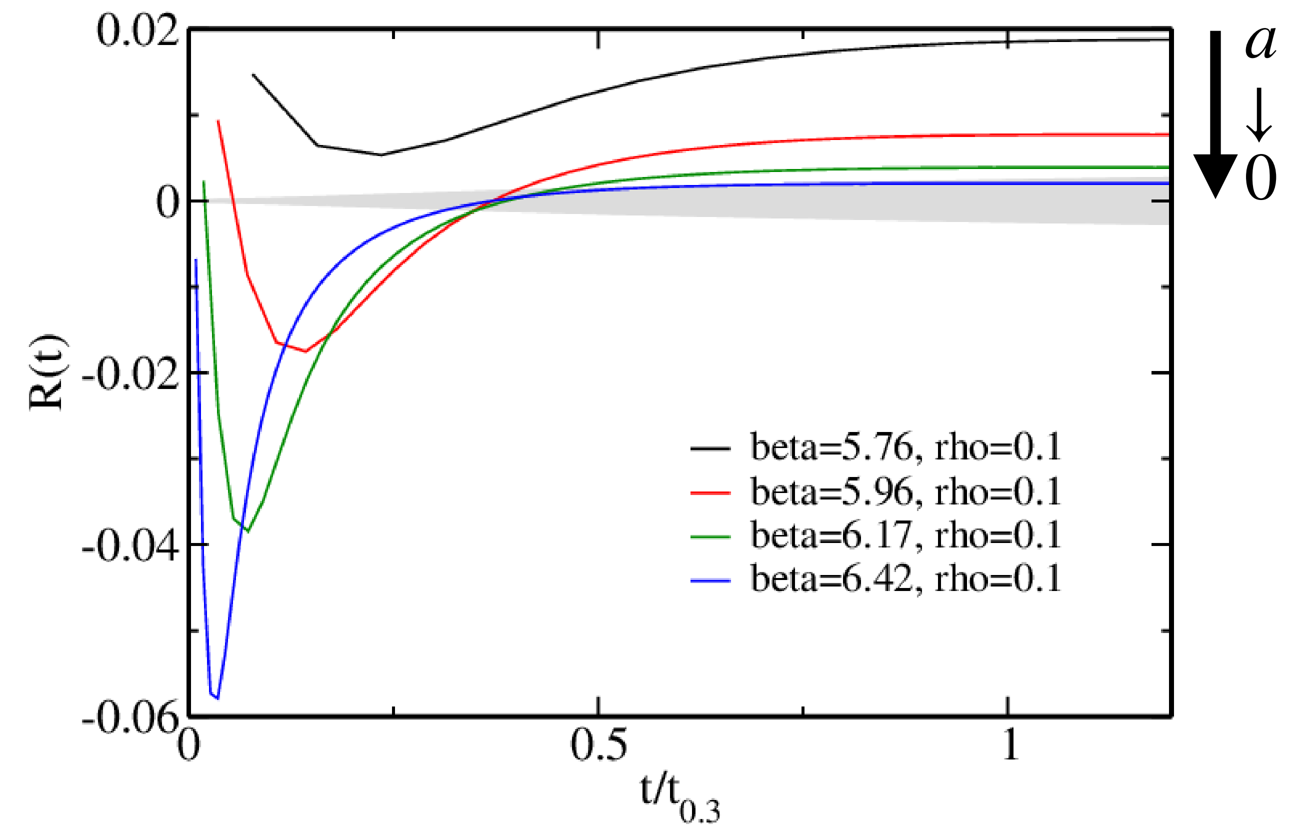
The gray band is noise/signal ratio of $X_{\mathrm{flow}}(t)$: $\Delta X_{\mathrm{flow}}/X_{\mathrm{flow}}$

$R(t) \to 0$ as $\rho, a \to 0$.

$\beta = 5.96$ fixed



$\rho = 0.1$ fixed

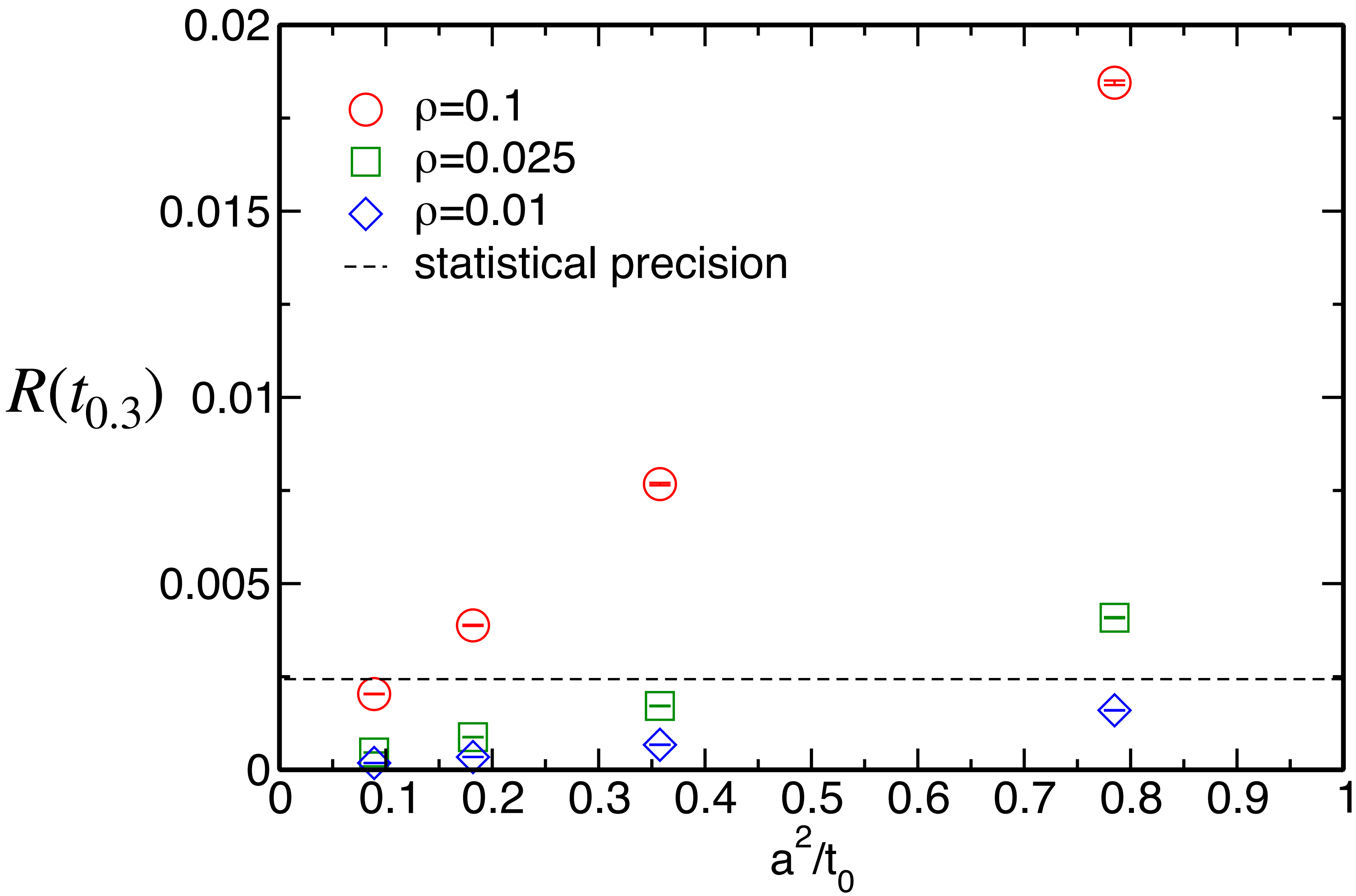


(We scale t with $t_{0.3}$ satisfying $X|_{t=t_{0.3}} = 0.3$)

a dependence of $R(t=t_{0.3})$

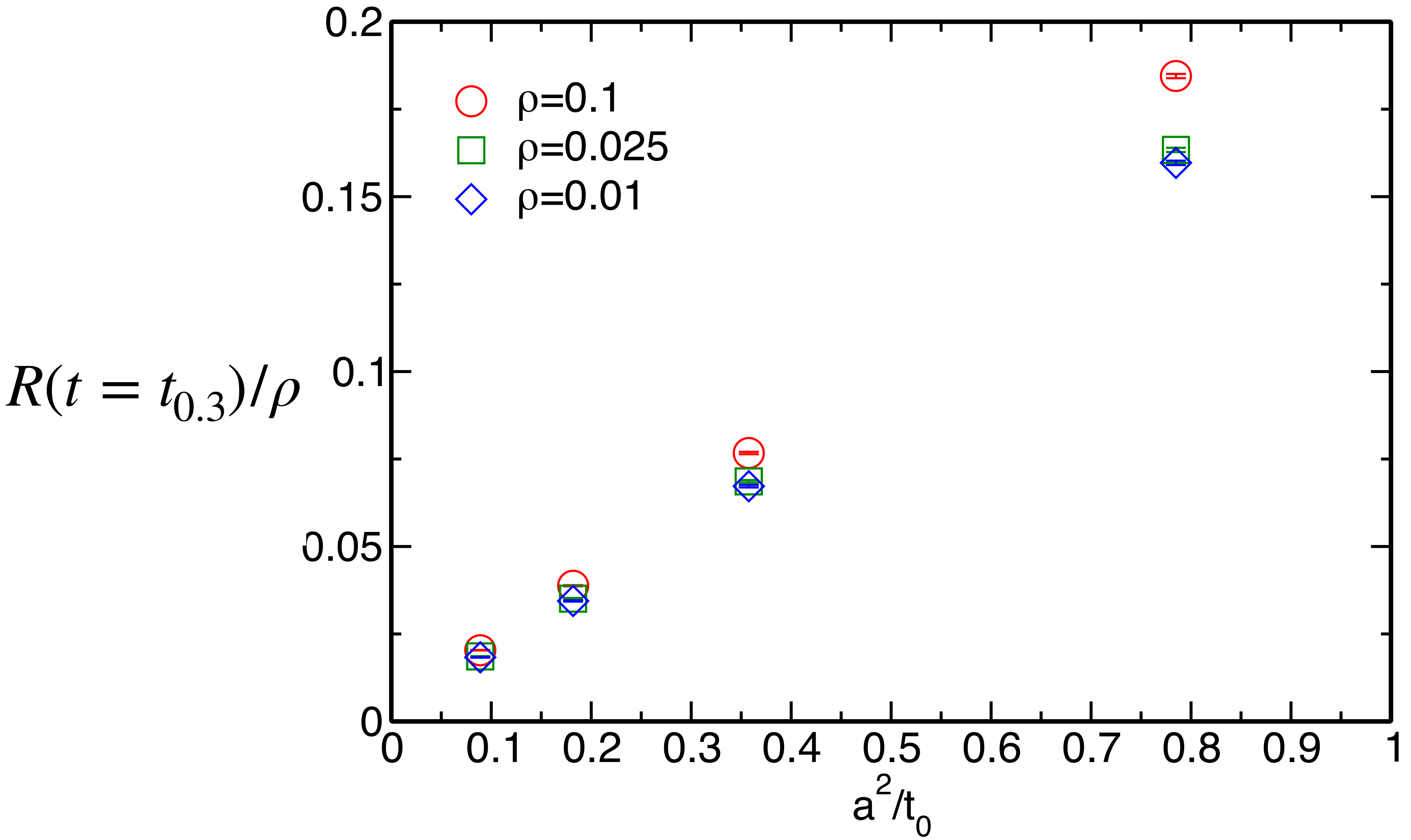
Evaluation of $R(t=t_{0.3})$ $\left(t_{0.3} \text{ satisfies } t^2\langle E\rangle|_{t=t_{0.3}} = 0.3\right)$

dashed line : $\Delta X_{\rm flow}/X_{\rm flow}$



a dependence of $R(t = t_{0.3})$

Evaluation of $R(t = t_{0.3})/\rho$ $\left(t_{0.3} \text{ satisfies } t^2\langle E\rangle\,|_{t=t_{0.3}} = 0.3 \right)$



Summary and future works

- The Wilson flow and the stout-link smearing is equivalent at $\rho \to 0$.
- We can regard the two methods are same within some numerical precision if we choose proper combinations.

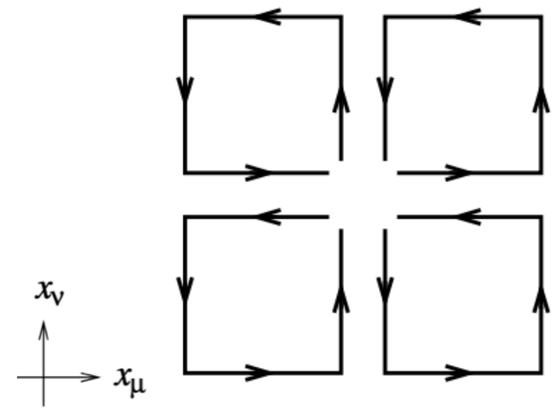
- Calculation with Stout-link smearing is a factor of $O(10)$ faster than Wilson flow → efficient simulations
- Perturbation theory in the gradient flow formalism can be use for the calculation of one loop quantities in lattice perturbation theory for smeared-link fermion action.

Backup

arXiv:2303.09938 **[hep-lat]**

Energy density $\langle E \rangle$

$G_{\mu\nu}(x,t) =$ Traceless Anti-Hermitian part of



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$$
\begin{aligned}
= \quad \frac{1}{4}\Big(& V_\mu(x,t)V_\nu(x+\hat{\mu},t)^\dagger V_\mu(x+\hat{\nu},t)V_\nu^\dagger(x,t) \\
+ & V_\nu^\dagger(x-\hat{\nu},t)V_\mu(x-\hat{\nu},t)V_\nu(x+\hat{\mu}-\hat{\nu},t)V_\mu^\dagger(x,t) \\
+ & V_\mu^\dagger(x-\hat{\mu},t)V_\nu^\dagger(x-\hat{\mu}-\hat{\nu},t)V_\mu(x-\hat{\mu}-\hat{\nu},t)V_\nu(x-\hat{\nu},t) \\
+ & V_\nu^\dagger(x,t)V_\mu(x-\hat{\mu}+\hat{\nu},t)V_\nu(x-\hat{\mu},t)V_\mu^\dagger(x-\hat{\mu},t) \quad \Big)_{AH}
\end{aligned}
$$

$E = \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}$: Energy density

Gauge conf.

TABLE I. Summary of the gauge ensembles: gauge coupling, lattice size ($L^3 \times T$), plaquette value, lattice spacing (a), spatial extent (La), the Sommer scale (r_0), the number of the gauge field configurations (N_{conf}), the number of thermalization sweeps (n_{therm}) and the number of update sweeps (n_{update}). All lattice spacings are set by the Sommer scale ($r_0 = 0.5$ fm) [20, 21].

$\beta = 6/g_0^2$	$L^3 \times T$	plaquette	a [fm]	La [fm]	r_0/a (Ref. [21])	N_{conf}	n_{therm}	n_{update}
5.76	$16^3 \times 32$	0.560938(9)	0.1486(7)	2.38	3.364(17)	100	5000	200
5.96	$24^3 \times 48$	0.589159(3)	0.1000(5)	2.40	5.002(25)	100	2000	200
6.17	$32^3 \times 64$	0.610867(1)	0.0708(3)	2.27	7.061(35)	100	2000	200
6.42	$48^3 \times 96$	0.632217(1)	0.0500(2)	2.40	10.00(5)	100	2000	200

t_0, w_0

TABLE III. Results of t_0/a^2 obtained from the stout smearing with three smearing parameters $\rho = 0.1,\ 0.025,\ 0.01$ and the Wilson flow ($\epsilon = 0.025$).

	t_0/a^2 (stout-link smearing)			t_0/a^2 (Wilson flow)
β	$\rho = 0.1$	$\rho = 0.025$	$\rho = 0.01$	$\epsilon = 0.025$
5.76	1.2502(30)	1.2690(30)	1.2722(31)	1.2741(31)
5.96	2.7744(62)	2.7919(62)	2.7949(62)	2.7968(62)
6.17	5.476(13)	5.494(13)	5.497(13)	5.499(13)
6.42	11.218(22)	11.236(23)	11.240(23)	11.242(23)

TABLE IV. Results of w_0/a obtained from the stout-link smearing with three smearing parameters $\rho = 0.1,\ 0.025,\ 0.01$ and the Wilson flow ($\epsilon = 0.025$).

	w_0/a (stout-link smearing)			w_0/a (Wilson flow)
β	$\rho = 0.1$	$\rho = 0.025$	$\rho = 0.01$	$\epsilon = 0.025$
5.76	1.1098(18)	1.1199(18)	1.1220(18)	1.1224(18)
5.96	1.6755(24)	1.6819(24)	1.6832(24)	1.6833(24)
6.17	2.3684(41)	2.3729(41)	2.3738(41)	2.3738(41)
6.42	3.4042(48)	3.4075(48)	3.4081(48)	3.4081(48)

Computational speed

Computational time for 100 step(single core、2.5GHz)

TABLE I. Run on a single core of Intel Xeon E5-2609 CPUs at 2.5GHz.

directions	stout-link smearing ($\rho = 0.025$)	Wilson flow ($\epsilon = 0.025$)		
		2nd-order RK	3rd-order RK	4th-order RK
space-time	136.80 [sec]	985.55 [sec]	1496.51 [sec]	2061.08 [sec]
space	90.39 [sec]	511.84 [sec]	763.51 [sec]	1074.07 [sec]