The chiral condensate at large N

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Introduction

Lattice strategies to study the large-N limit can be broadly divided into:

- Standard approach: extended lattices with periodic boundary conditions + extrapolation from $N < 10$
- **Twisted Eguchi–Kawai (TEK) reduced models**: single-site lattice with twisted boundary conditions $\Longrightarrow$ $N > 100$, practically works directly at $N = \infty$

Complementary approaches: agreement among them is highly non-trivial, standard approach necessary for sub-leading effects in $1/N$.

Many large-N calculations within TEK models:

- String tension σ [González-Arroyo et al., 2013; 1206.0049]
- QCD running coupling [García Pérez, 2014; 1412.0941]
- QCD scale Λ_{QCD} [see P. Butti's talk]
- Meson masses [García Pérez et al., 2020; 2011.13061]
- Ongoing studies with adjoint fermions at large-N [Butti et al., 2205.03166]

The chiral condensate

Computation of a new observable within the TEK model:
the chiral condensate.

$$\Sigma \equiv -\lim_{m\to 0}\lim_{V\to\infty}\langle \overline{u}u\rangle\,, \qquad m_u = m_d \equiv m.$$

Many QCD computations for various N_f in the last 10 years,
but just few large-N estimations (all using just 1 lattice spacing)
[Narayanan & Neuberger, 2003; hep-lat/0405025] [Hernandez et al., 2019; 1907.11511].

In the 't Hooft limit $1/N \to 0$ *and* $N_f/N \to 0$:

$$\Sigma(N) = N\left[\overline{\Sigma} + O\left(\frac{1}{N^2}\right)\right]$$

Outline of our work

- Solid computation of the large-N limit of Σ/N for $N_f = 0$:
  4 values of lat. spac. a and 3 values of m_π for each a
  $\Longrightarrow$ **controlled continuum and chiral extrapolations**.

- Strategy: Giusti–Lüscher method [arXiv:0812.3638]
  to extract Σ from mode number of the Dirac operator.

TEK model: Wilson action

Lattice TEK action: one-site Wilson plaquette action with inverse 't Hooft coupling b and twisted boundary conditions, no dynamical quarks.

$$S_{\mathrm{W}}[U] = -Nb \sum_{n,\nu\neq\mu} \mathrm{Tr}\left\{U_\mu(n) U_\nu(n+a\hat{\mu}) U_\mu^\dagger(n+a\hat{\nu}) U_\nu^\dagger(n)\right\}$$

Reduction

- $U_\mu(n) \longrightarrow U_\mu$ (one site $\implies$ $L=1$, only $d=4$ links)
- $U_\mu(n+a\hat{\nu}) = \Gamma_\nu U_\mu \Gamma_\nu^\dagger$ (twisted boundary conditions)
- $U_\mu \longrightarrow U_\mu \Gamma_\mu$ (change of variables)

$$S_{\mathrm{TEK}}[U] = -Nb \sum_{\nu\neq\mu} z_{\nu\mu} \mathrm{Tr}\left\{U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right\}$$

- Twist-eaters $\Gamma_\mu \Gamma_\nu = z_{\nu\mu} \Gamma_\nu \Gamma_\mu$ with twist factor $z_{\nu\mu} = \exp\left\{2\pi i k \varepsilon_{\nu\mu}/\sqrt{N}\right\}$
- Effective box size $L = \sqrt{N}$
- Large-N volume independence: for any given closed path $\mathcal{P}$

$$z(\mathcal{P}) \langle W(\mathcal{P}) \rangle_{\mathrm{TEK}} \underset{N\to\infty}{\longrightarrow} \langle W(\mathcal{P}) \rangle_{\ell\to\infty; N\to\infty}$$

TEK model: Wilson fermions

We will consider 1 valence **fundamental** quark flavor.
We will use the **Wilson discretization**.

For fermions, calculation of reduction is much more involved, see, e.g., [González-Arroyo & Okawa, 2015; arXiv:1510.05428].
Here we just give directly our TEK discretized Dirac–Wilson operator:

$$D_{\mathrm{W}}^{(\mathrm{TEK})} = \frac{1}{2\kappa} - \frac{1}{2}\sum_{\mu=0}^{d-1}\left[(\mathbb{I}+\gamma_\mu)\otimes U_\mu \otimes \Gamma_\mu^* + \mathrm{h.c.}\right].$$

For the purpose of computing the mode number, we will solve numerically the following eigenproblem using the `ARPACK` library:

$$Q_{\mathrm{W}} u_\lambda = \lambda u_\lambda, \qquad \lambda \in \mathbb{R},$$

where we have introduced the Hermitian operator

$$Q_{\mathrm{W}} \equiv \gamma_5 D_{\mathrm{W}}^{(\mathrm{TEK})}, \qquad Q_{\mathrm{W}}^\dagger = Q_{\mathrm{W}}.$$

Chiral condensate from the mode number

Banks–Casher relates the chiral condensate with spectral density in the origin:

$$\frac{\Sigma}{\pi} = \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty} \rho(\lambda, m)$$

The mode number is essentially equivalent to ρ,
but is more amenable to be computed on the lattice:

$$\begin{aligned}\langle \nu(M)\rangle &\equiv \langle \# \ |i\lambda + m| \leq M\rangle \\ &= V\int_{-\Lambda}^{\Lambda} \rho(\lambda, m) d\lambda, \qquad \Lambda^2 \equiv M^2 - m^2.\end{aligned}$$

- Banks–Casher implies linear rise of $\langle \nu(M)\rangle$ close to $M = m$:

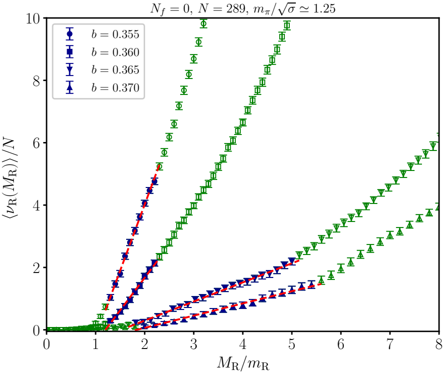
$$\langle \nu(M)\rangle = \frac{2}{\pi} V\Sigma\Lambda + o(\Lambda) = \frac{2}{\pi} V\Sigma M + o(M).$$

- **Giusti–Lüscher method**: obtain Σ from slope of mode number as

$$\Sigma(m) = \frac{\pi}{2V}\sqrt{1 - \frac{m^2}{M^2}}\left[\frac{\partial\langle \nu(M)\rangle}{\partial M}\right] \longleftarrow \text{slope of } \langle \nu(M)\rangle \text{ vs } M.$$

Implementation of Giusti–Lüscher method:

- Solve numerically $(\gamma_5 D_{\rm W})u_\lambda = \lambda u_\lambda$
- Count modes below treshold M to obtain $\langle \nu(M) \rangle$
- Slope: linear best fit of $\langle \nu(M) \rangle$ vs M close to $M/m = 1$
- $\Sigma = \frac{\pi}{2V}\sqrt{1 - \frac{m^2}{M^2}}\left[\frac{d\langle\nu(M)\rangle}{dM}\right]$ $\longleftarrow$ slope of $\langle \nu(M) \rangle$ vs M from linear fit



Renormalization:

$$\langle \nu \rangle = \langle \nu_{\rm R} \rangle\,, \qquad M_{\rm R} = M/Z_{\rm P}$$
$$\lambda_{\rm R} = \lambda/Z_{\rm P}.$$

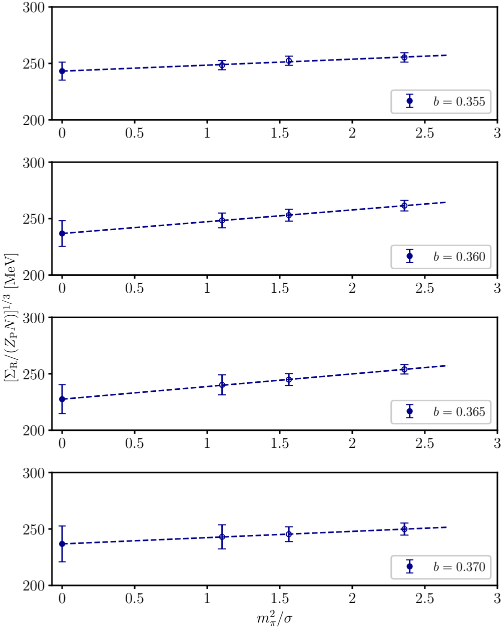
We know $Z_{\rm A} m_{\rm PCAC} = Z_{\rm P} m_{\rm R}$ on our ensembles

$\Longrightarrow$ we count $\#\ |\lambda_{\rm R}|/m_{\rm R} \leq M_{\rm R}/m_{\rm R}$.

From the fit of $\langle \nu_{\rm R} \rangle / N$ we extract the RG-invariant quantity $\Sigma_{\rm R} m_{\rm R}/(\sigma^2 N)$.

Using $Z_{\rm P} m_{\rm R}$ and $\sqrt{\sigma}$, we finally obtain the bare condensate $\Sigma_{\rm R}/(\sigma^{3/2} N Z_{\rm P})$.

Chiral limit at fixed lattice spacing



Chiral Perturbation Theory predicts:

$$\Sigma(m) = \Sigma + k\, m$$

$$\longrightarrow$$

Our results perfectly described by:

$$\Sigma(m_\pi^2, b) = \Sigma(b) + \tilde{k}(b)\, m_\pi^2$$

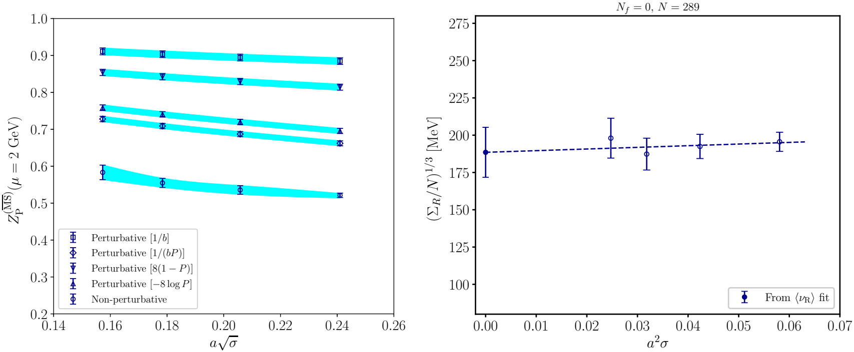
Perfect agreement with expectations.

Plot on the left refers to **bare** quantity $\Sigma/N = \Sigma_{\mathrm{R}}/(N Z_{\mathrm{P}})$.

Conversion to MeV units done using conventional value $\sqrt{\sigma} = 440$ MeV.

Continuum limit

We do not have a calculation of Z_{P} alone from the TEK model.
Non-perturbative large-N results for Z_{P} in the range of a used in this work can be found in [Castagnini, 2015; inspirehep/1411974]. Significant deviations from perturbative estimates obtained from various improved couplings.



Continuum limit assuming $O(a^2)$ corrections $\Longrightarrow$ $\Sigma_{\mathrm{R}}/N = [189(17)\ \mathrm{MeV}]^3$

FLAG21 SU(3) $N_f = 2$: $\Sigma_{\mathrm{R}} = [266(10)\ \mathrm{MeV}]^3$ $\Longrightarrow$ $\Sigma_{\mathrm{R}}/N = [184(7)\ \mathrm{MeV}]^3$

Conclusions

Take-home messages

- Computation of the large-N chiral condensate from TEK models using the Giusti–Lüscher spectral method for $N = 289$, 4 lattice spacings and 3 pion masses each
- Controlled continuum and chiral extrapolations lead to $\Sigma_{\rm R}/N = [189(17)\ \mathrm{MeV}]^3$, which is in remarkable agreement with the FLAG21 world-average for 2-flavor QCD $\Sigma_{\rm R}/N = [184(7)\ \mathrm{MeV}]^3$ when using $\sqrt{\sigma}$ to set the scale
- Our calculation suggests that $1/N^2$ corrections are small and $N = 3$ is already very close to $N = \infty$. Such conclusion fits very well with other large-N calculations pointing towards the same scenario

Future outlooks

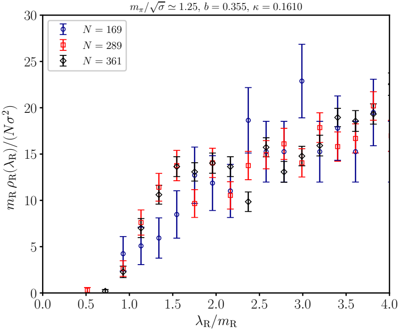
- Compare with calculation from quark mass dependence of m_π
- Extend calculation to the case of adjoint fermions (interesting for BSM phenomenology)

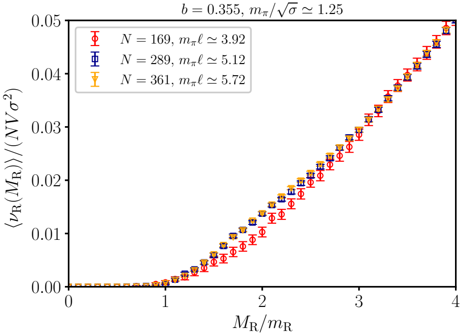
Back-up slides

Check of finite-N effects

Exploring $m_\pi \ell = m_\pi a\sqrt{N} \simeq 3.9, 5.1, 5.7$ we observe Finite Size Effects (FSEs), i.e., finite-N effects, in the spectral density ρ just in the smallest bins, while plateaus are perfectly agreeing

$\Longrightarrow$ no significant FSEs in the slope of $\langle \nu \rangle / V$, and thus in the condensate.





$$N = 169 \longrightarrow [\Sigma_R/(N Z_P)]^{1/3} = 254(9)\ \text{MeV}$$
$$N = 289 \longrightarrow [\Sigma_R/(N Z_P)]^{1/3} = 254(2)\ \text{MeV}$$
$$N = 361 \longrightarrow [\Sigma_R/(N Z_P)]^{1/3} = 256(3)\ \text{MeV}$$