Nuclear Science Computing Center at CCNU

Microscopic Encoding of Macroscopic Universality: Scaling properties of Dirac Eigenspectra near QCD Chiral Phase Transition

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based on arXiv: 2305.10916,

in collaboration with

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2023 LATTICE

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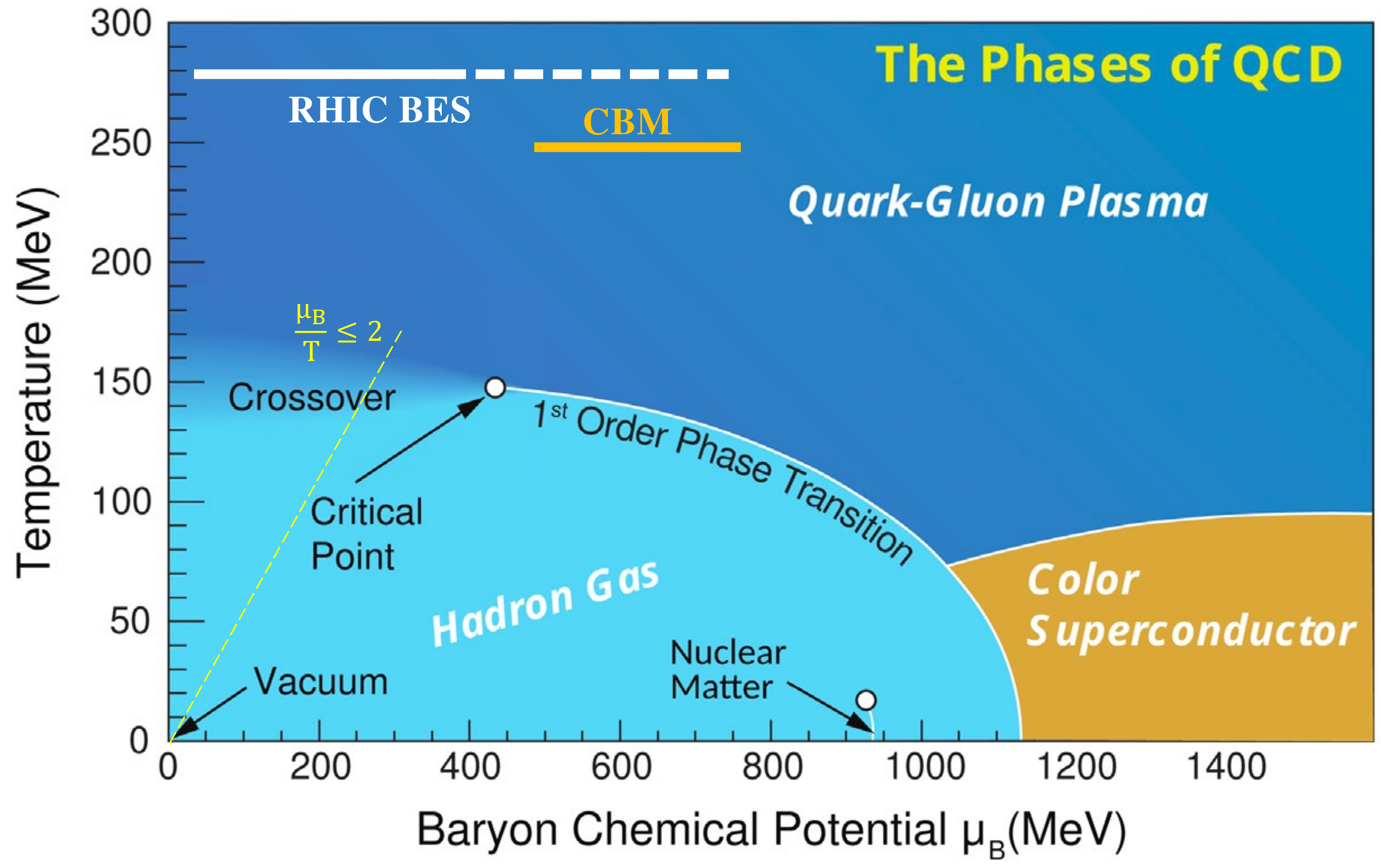
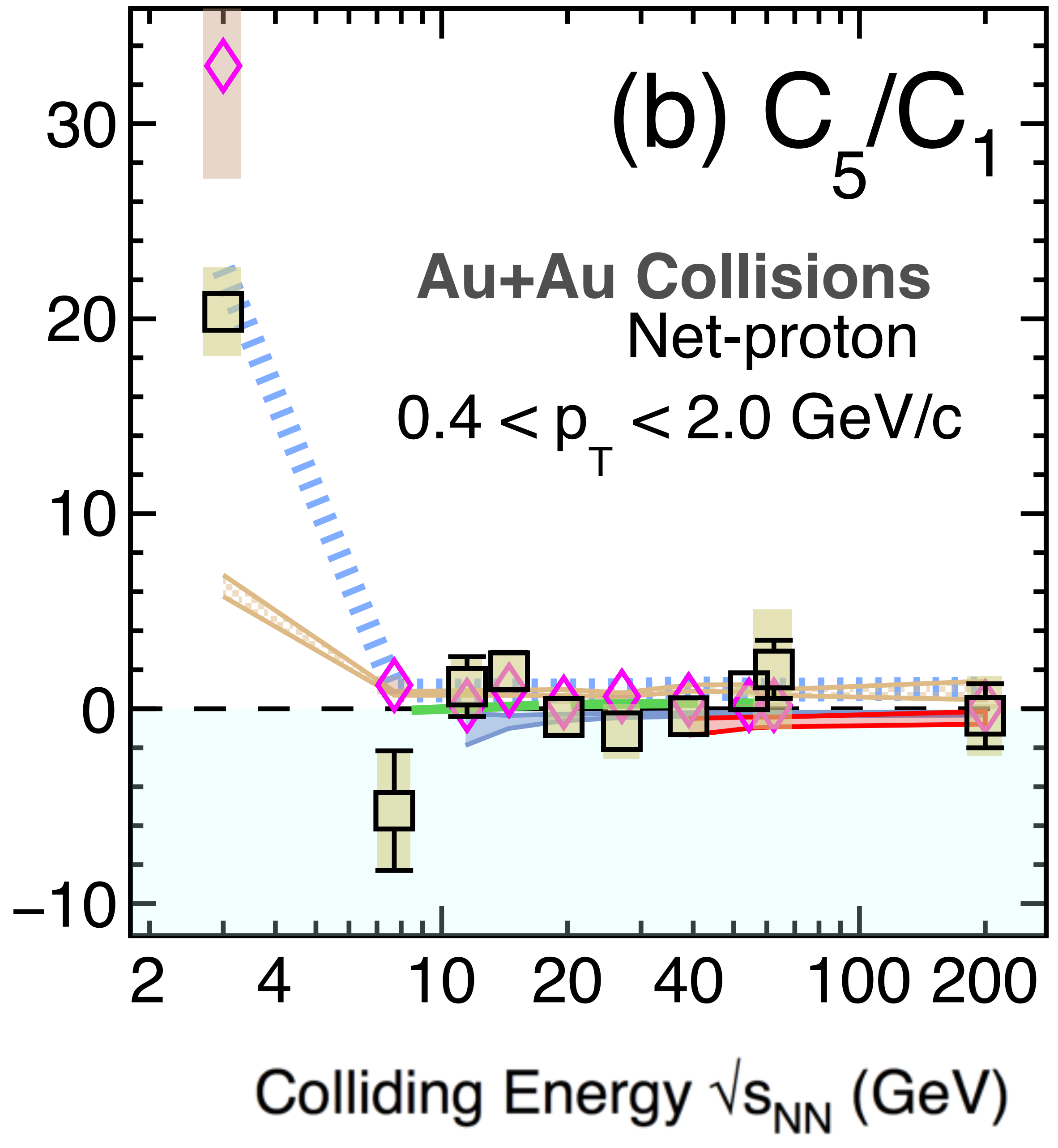
Search for Criticality in QCD

Exploration of QCD phase diagram



D. Almaalol et al., arXiv:2209.05009

Searching for signatures of criticality in Macroscopic quantities



STAR, Phys. Rev. Lett. 130, 082301 (2023)

How criticality at Macroscale arises from Microscopic *d.o.f* of QCD ?

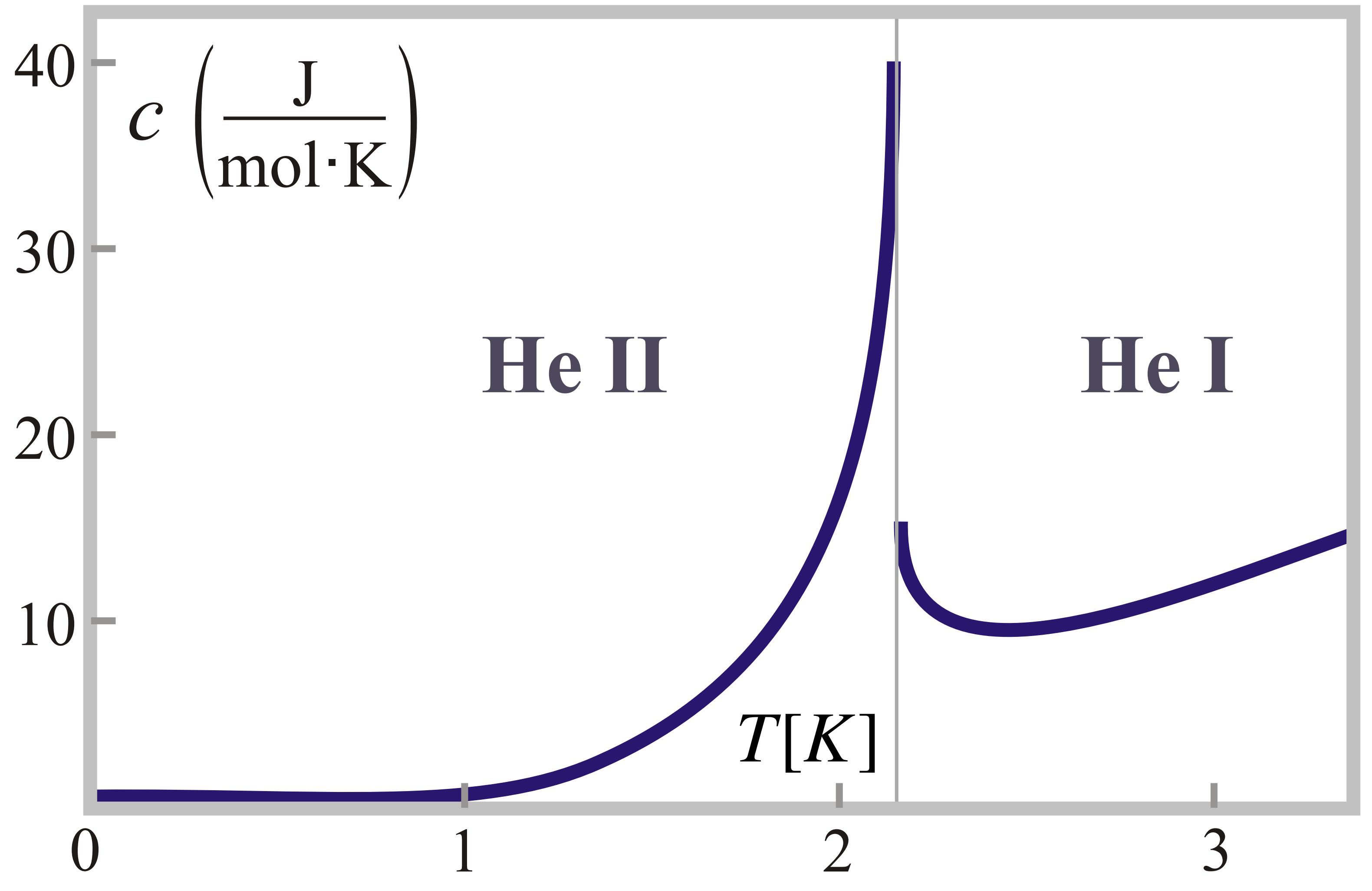
Way to Microscopic Origin of Criticality

Effective Hamiltonian approach

hard to understand origin of criticality

Liquid to superfluid λ-transition in ^{4}He

d.o.f : electrons & photons



First-principle calculations of QCD

possible to understand origin of criticality

Chiral phase transition in the chiral limit of light quark

d.o.f : quarks & gluons

Chiral order parameter : $\langle \bar{\psi}\psi \rangle = \int_0^\infty \frac{4m\,\rho(\lambda, m)}{\lambda^2 + m^2} d\lambda$

$$\rho(\lambda, m) = \frac{T}{V}\langle \rho_U(\lambda) \rangle = \frac{T}{V}\langle \sum_j \delta(\lambda - \lambda_j) \rangle \quad \text{with} \quad \not{D}\psi_j = i\lambda_j \psi_j$$

$\langle \bar{\psi}\psi \rangle \Longleftrightarrow \rho_U(\lambda)$: ***energy spectra of massless quarks***

This talk: A first lattice QCD-based understanding of Microscopic Origin of Criticality in QCD !

From Quark Energy Spectra to its Cumulants

$$\langle \bar{\psi}\psi \rangle = \int_0^\infty \frac{4m\ \rho(\lambda, m)}{\lambda^2 + m^2}\, d\lambda = \frac{T}{V}\int_0^\infty \left\langle \frac{4m\ \rho_U(\lambda)}{\lambda^2 + m^2} \right\rangle d\lambda$$

H.-T. Ding et al., arXiv:2305.10916

1st order cumulant :

$$\langle \bar{\psi}\psi \rangle = \frac{T}{V}\langle 2\ \mathrm{Tr}(\not{D}[U] + m)^{-1} \rangle$$

1-point correlation of quark energy spectra $\frac{4m\ \rho_U(\lambda)}{\lambda^2 + m^2}$

Generalization : from ***generating functional*** of cumulants

— ***Generating functional*** : $\mathbb{G}\,(m;\epsilon) = \ln \left\langle \exp\left\{-m\ \bar{\psi}\psi(\epsilon)\right\}\right\rangle_0 = \ln \left\langle \exp \left\{ -m \int_0^\infty P_U(\lambda;\epsilon)\, d\lambda \right\} \right\rangle_0$

$\langle \cdots \rangle_0$: average over QCD partition function in the chiral limit

Probe operator with valance quark mass ϵ :
$\bar{\psi}\psi(\epsilon) \equiv 2\ \mathrm{Tr}(\not{D}[U] + \epsilon)^{-1}$

Defining $P_U(\lambda;\epsilon) \equiv \frac{4\epsilon\rho_U(\lambda)}{\lambda^2 + \epsilon^2}$

— ***n-th order cumulant of*** $\bar{\psi}\psi$: $\mathbb{K}_n[\bar{\psi}\psi] = \frac{T}{V}(-1)^n \frac{\partial^n \mathbb{G}\,(m;\epsilon)}{\partial m^n}\bigg|_{\epsilon=m}$

$$\mathbb{K}_n[\bar{\psi}\psi(m)] = \int_0^\infty K_1[P_U(\lambda_1;m), P_U(\lambda_2;m), \cdots, P_U(\lambda_n;m)] \prod_{i=1}^{n} d\lambda_i$$

$K_1[X_1, X_2, \cdots, X_n]$ denotes 1st order joint cumulant of n-variables

Microscopic Encoding of Macroscopic Criticality

H.-T. Ding et al., arXiv:2305.10916

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1;m), P_U(\lambda_2;m), \cdots, P_U(\lambda_n;m)] \prod_{i=1}^{n} d\lambda_i \equiv \int_0^\infty P_n(\lambda)\, d\lambda$$

n-th order cumulant of the chiral order parameter

n-point correlation of the quark energy spectra

Chiral condensate : $\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V}\langle\bar{\psi}\psi(m)\rangle = \int_0^\infty P_1(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m^{1/\delta} f_1(z)$

$$z = z_0\left(\frac{m_l}{m_s}\right)^{-\frac{1}{\beta\delta}} \frac{T-T_c}{T_c}$$

$$f_1(z) \equiv f_G(z)$$

$$f_2(z) \equiv f_\chi(z)$$

Disconnected susceptibility : $\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V}\left\langle\left[\bar{\psi}\psi(m) - \langle\bar{\psi}\psi(m)\rangle\right]^2\right\rangle = \int_0^\infty P_2(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m^{1/\delta-1} f_2(z)$

……

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta-n+1} f_n(z)$$

How criticality of $\mathbb{K}_n[\bar{\psi}\psi]$ arises from $P_n(\lambda)$?

Microscopic Encoding of Macroscopic Criticality

Hints from the chiral limit :

$$P_U(\lambda; m) \equiv \frac{4m\rho_U(\lambda)}{\lambda^2 + m^2}$$

$$P_U(\lambda; m \to 0) = 2\pi\rho_U(\lambda)\delta(\lambda)$$

Generalized Banks-Casher relation :

$$\lim_{m\to 0} P_n(\lambda) = (2\pi)^n K_1[\rho_U(\lambda), \underbrace{\rho_U(0), \cdots, \rho_U(0)}_{(n-1)\text{ terms}}]\delta(\lambda)$$

$$\Longrightarrow \lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]$$

$n = 1$ back to Banks-Casher relation !

Criticality in $\lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi]$ must arise from universal behaviors of ***λ-independent*** $\mathbb{K}_n[\rho_U(0)]$

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty P_n(\lambda)\, d\lambda \xrightarrow{\text{Around } T_c} \sim m_l^{1/\delta - n + 1} f_n(z)$$

Conjecture: $P_n(\lambda) = m^{1/\delta - n + 1} f_n(z) g_n(\lambda)$

Scaling arise from $P_n(\lambda)$ at ***deep infrared*** λ region

Include ***all*** system-specific λ-dependence

H.-T. Ding et al., arXiv:2305.10916

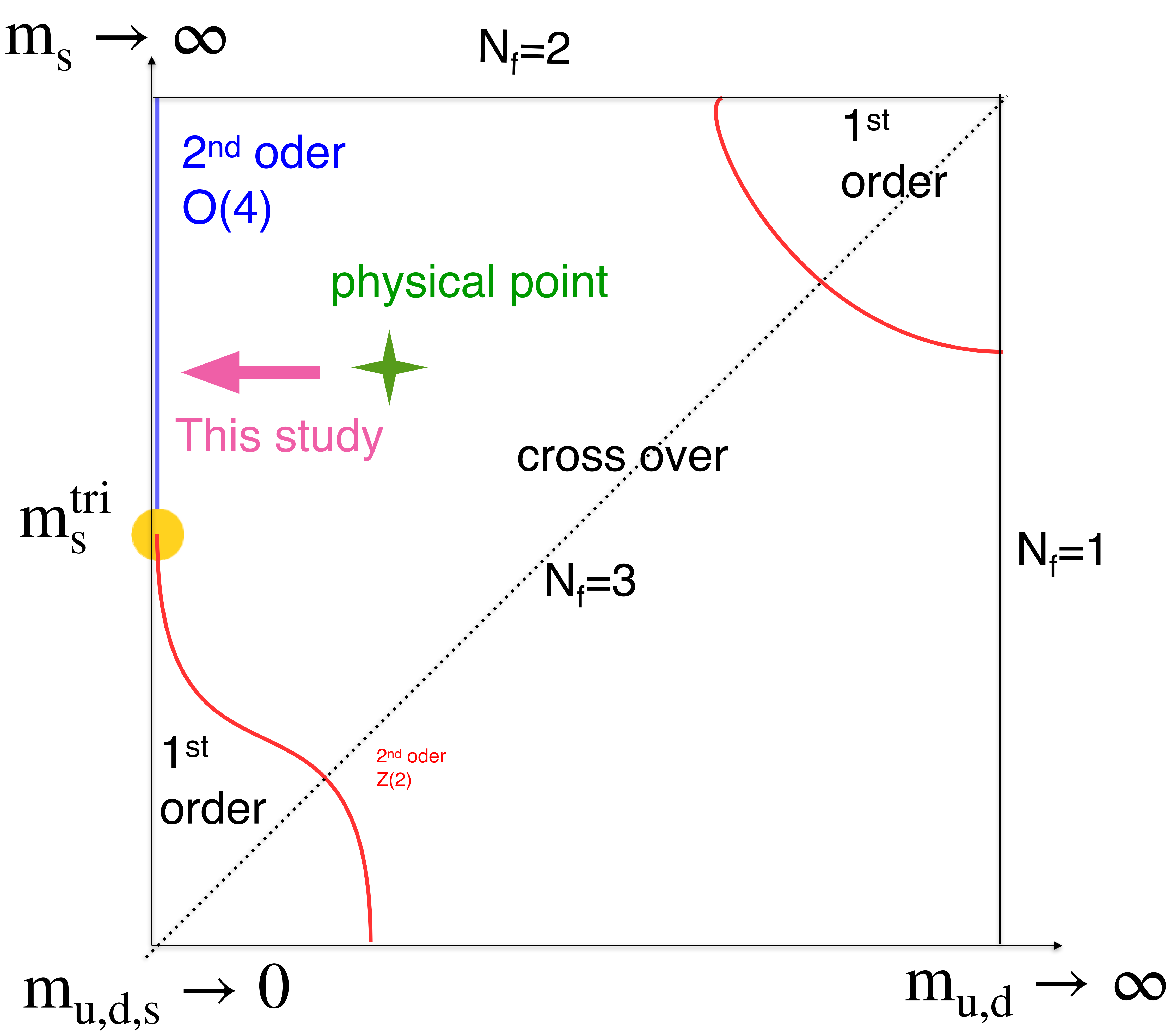
Lattice Setup

- Actions: Highly improved staggered quarks and tree-level Symanzik gauge action
- Lattice size: $N_\tau = 8, \;\; N_\sigma = 32, 40, 56$
- Quark mass: $m_s^{\text{phy}}/m_l = 27, 40, 80, 160$

  $(m_\pi \approx 140, 110, 80, 55 \text{ MeV})$
- Temperatures: $T \in (135, 176)$ MeV
- $\rho_U(\lambda)$ computed via Chebyshev filtering technique

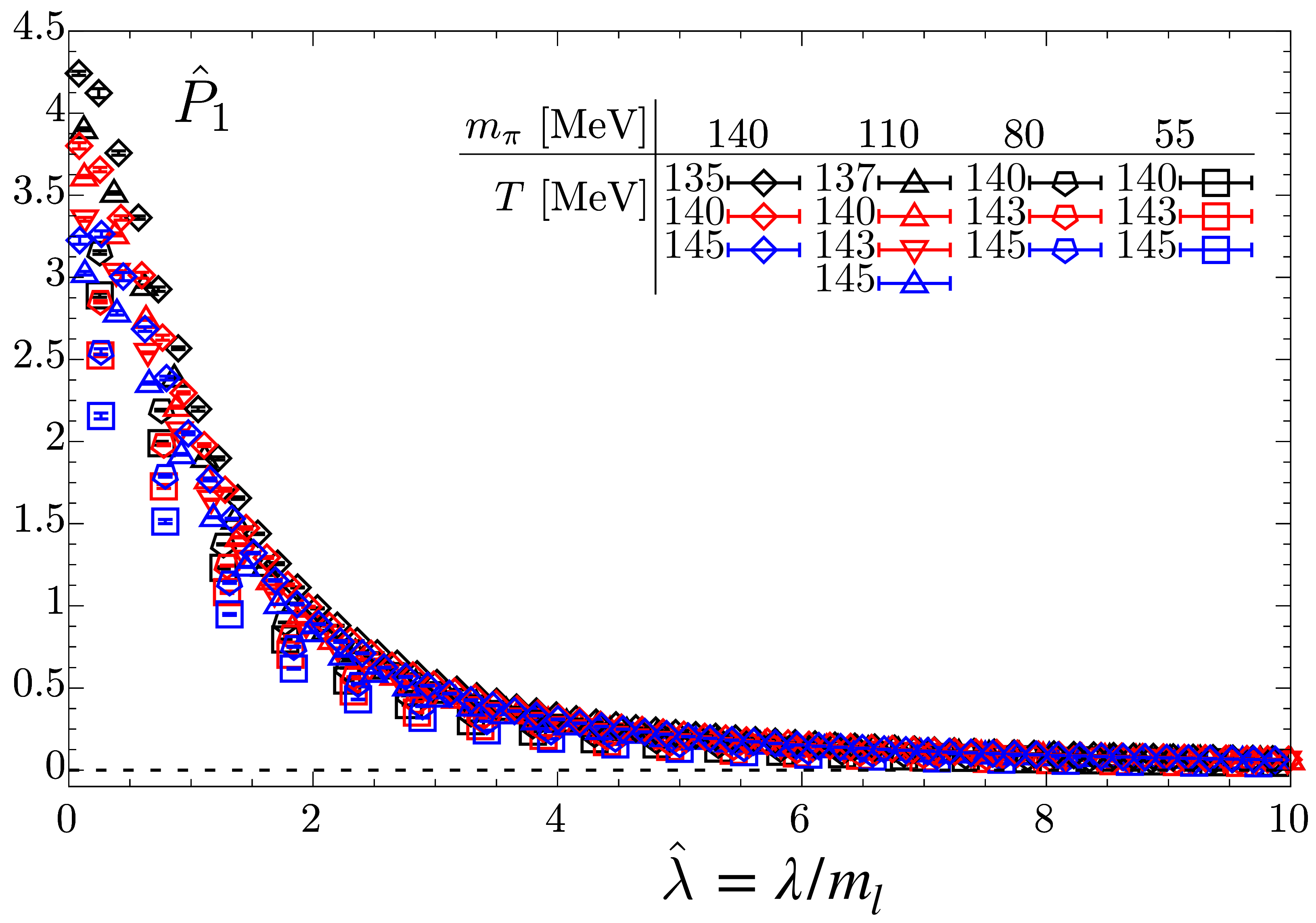
  H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)
- HotQCD configurations; Measurements carried out on NSC^3 at CCNU & BNL

Nuclear Science Computing Center at CCNU

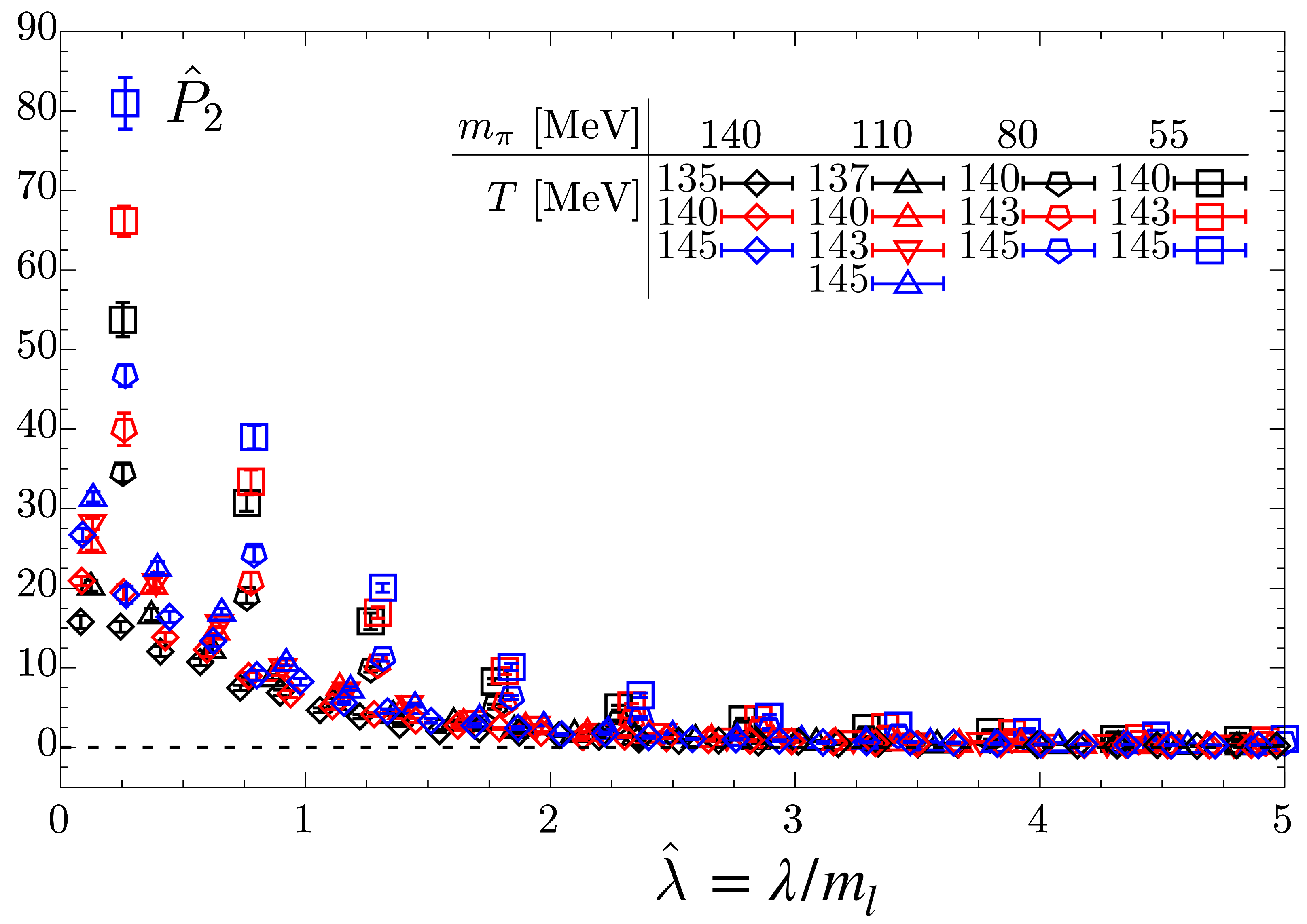


$P_n(\lambda)$ around T_c

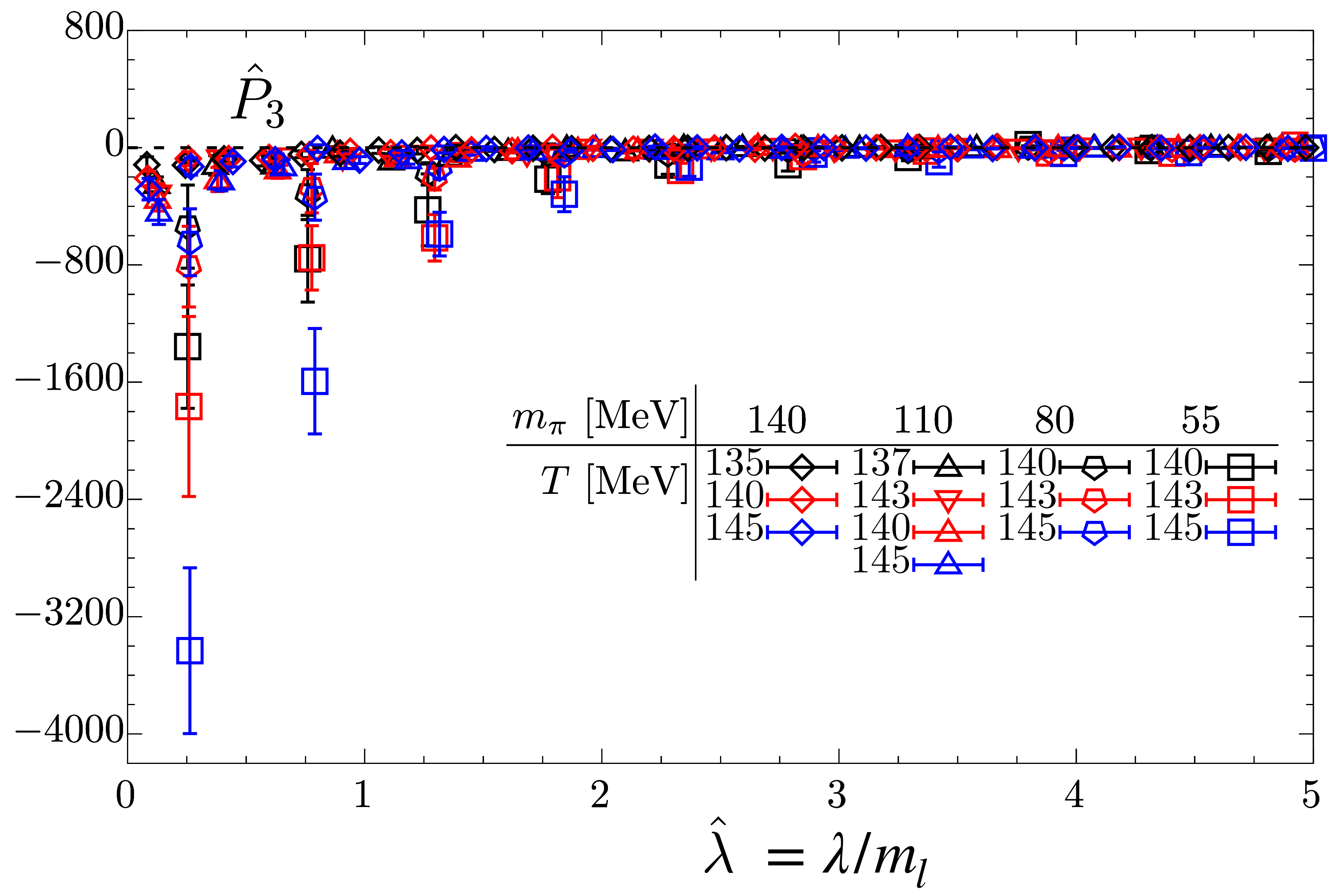
$\hat{P}_1(\hat{\lambda}) = m_s^2(m_l/m_s)P_1(\lambda)/T_c^4$



$\hat{P}_2(\hat{\lambda}) = m_s^3(m_l/m_s)P_2(\lambda)/T_c^4$



$\hat{P}_3(\hat{\lambda}) = m_s^4(m_l/m_s)P_3(\lambda)/T_c^4$



$\hat{P}_1(\hat{\lambda}), \hat{P}_2(\hat{\lambda})$ and $\hat{P}_3(\hat{\lambda})$

— Infrared lambda region dominates;

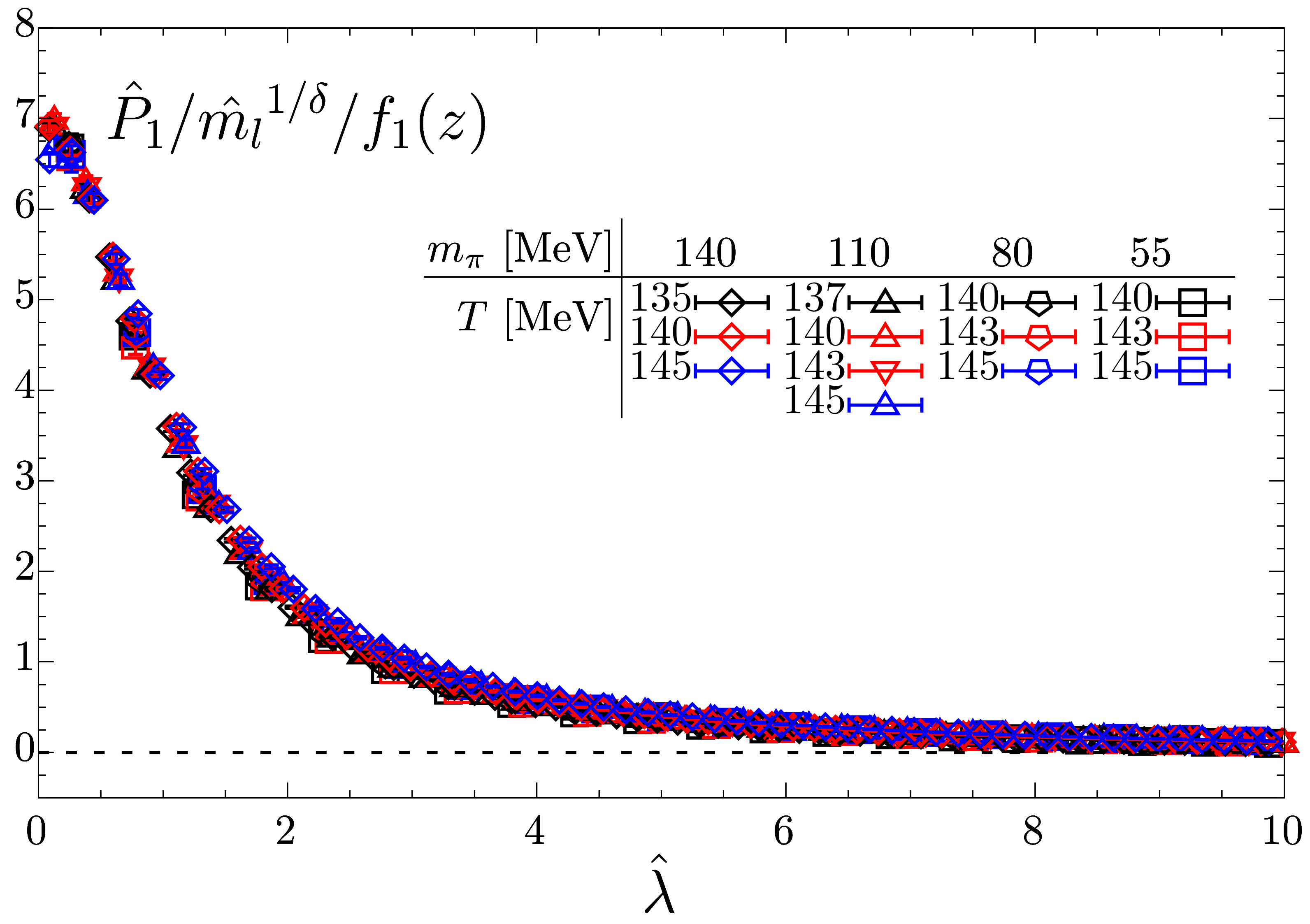
— Significant dependence on quark mass and temperature

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta - n + 1} f_n(z) g_n(\hat{\lambda})$

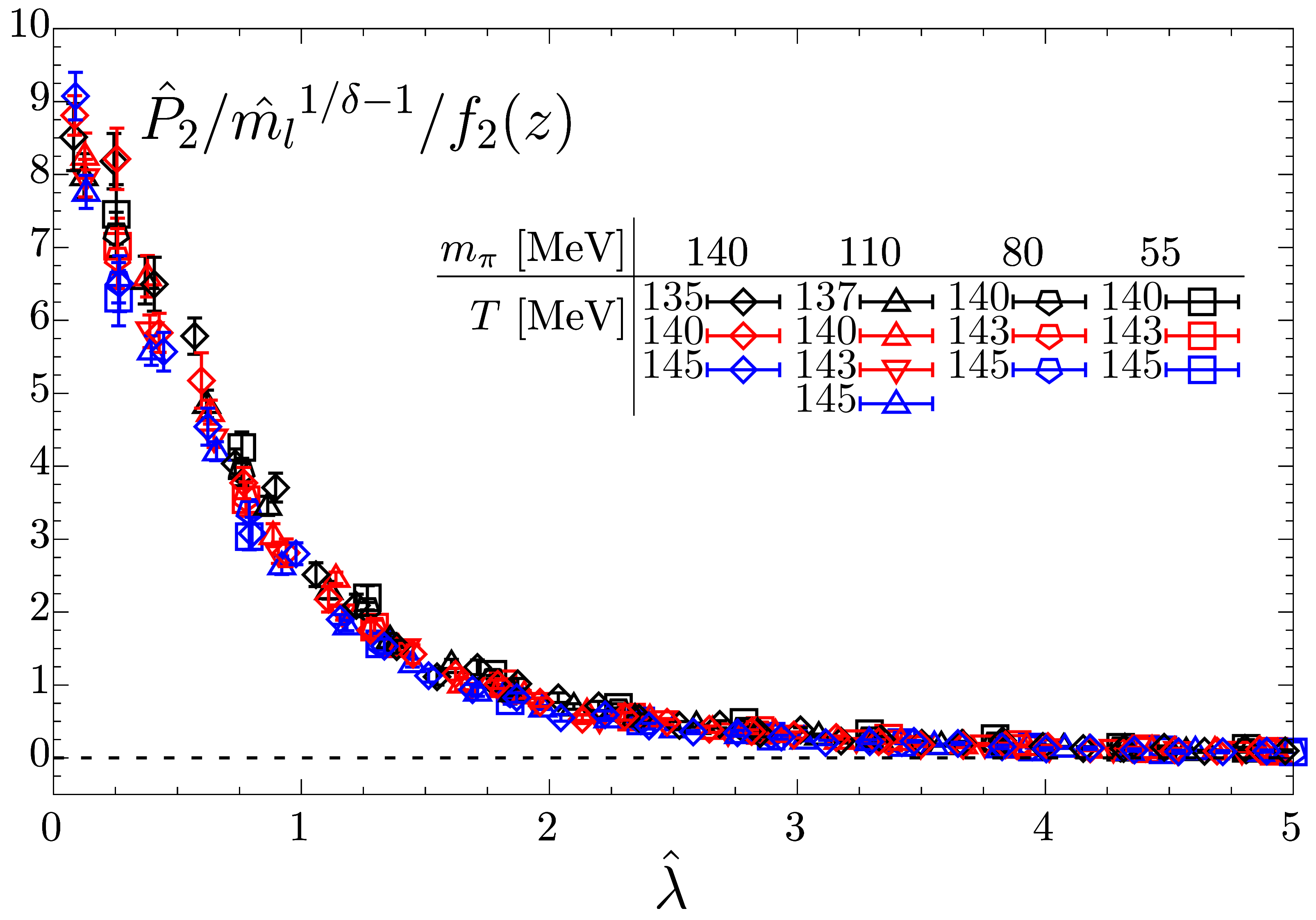
Rescaled $P_n(\lambda)$ around T_c

Conjecture: $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta-n+1} f_n(z) g_n(\hat{\lambda})$

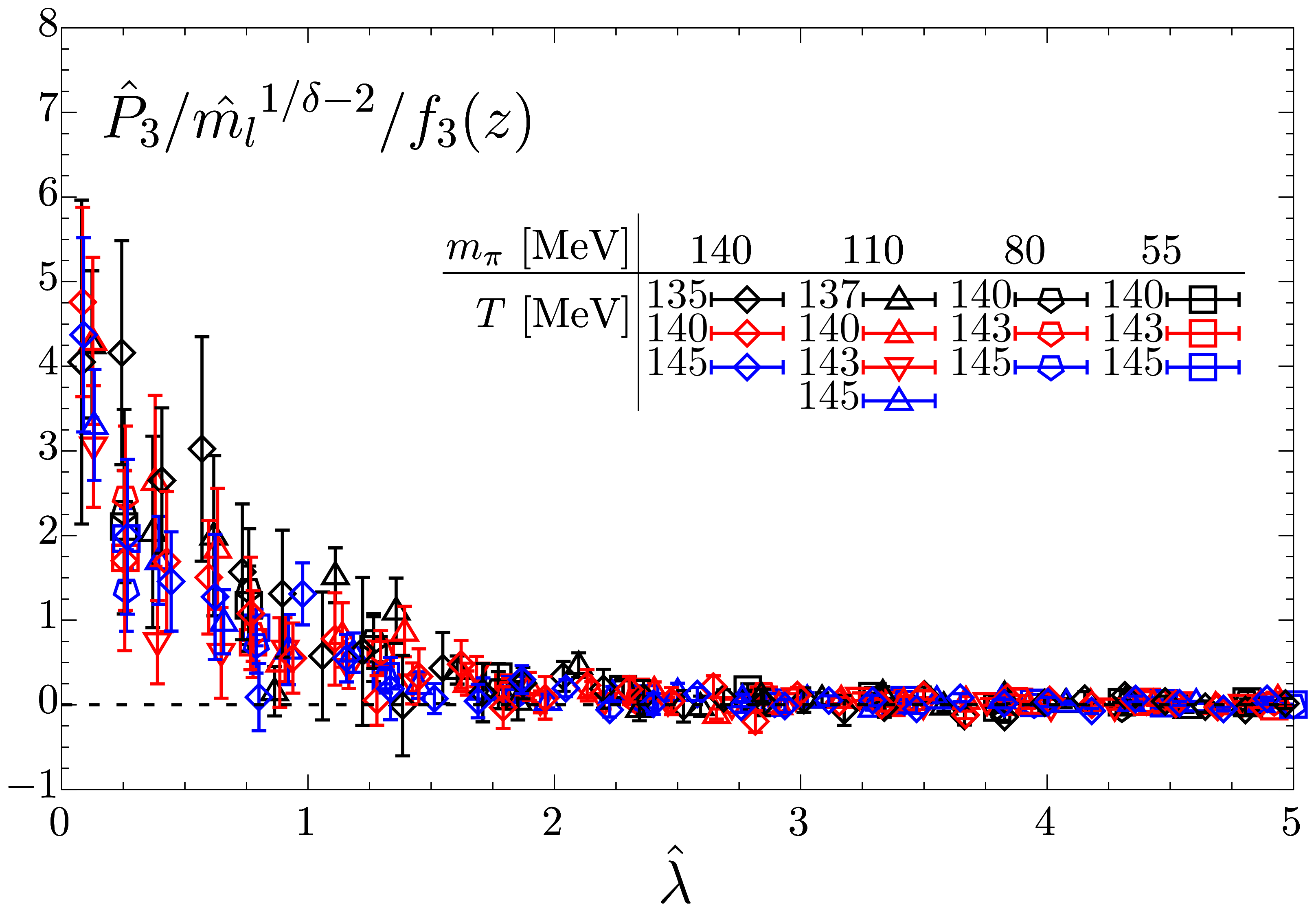
$\hat{P}_1(\hat{\lambda})/(m_l/m_s)^{1/\delta}/f_1(z)$



$\hat{P}_2(\hat{\lambda})/(m_l/m_s)^{1/\delta-1}/f_2(z)$



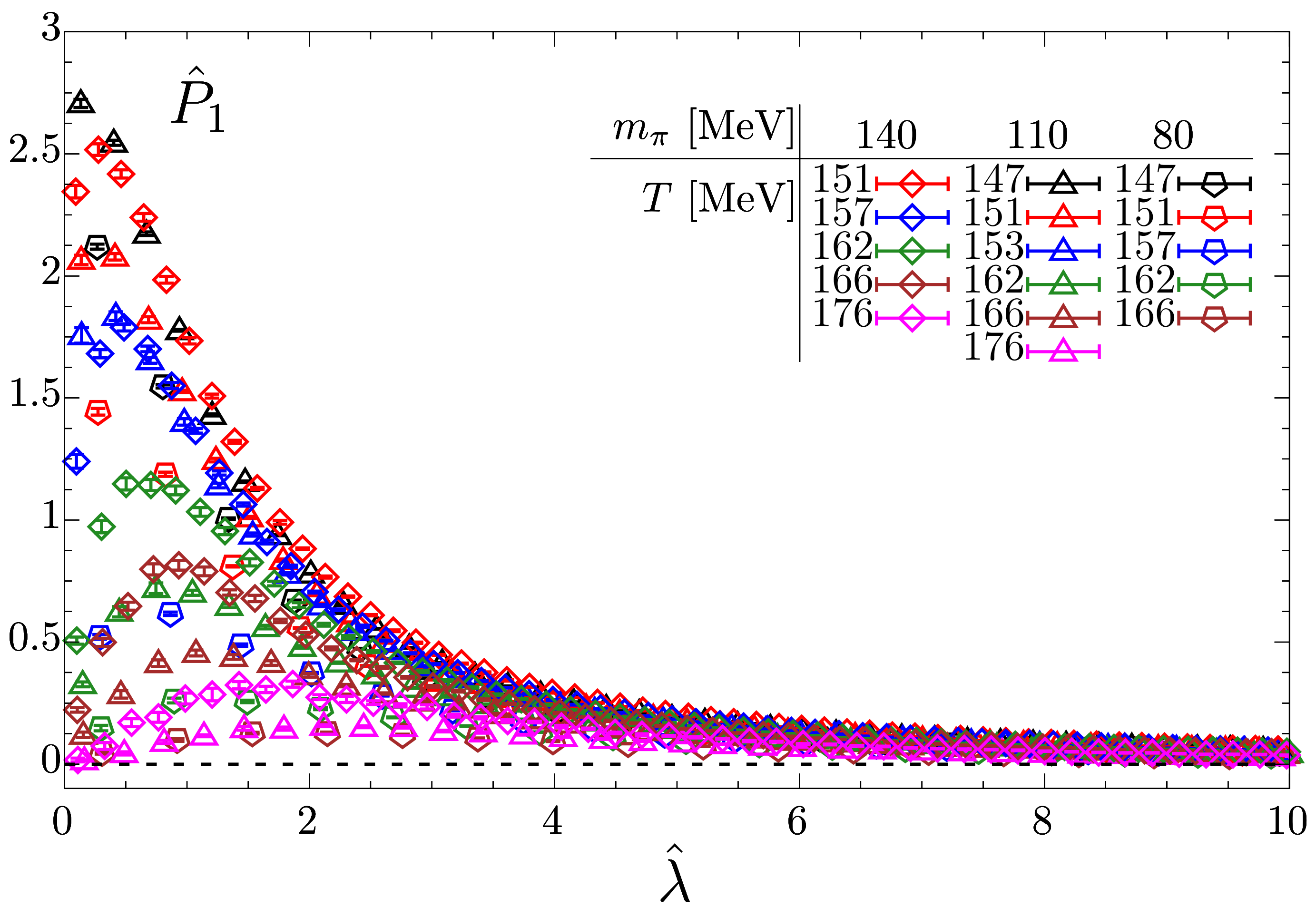
$\hat{P}_3(\hat{\lambda})/(m_l/m_s)^{1/\delta-2}/f_3(z)$

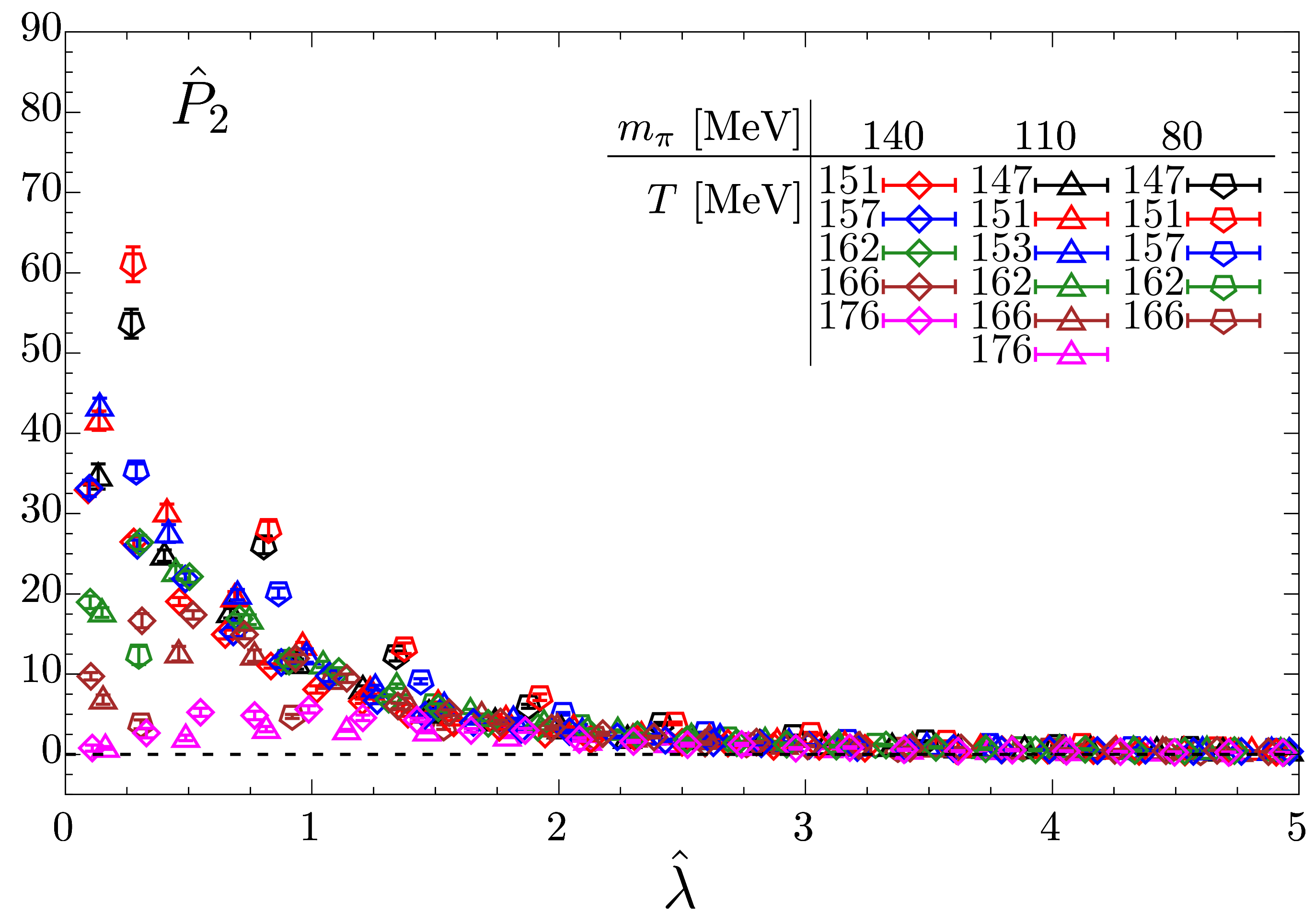


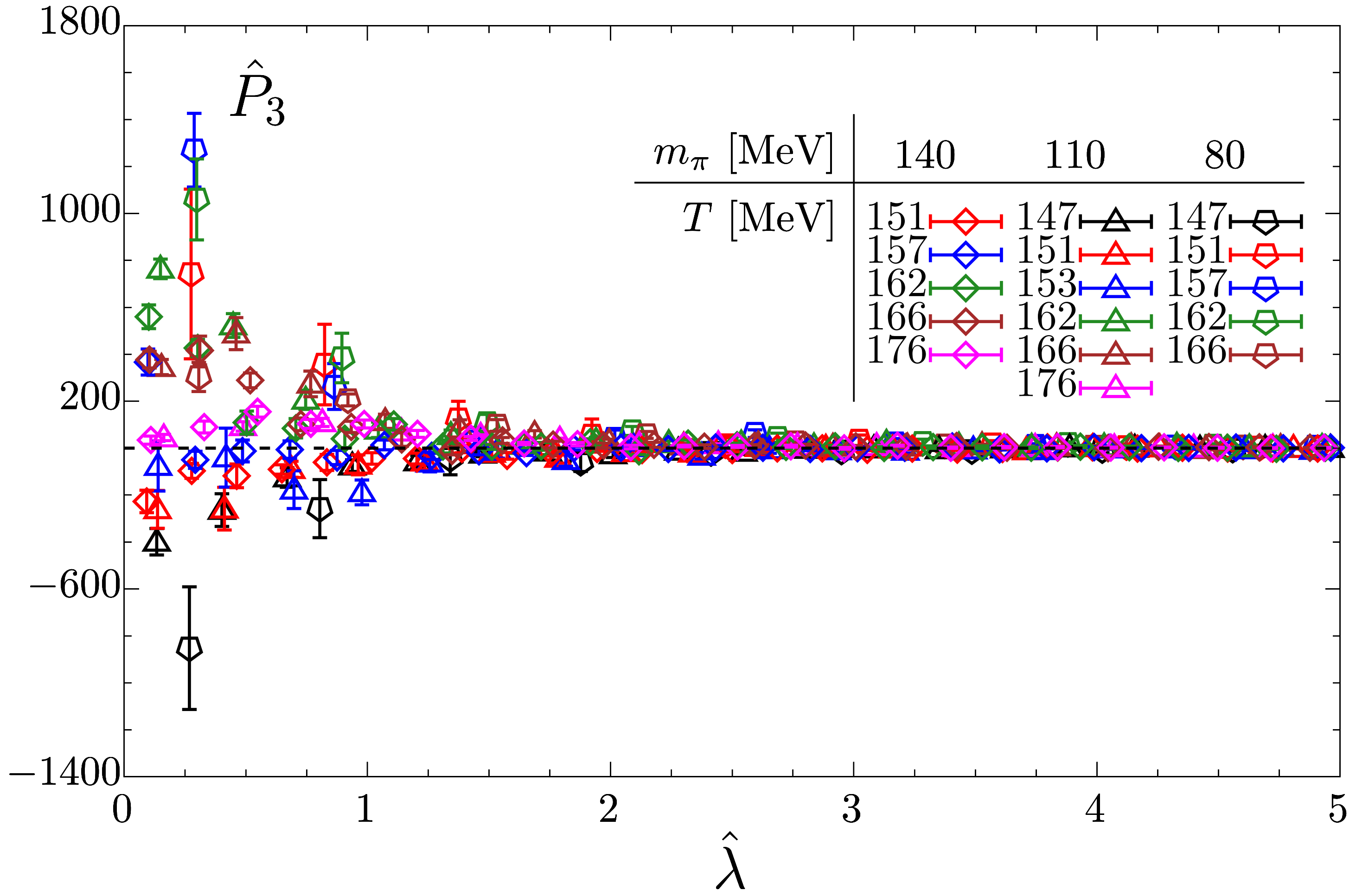
$z = z_0(m_l/m_s)^{-\frac{1}{\beta\delta}}(T - T_c)/T_c$: $O(2)$ scaling parameters adopted from [S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)]

- In the vicinity of T_c , $\hat{P}_n(\hat{\lambda}) = (m_l/m_s)^{1/\delta-n+1} f_n(z) g_n(\hat{\lambda})$
- Scaling behaviors in $\hat{P}_n(\hat{\lambda})$ extend up to physical light quark mass

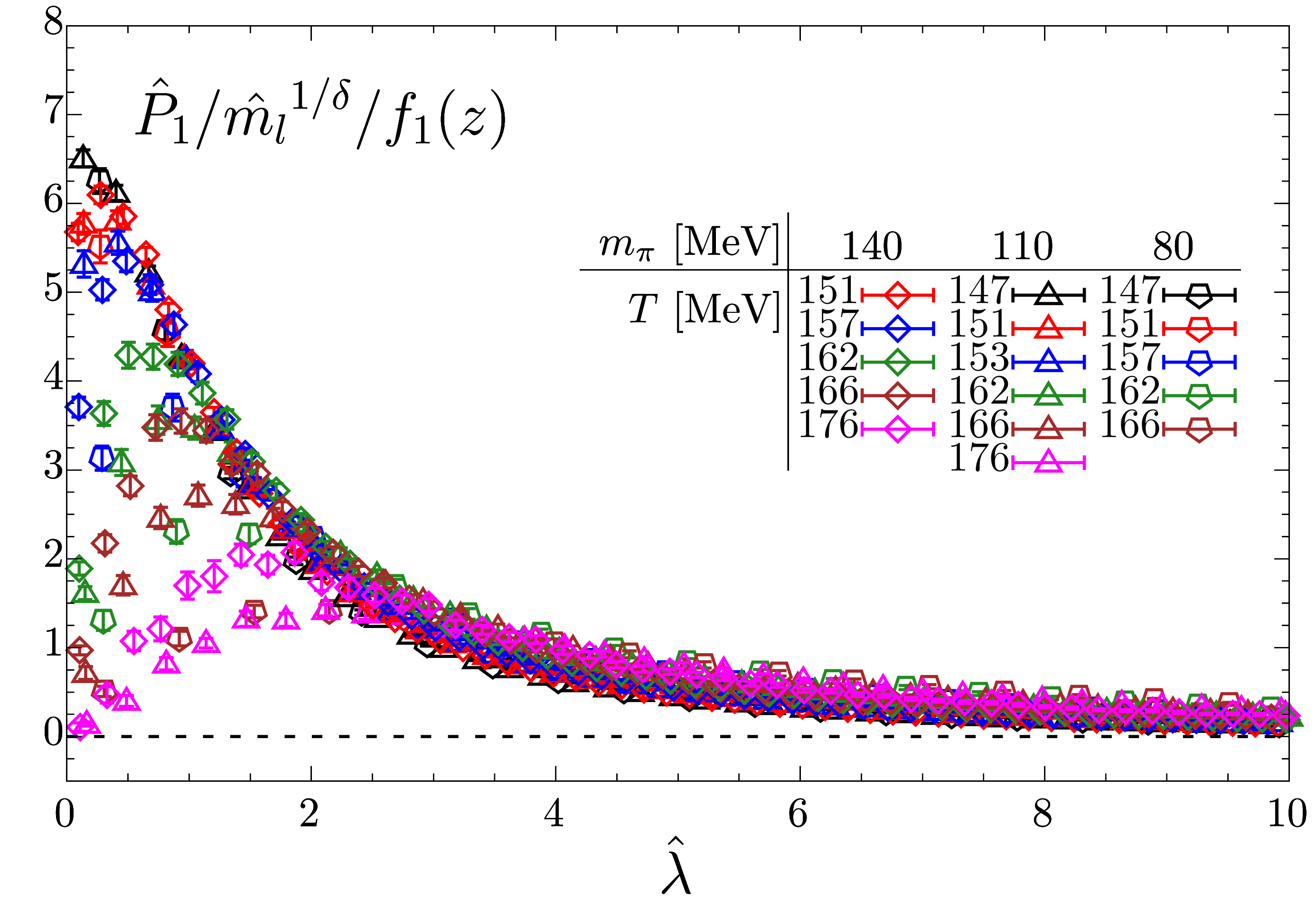
$P_n(\lambda)$ and Rescaled $P_n(\lambda)$ away from T_c

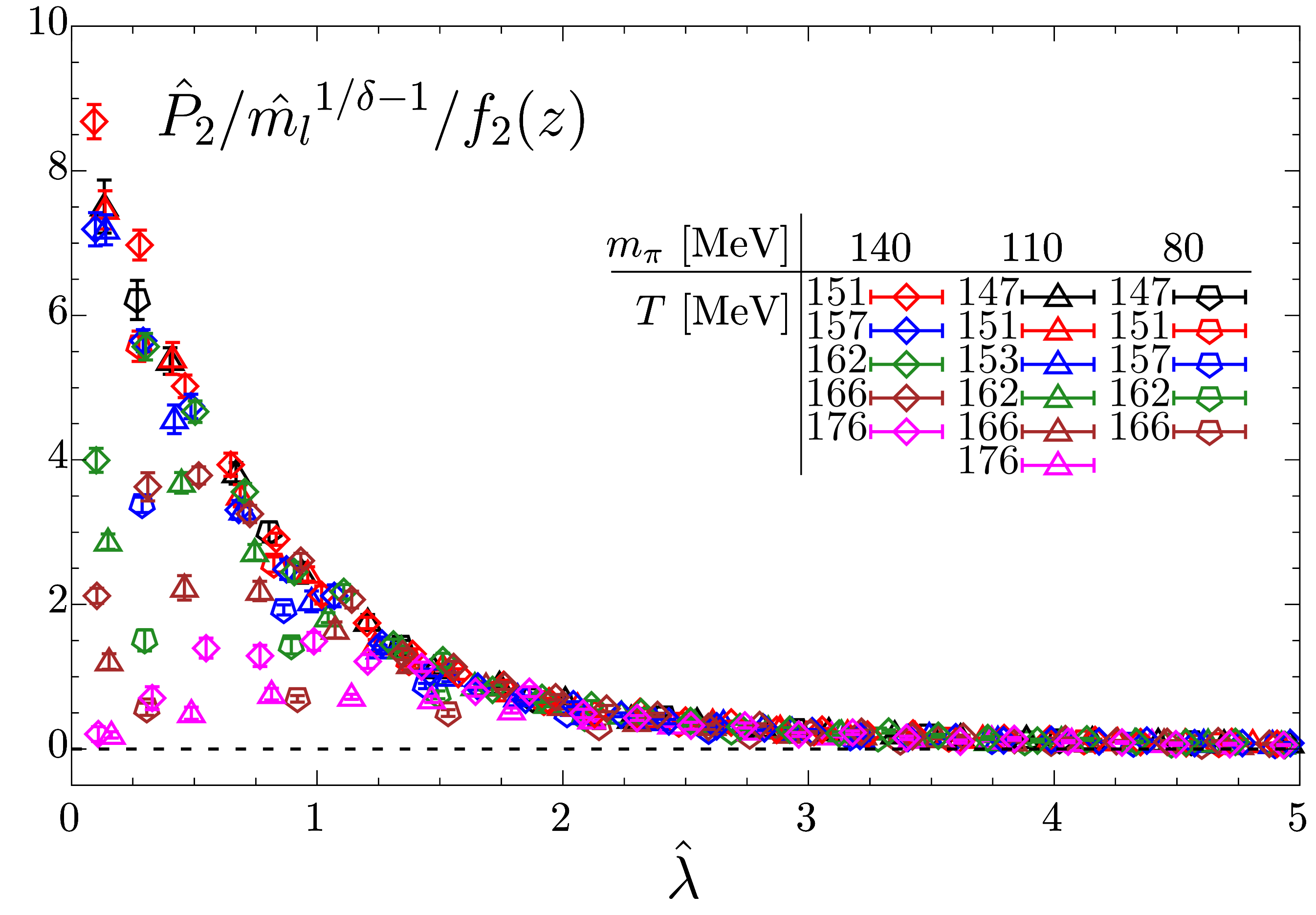


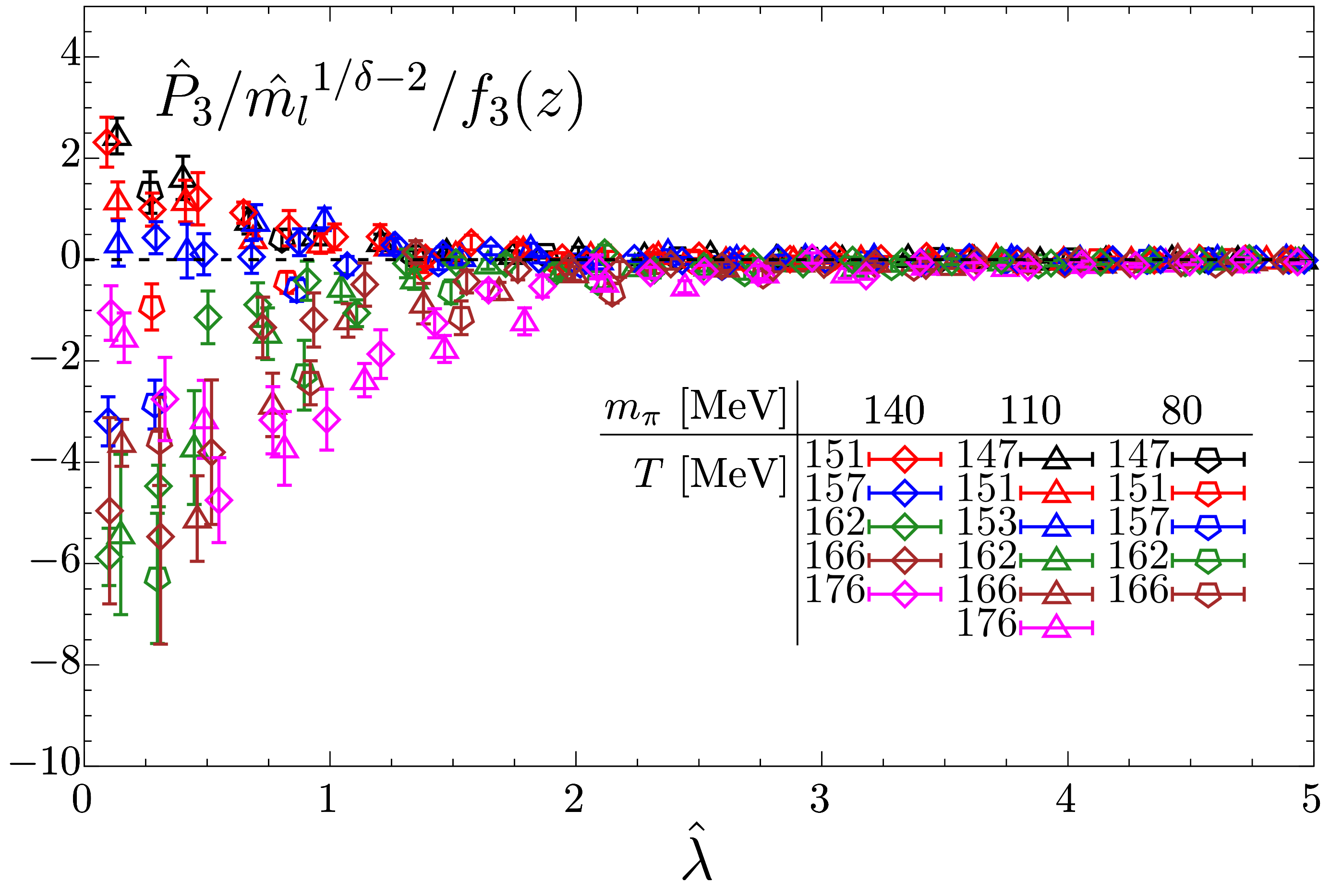




Away from T_c , no scaling behaviors are observed in $\hat{P}_n(\hat{\lambda})$







Summary

☑ We establish a novel relation

$$\mathbb{K}_n[\bar{\psi}\psi] = \int_0^\infty K_1[P_U(\lambda_1; m), P_U(\lambda_2; m), \cdots, P_U(\lambda_n; m)] \prod_{i=1}^{n} d\lambda_i \equiv \int_0^\infty P_n(\lambda)\ d\lambda$$

n-th order cumulant of the chiral condensate

n-point correlation of the quark energy spectra

☑ A generalization of the Banks-Casher relation is obtained:

$$\lim_{m\to 0} \mathbb{K}_n[\bar{\psi}\psi] = (2\pi)^n \mathbb{K}_n[\rho_U(0)]$$

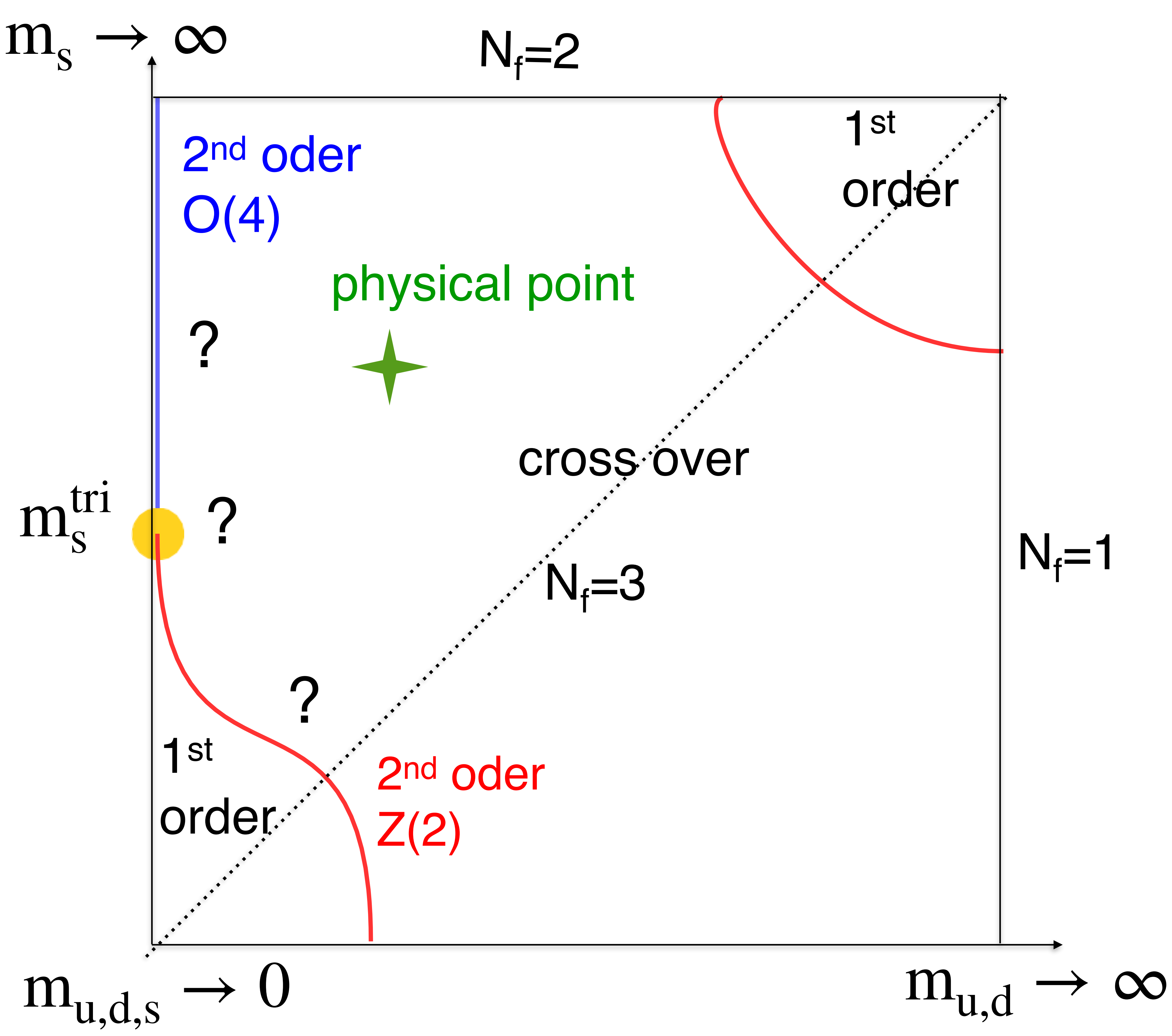
☑ Microscopic encoding of macroscopic criticality

$$P_n(\lambda) = m_l^{1/\delta - n+1} f_n(z) g_n(\lambda)$$

☑ Universal behaviors manifested in microscopic energy levels of QCD extend up to physical light quark masses

Backup

Symmetry and chiral phase transition



Columbia plot:
QCD phase diagram
in quark mass plane

- Symmetry of QCD:

In the classical and chiral limit: $SU(2)_L \otimes SU(2)_R \otimes U(1)_V \otimes U(1)_A$

Spontaneous chiral symmetry breaking ⇓ Quantum axial anomaly

Quantized: $SU(2)_V \otimes U(1)_V$

- Restoration of chiral symmetry:
  - At physical point: crossover at $T_{pc} = 156.5(1.5)$ MeV

    A. Bazavov et al., [HotQCD], PLB 795 (2019) 15
    S. Borsanyi et al., PRL 125 (2020) 052001

  - In light chiral limit: chiral phase transition at $T_c = 132^{+3}_{-6}$ MeV

    H.-T. Ding et al., [HotQCD], PRL 123 (2019) 062002

- Role of $U(1)_A$ symmetry in chiral phase transition:
  - Broken: 2^{nd} order O(4) phase transition
  - Effective restored: 1^{st} or 2^{nd} order $U(2)_L \otimes U(2)_R / U(2)_V$ phase transition

    Pisarski & Wilczek, PRD 29 (1984) 338
    Pelissetto & Vicari, PRD 88 (2013) 105018
    Grahl, PRD 90 (2014) 117904

Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

Mode number : $n_{[s,t]} \approx \frac{1}{n_v} \sum_{k=1}^{n_v} \left[\sum_{j=0}^{p} g_j^p \gamma_j v_k^T T_j(A) v_k \right]$

T_j : Chebyshev polynomial

γ_j & g_j^p : expansion coefficients

n_v : number of random vectors

p : number of polynomial orders

eigenvalue spectrum : $\rho_U(\lambda) = \frac{1}{4} \frac{n_{[\lambda-\delta/2,\lambda+\delta/2]}}{2\delta\lambda}$

$1/4$: Staggered Fermion Discretization Scheme

$1/2$: positive and negative eigenvalue pairs

$\delta\lambda$: bin-size

H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

Yu Zhang, Lattice 19', arXiv: 2001.05217

Cossu et al., arXiv: 1601.00744

Reproduction of Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$ via $P_n(\lambda)$

$\mathbb{K}_1[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_1[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_1(\lambda)\, d\lambda$ $\quad$ $\mathbb{K}_2[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_2[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_2(\lambda)\, d\lambda$ $\quad$ $\mathbb{K}_3[\bar{\psi}\psi] = \frac{T}{V}\mathbb{K}_3[2\ \mathrm{Tr}M^{-1}] = \int_0^\infty P_3(\lambda)\, d\lambda$

Open symbols: computation via inversions of the fermion matrix $\mathrm{Tr}M^{-1}$

Filled symbols: reconstructed from $P_n(\lambda)$

Cumulants related to $P_n(\lambda)$ can successfully reproduce their corresponding results from inverse fermion matrix

Criticality in Macroscopic Cumulants $\mathbb{K}_n[\bar{\psi}\psi]$

$O(2)$ scaling with $\beta = 0.349$, $\delta = 4.78$, $z_0 = 1.83(9)$, $T_c(N_\tau = 8) = 144.2(6)$ MeV

Parameters adopted from: S. Ejiri et al., Phys. Rev. D 80, 094505 (2009); D. A. Clarke et al., Phys. Rev. D 103, L011501 (2021)

- For $|z/z_0| \lesssim 0.2$, $K_n(z)/K_n(z=0)$ with $n = 1,2,3$ can be well described by $O(2)$ scaling function $\dfrac{f_{n-1}(z)}{f_{n-1}(z=0)}$
- For $|z/z_0| \lesssim 0.2$, K_n rescaled by $H^{1/\delta-n+1}$ show small quark dependence

$\partial^n \rho / \partial m_l^n$ and Quark Energy Spectra

Eigenvalue spectrum for (2+1)-flavor QCD:

$$\rho\left(\lambda, m_l\right) = \frac{T}{VZ[U]} \int D[U] e^{-S_G[U]} \det\left[\not{D}[U] + m_s\right] \times \left(\det\left[\not{D}[U] + m_l\right]\right)^2 \rho_U(\lambda)$$

Eigenvalue spectrum for a given configuration:

$$\rho_U(\lambda) = \sum_j \delta\left(\lambda - \lambda_j\right)$$

m_l dependence enters ρ:

$$\det\left[\not{D}[U] + m_l\right] = \prod_j \left(+\mathrm{i}\lambda_j + m_l\right)\left(-\mathrm{i}\lambda_j + m_l\right)$$

$$= \exp\left(\int_0^\infty \mathrm{d}\lambda \rho_U(\lambda) \mathrm{In}\left[\lambda^2 + m_l^2\right]\right)$$

Partition function:

$$Z[U] = \int D[U] e^{-S_G[U]} \det\left[\not{D}[U] + m_s\right] \times \left(\det\left[\not{D}[U] + m_l\right]\right)^2$$

Mass derivative of $\rho(\lambda, m_l)$:

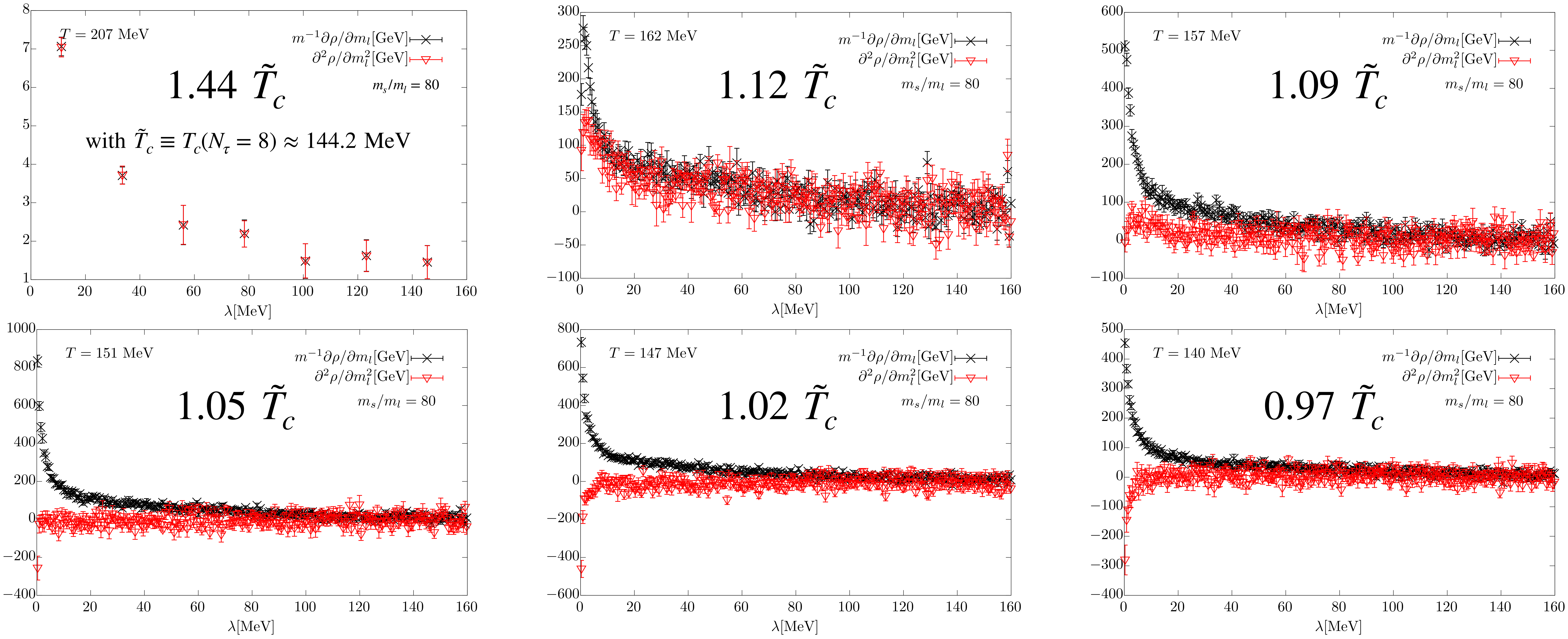
$$\frac{V}{T} \frac{\partial \rho(\lambda, m_l)}{\partial m_l} = \int_0^\infty \mathrm{d}\lambda_2 \, \frac{4 m_l C_2}{\lambda_2^2 + m_l^2} \qquad C_2\left(\lambda, \lambda_2\right) = \left\langle \rho_U(\lambda) \rho_U\left(\lambda_2\right)\right\rangle - \left\langle \rho_U(\lambda)\right\rangle \left\langle \rho_U\left(\lambda_2\right)\right\rangle$$

H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

Temperature dependence of $\partial^n \rho / \partial m_l^n$ with $n = 1,2$

W.-P. Huang, Lattice 21', arXiv: 2112.00318
H.-T. Ding et al., Phys. Rev. Lett. 126, 082001 (2021)

$$\frac{\partial \rho}{\partial m} = \frac{T}{V}\int_0^\infty \frac{4mK_1[\rho_U(\lambda),\rho_U(\lambda_2)]}{\lambda_2^2+m^2}d\lambda_2\,, \quad \frac{\partial^2 \rho}{\partial m^2} = \frac{T}{V}\int_0^\infty \frac{(4m)^2K_1[\rho_U(\lambda),\rho_U(\lambda_2),\rho_U(\lambda_3)]}{(\lambda_2^2+m^2)(\lambda_3^2+m^2)}d\lambda_2 d\lambda_3 + \frac{T}{V}\int_0^\infty \frac{4(\lambda_2^2-m^2)K_1[\rho_U(\lambda),\rho_U(\lambda_2)]}{(\lambda_2^2+m^2)^2}d\lambda_2$$



$\partial^2\rho/\partial m^2 \neq m^{-1}\partial\rho/\partial m$, Dilute Instanton Gas Approximation becomes invalid as getting closer to T_c