SymEFT predictions for local fermion bilinears

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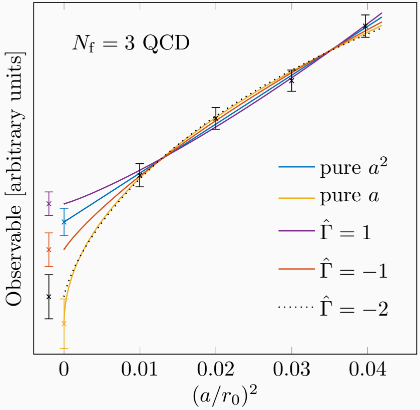
Motivation: Continuum extrapolation



In an asymptotically free theory, like QCD, leading lattice artifacts are of the form (up to factors of $\log \bar{g}(1/a)$)

$$\frac{\mathcal{P}(a)}{\mathcal{P}(0)} = 1 + a^{n_\mathrm{min}} \sum_i [\bar{g}^2(1/a)]^{\hat\Gamma_i} c_i + \mathrm{O}(a^{n_\mathrm{min}+1}, a^{n_\mathrm{min}} \bar{g}^{2\hat\Gamma_i+2}(1/a), \ldots)$$

$\hat\Gamma_i$ can be negative and distinctly nonzero

$\Rightarrow$ impact on convergence.

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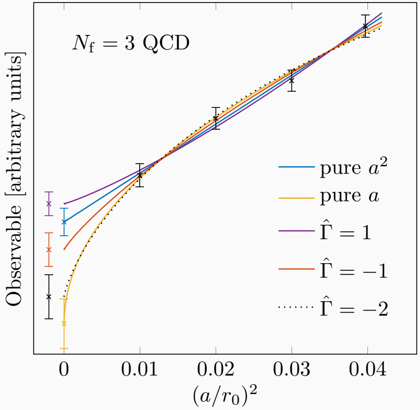

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Warning example: 2d O(3) non-linear sigma model **$\min \hat{\Gamma}_i = -3$** [Balog et al., 2009, 2010]

$\Rightarrow$ Compute $\hat{\Gamma}_i$ in QCD to gain better control over continuum extrapolation.

Symanzik Effective Theory (SymEFT) [Symanzik, 1980, 1981, 1983a,b]

Spectral quantities get corrections from the lattice action

$$\Delta S = a^{n_{\min}} \int \mathrm{d}^4 x \sum_i \bar{\omega}_i(g_0)\mathcal{O}_i(x) + \dots$$

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Example: hadron masses

$$am^X(a) = \lim_{t\to\infty} \log \frac{C^X_{2\text{pt}}(t)}{C^X_{2\text{pt}}(t+a)}$$

$$\frac{m^X(a)}{m^Y(a)} = \frac{m^X}{m^Y}\left\{1 - a^{n_{\min}} \sum_i \hat{c}_i [2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}_i^{\mathcal{B}}} \left(\frac{m^X_{i;\mathrm{RGI}}}{m^X} - \frac{m^Y_{i;\mathrm{RGI}}}{m^Y}\right) + \dots\right\}$$

where $\hat{\Gamma}_i^{\mathcal{B}} = (\gamma_0^{\mathcal{B}})_i/(2b_0) + n_i^!$ can be obtained from 1-loop running of the operators $\mathcal{O}_i$

$$\mu \frac{\mathrm{d}\mathcal{O}_{i;\overline{\mathrm{MS}}}(\mu)}{\mathrm{d}\mu} = -(\gamma_0^{\mathcal{O}})_{ik}\bar{g}^2(\mu)\mathcal{O}_{k;\overline{\mathrm{MS}}}(\mu)$$

and a change of basis $\mathcal{O} \to \mathcal{B}$ s.t. $\gamma_0^{\mathcal{B}}$ is diagonal.

Leading powers $\hat{\Gamma}_i^{\mathcal{B}}$ for pure gauge, Wilson and GW quarks [NH et al., 2020, 2022; NH, 2023]

Symanzik Effective Theory (SymEFT) [Symanzik, 1980, 1981, 1983a,b]

Local fields are more complicated with two contributions

Discretised local field: $$\Delta J(x) = a^{n_{\mathrm{min}}} \sum_i \bar{\nu}_i(g_0) J_i(x) + \ldots\,,$$

Lattice action: $$\Delta S = a^{n_{\mathrm{min}}} \int \mathrm{d}^4 x \sum_i \bar{\omega}_i(g_0) Q_i(x) + \ldots\,, \quad Q = \mathcal{O} \cup \mathcal{E}$$

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Example: pion decay constant (Contact-terms will affect matching coefficients d_i)

$$\frac{Z_\mathrm{A}(a\mu)\langle 0|\mathrm{A}_0(x;a)|\pi(\mathbf{0})\rangle}{\lim_{a'\searrow 0}[M_\pi f_\pi](a')} = 1 + a^{n_{\min}} \sum_i d_i \frac{\langle 0|(\mathrm{A}_0)_{i;\overline{\mathrm{MS}}}(x)|\pi(\mathbf{0})\rangle}{\lim_{a'\searrow 0}[M_\pi f_\pi](a')} + \left(\begin{smallmatrix}\text{corrections}\\ \text{from } Z_\mathrm{A}\end{smallmatrix}\right)$$

$$-a^{n_{\min}} \sum_i c_i \int \mathrm{d}^4 y \frac{\langle 0|\mathrm{A}_0(x) Q_{i;\overline{\mathrm{MS}}}(y)|\pi(\mathbf{0})\rangle_c}{\lim_{a'\searrow 0}[M_\pi f_\pi](a')} + \dots$$

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Example: pion decay constant $(\mathrm{A}_0)_i \to (\mathcal{A}_0)_i$ s.t. $\hat{\Gamma}_i^{\mathrm{A}} = \frac{(\gamma_0^{\mathcal{A}})_i}{2b_0} + n_i^{\mathrm{I}}$

$$\frac{Z_{\mathrm{A}}(a\mu)\langle 0|\mathrm{A}_0(x;a)|\pi(\mathbf{0})\rangle}{\lim_{a'\searrow 0}[M_\pi f_\pi](a')} = 1 + a^{n_{\min}} \sum_i \hat{d}_i [2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}_i^{\mathrm{A}}} \frac{\langle 0|(\mathcal{A}_0)_{i;\mathrm{RGI}}(x)|\pi(\mathbf{0})\rangle}{\lim_{a'\searrow 0}[M_\pi f_\pi](a')} + \left(\substack{\text{corrections}\\ \text{from } Z_{\mathrm{A}}}\right)$$

$$- a^{n_{\min}} \sum_i \hat{c}_i [2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}_i^{\mathcal{B}}} \int \mathrm{d}^4 y \frac{\langle 0|\mathrm{A}_0(x)\mathcal{B}_{i;\mathrm{RGI}}(y)|\pi(\mathbf{0})\rangle_{c\,\text{contact div. subtracted}}}{\lim_{a'\searrow 0}[M_\pi f_\pi](a')} + \dots$$

Matching coefficients for local fields

Renormalisation of contact term divergences in the SymEFT can shift tree-level matching coefficients of the the local field.

$$\begin{pmatrix} a^{n_{\min}} J^{(n_{\min})}(x) \\ a^{n_{\min}} \tilde{Q}(0) J(x) \\ J(x) \end{pmatrix}_{\overline{\mathrm{MS}}} = \begin{pmatrix} Z^{J^{(n_{\min})}} & 0 & 0 \\ Z^{QJ} & Z^{Q} Z^{J} & 0 \\ 0 & 0 & Z^{J} \end{pmatrix} \begin{pmatrix} a^{n_{\min}} J^{(n_{\min})}(x) \\ a^{n_{\min}} \tilde{Q}(0) J(x) \\ J(x) \end{pmatrix}$$

Z^{QJ} will affect the matching coefficients of fields in $J^{(n_{\min})}$ when diagonalising this mixing matrix.

$\Rightarrow$ Relying solely on classical a-expansion leads to incomplete tree-level matching.
But combining classical a-expansion with full off-shell mixing ensures correct matching!

Example: Axial vector basis

O(*a*): [Lüscher et al., 1996; Bhattacharya et al., 2004, 2006]

$$(\mathrm{A}_\mu^{kl})_1^{(1)} = \partial_\mu \mathrm{P}^{kl}, \qquad (\mathrm{A}_\mu^{kl})_2^{(1)} = \frac{m_{k+l}}{2}\mathrm{A}_\mu^{kl}, \qquad (\mathrm{A}_\mu^{kl})_3^{(1)} = \mathrm{tr}(M)\mathrm{A}_\mu^{kl},$$

O(*a*²):

$$(\mathrm{A}_\mu^{kl})_1^{(2)} = \delta_{\mu\rho\lambda}\bar{q}_k\gamma_\rho\gamma_5\overleftrightarrow{D}_\lambda^2 q_l, \qquad (\mathrm{A}_\mu^{kl})_2^{(2)} = \bar{q}_k\gamma_\rho\tilde{F}_{\rho\mu}q_l, \qquad (\mathrm{A}_\mu^{kl})_3^{(2)} = m_{k-l}\bar{q}_k(\overleftarrow{D}_\mu - D_\mu)\gamma_5 q_l,$$

$$(\mathrm{A}_\mu^{kl})_4^{(2)} = \frac{\delta_{kl}\delta_{\mu\nu\rho\sigma}}{g^2}\mathrm{tr}(D_\nu F_{\rho\lambda}\tilde{F}_{\sigma\lambda}), \quad (\mathrm{A}_\mu^{kl})_5^{(2)} = \frac{\delta_{kl}}{g^2}\mathrm{tr}(D_\rho F_{\rho\lambda}\tilde{F}_{\mu\lambda}), \quad (\mathrm{A}_\mu^{kl})_6^{(2)} = \delta_{\mu\rho\lambda}\partial_\rho^2\mathrm{A}_\lambda^{kl},$$

$$(\mathrm{A}_\mu^{kl})_7^{(2)} = \partial^2\mathrm{A}_\mu^{kl}, \qquad (\mathrm{A}_\mu^{kl})_8^{(2)} = \frac{m_k^2 + m_l^2}{2}\mathrm{A}_\mu^{kl}, \qquad (\mathrm{A}_\mu^{kl})_9^{(2)} = \frac{\delta_{kl}}{g^2}\partial_\mu\mathrm{tr}(F_{\nu\rho}\tilde{F}_{\nu\rho}),$$

$$(\mathrm{A}_\mu^{kl})_{9+j}^{(2)} = \frac{m_{k+l}}{2}(\mathrm{A}_\mu^{kl})_j^{(1)}, \qquad (\mathrm{A}_\mu^{kl})_{12+j}^{(2)} = \mathrm{tr}(M)(\mathrm{A}_\mu^{kl})_j^{(1)}, \qquad (\mathrm{A}_\mu^{kl})_{16}^{(2)} = \mathrm{tr}(M^2)\mathrm{A}_\mu^{kl},$$

Additional powers of $\bar{g}^2(1/a)$ for local fields

For each local field J, we have yet another set of higher dimensional operators introducing additional powers

$$a^{n_{\min}}[2b_0\bar{g}^2(1/a)]^{\hat{\gamma}_i^{\mathcal{J}}+n_i^{\mathrm{I}}}, \quad \hat{\gamma}_i^{\mathcal{J}} = \frac{(\gamma_0^{\mathcal{J}})_i - \gamma_0^{J}}{2b_0}, \quad \mu\frac{\mathrm{d}\mathcal{J}_{i;\overline{\mathrm{MS}}}(\mu)}{\mathrm{d}\mu} = -(\gamma_0^{\mathcal{J}})_i\bar{g}^2(\mu)\mathcal{J}_{i;\overline{\mathrm{MS}}}(\mu) + \mathrm{O}(\bar{g}^4(\mu)),$$

where $n_i^{\mathrm{I}} \geq 0$ depends on improvement and we assume that the 1-loop operator mixing is diagonalisable. $\hat{\gamma}_i^{\mathcal{J}}$ depends only on quantum numbers but not chosen discretisation.

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where $n_i^{\mathrm{I}} \geq 0$ depends on improvement and we assume that the 1-loop operator mixing is diagonalisable. $\hat{\gamma}_i^{\mathcal{J}}$ depends only on quantum numbers but not chosen discretisation.

$\Rightarrow$ Beware of continuum fields with large 1-loop anomalous dimensions, i.e., $\gamma_0^J \gtrsim 2b_0$!

Leading powers $a^{n_{\min}}[2b_0\bar{g}^2(1/a)]^{\hat{\gamma}_i+n_i^{\mathrm{I}}}$

massive   only if $k = l$

$\mathcal{J}^{kl}$ at $\mathbf{O}(a)$	N_{f}	$\hat{\gamma}^{\mathcal{J}}$
scalar ($k \neq l$ only)	2	0.414
	3	0.444
pseudo-scalar	2	−0.586, 0.414
	3	−0.556, 0.444
vector	2	0.138, 0.414
	3	0.148, 0.444
axial-vector	2	−0.414, 0.414
	3	−0.444, 0.444
tensor	2	−0.138, 0.414
	3	−0.148, 0.444

+ Powers from SymEFT action! [NH et al., 2022; NH, 2023]

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$\mathcal{J}^{kl} \sim \frac{\delta_{kl}}{g_0^2}\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$

Contact term with $i\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi$ gives rise to TL contributions.

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$\mathcal{J}^{kl}_\mu \sim \partial_\mu \mathrm{P}^{kl}$, $\mathcal{J}^{kl}_{\mu\nu} \sim \partial_\mu \mathrm{V}^{kl}_\nu - \partial_\nu \mathrm{V}^{kl}_\mu$

Likely present at TL for discretisations spanning multiple lattice sites.

Opposite chirality.

$\Rightarrow$ Suppressed for light quarks **in finite volume**. May still impact $\mathrm{O}(a^2)$.

$\mathcal{J}^{kl}_{\mu\nu}$ can arise at TL via contact terms with $i\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi$.

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Leading powers $a^{n_{\min}}[2b_0\bar{g}^2(1/a)]^{\hat{\gamma}_i+n_i^{\mathrm{I}}}$

massive   only if $k=l$ massive, only if $k\neq l$ and non-degenerate

$\mathcal{J}^{kl}$ at $\mathrm{O}(a^2)$	N_f	$\hat{\gamma}^{\mathcal{J}}$
scalar ($k\neq l$ only)	2	0, 0.483, 0.828
	3	0, 0.519, 0.889
pseudo-scalar	2	−0.172, 0, 0.483, 0.828
	3	−0.111, 0, 0.519, 0.889
vector	2	0, 0.368, 0.552, 0.575, 0.828
	3	0, 0.395, 0.593, 0.617, 0.889
axial-vector	2	−1, 0, 0.368, 0.506, 0.552, 0.559, 0.575, 0.828, 1.085
	3	−1, 0, 0.395, 0.593, 0.595, 0.617, 0.889, 1.244
tensor	2	0, 0.276, 0.46, 0.563, 0.69, 0.828
	3	0, 0.296, 0.494, 0.605, 0.741, 0.889

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$\mathcal{J}^{kl} \sim m\frac{\delta_{kl}}{g_0^2}\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$, $\mathcal{J}_\mu^{kl} \sim \frac{\delta_{kl}}{g_0^2}\partial_\mu\mathrm{tr}(F_{\nu\rho}\tilde{F}_{\nu\rho})$
Massive / total divergence version but otherwise the same as $\mathrm{O}(a)$ term.

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Conclusion

- $\mathbf{O}(\boldsymbol{a})$**:** Axial and tensor have negative powers in the coupling also for non-singlets but terms have opposite chirality. Unhandled, those terms may severely worsen $\mathrm{O}(a^2)$.
- $\mathbf{O}(\boldsymbol{a}^2)$**:** Negative powers in the coupling only for flavours $k = l$ (if fully $\mathrm{O}(a)$ improved).
- Finding the minimal bases, split properly into on-shell and EOM-vanishing is a very tedious task.
- Computing 1-loop mixing including SymEFT action contact terms can be automated to a high degree.[1] $\Rightarrow$ Affects tree-level matching.
- Do not forget about SymEFT action $\Rightarrow$ Unfortunately many more powers relevant.
- Nowadays, lattice QCD simulations reach lattice spacings $a \sim 0.04\,\mathrm{fm}$, i.e., $\alpha_{\overline{\mathrm{MS}}}^{\text{5-loop}}(1/a) \sim 0.21$.

[1] https://github.com/nikolai-husung/Symanzik-QCD-workflow

Restrictions

- Assumes lattice actions that fulfil at least symmetry-constraints of Wilson-type quarks.
- Only applicable to local fields, i.e., integrated correlation functions get (additional) contributions including log-enhanced cut-off effects [Cè et al., 2021; Sommer et al., 2023].
  $\Rightarrow$ "Window quantities" away from short-distances should be accessible!
- Results incomplete to account for **off-shell renormalisation** in, e.g., RI/(S)MOM.
  $\Rightarrow$ Cannot (trivially) ignore EOM operators anymore.
  $\Rightarrow$ Some schemes even have intrinsic gauge-choice dependence!

What remains to be done (at the very least)

- Other local fields of interest, e.g., energy-momentum tensor, 4-quark operators, ...
- $\hat{\gamma}_i^{\mathcal{J}} = \frac{(\gamma_0^{\mathcal{J}})_i - \gamma_0^J}{2b_0}$, if γ_0^J is very positive such local fields should probably be checked first, e.g., 4-quark operators in [Ciuchini et al., 1998]. No guarantee for any local field!
- **Further complication** if the continuum local fields mix under renormalisation.

- YM Gradient flow in full QCD. Also flowed fermion fields?
- (Unrooted) staggered quark action.
  $\Rightarrow$ Generalisation to rooted should be possible with certain conjectures.

Far from complete:

- QCD+QED: enlarges the minimal operator basis further.
- Combine lattice χPT formulae with powers obtained here?
- Any further insight into smearing?
- More exotic improvement, e.g., adding Pauli-Villars fields, Anna Hasenfratz' parallel talk.
- ...

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