A pion decay constant in the multi-flavor Schwinger model

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Schwinger model [1]

- Represents two-dimensional QED.
- Shares properties with QCD: confinement, chiral symmetry breaking, topology [2].
- For N_f massless fermions, it has been shown in the framework of bosonization that a boson with mass $M_\eta = \sqrt{N_f g^2/\pi}$, together with $N_f - 1$ massless bosons ("pions") appear [3,4].

[1] J. Schwinger, *Phys. Rev.* **125** (1962). J. Schwinger, *Phys. Rev.* **128** (1962).

[2] S. R. Coleman et al, *Ann. Phys.* **93** (1975).

[3] L. V. Belvedere et al, *Nucl. Phys. B* **153** (1979).

[4] J. Hetrick, Y. Hosotani and S. Iso, *Phys. Lett. B* **350** (1995).

- Some works assume $N_f^2 - 1$ pions, which matches the number of Nambu-Goldstone bosons of the breaking pattern $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \to \mathrm{SU}(N_f)$ [5].

- The "pion" decay constant has been considered in the two-flavor Schwinger model before through the relation [6]

$$\langle 0|\partial^\mu J_\mu^5(0)|\pi(p)\rangle = M_\pi^2 F_\pi,$$

  where J_μ^5 is the axial current and M_π is the "pion" mass. Ref. [6] obtains, for two degenerate fermions in a light-cone formulation,

$$F_\pi(m) = 0.394518(14) + 0.040(1)m/g.$$

- We are interested in F_π in the chiral limit.

[5] C. Gattringer and E. Seiler, *Annals Phys.* **233** (1994).

[6] K. Harada et al, *Phys. Rev. D* **49** (1994).

Other ways of determining F_π

- Ref. [7] shows that

$$\Sigma = \frac{M_\pi^2}{4\pi m}.$$

On the other hand, the Gell-Mann–Oakes–Renner relation in QCD reads

$$F_\pi^2(m) = \frac{2m}{M_\pi^2}\Sigma.$$

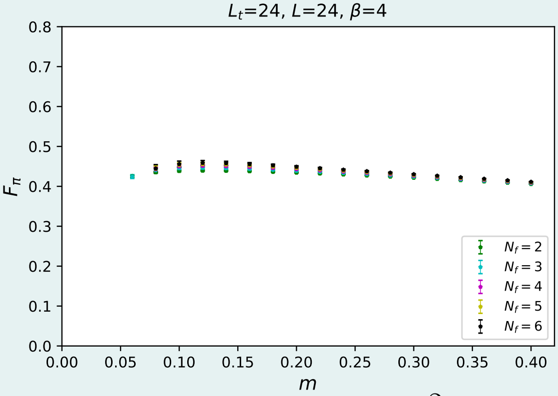
If we push the analogy with the Schwinger model further, we obtain

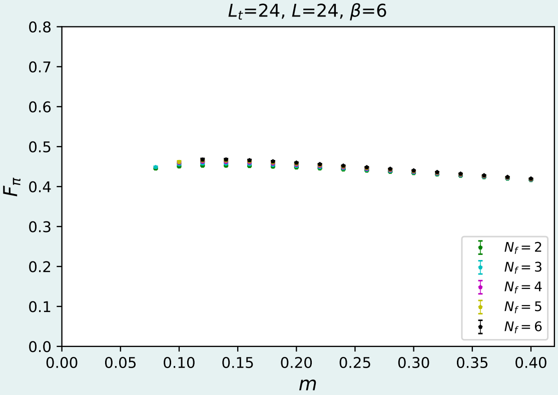
$$F_\pi = \frac{1}{\sqrt{2\pi}} \simeq 0.3989.$$

[7] Y. Hosotani and R. Rodriguez, *J. Phys. A* **31** (1998).

GMOR relation

Reweighted overlap-hypercube fermions, $\beta = \frac{1}{g^2}$





$$F_\pi^2(m) = \frac{2m}{M_\pi^2}\Sigma, \quad F_\pi(0) \approx 0.4, \text{ for } N_f = 2, \ldots, 6.$$

- The Witten-Veneziano formula [8] has been shown to be exact in the Schwinger model in the chiral limit [9]

$$M_\eta^2 = \frac{2N_f}{F_\eta^2}\chi_t^q,$$

where

$$M_\eta^2 = \frac{N_f g^2}{\pi}, \quad \chi_t^q = \frac{g^2}{4\pi^2}.$$

Then

$$F_\eta = \frac{1}{\sqrt{2\pi}}.$$

In large-N_c QCD both F_η and F_π are asymptotically equal. If we push, once again, the analogy between QCD and the Schwinger model we obtain

$$F_\pi = \frac{1}{\sqrt{2\pi}}.$$

[8] E. Witten, *Nucl. Phys. B*, **156** (1979). G. Veneziano, *Nucl. Phys. B* **159** (1979).

[9] E. Seiler and I. O. Stamatescu, *MPI-PAE-PTh-10-87*. Seiler, *Phys. Lett. B* **525** (2002).

δ-regime

- We perform simulations in the δ-regime: $L_t \gg \frac{1}{M_\pi} \gtrsim L$.
- The small spatial volume allows one to consider the model as a quasi one-dimensional system, approximated by a quantum mechanical rotor [10].
- The pion has a residual mass in the chiral limit

$$m \to 0 \Rightarrow M_\pi \to M_\pi^R = \frac{N_\pi}{2\Theta_{\text{eff}}} > 0,$$

where Θ_{eff} is the effective moment of inertia.

[10] H. Leutwyler, *Phys. Lett. B* **189** (1987).

- Leutwyler computed $\Theta_{\text{eff}} = F_\pi^2 L^3$ to leading order.
- Hasenfratz and Niedermayer [11] computed the moment of inertia to next-to-leading order in $d > 2$

$$\Theta_{\text{eff}} = F_\pi^2 L^{d-1} \left[1 + \frac{N_\pi - 1}{2\pi F_\pi^2 L^{d-2}} \left(\frac{d-1}{d-2} + \ldots \right) + \cdots \right].$$

  This assumes the existence of Nambu-Goldstone bosons.
- In two dimensions we conjecture that

$$\Theta_{\text{eff}} \simeq F_\pi^2 L \quad \rightarrow \quad M_\pi^R \simeq \frac{N_\pi}{2F_\pi^2 L}.$$

  For the moment we consider $N_\pi = N_f - 1$.

[11] P. Hasenfratz and F. Niedermayer, *Z. Phys. B* **92** (1993).

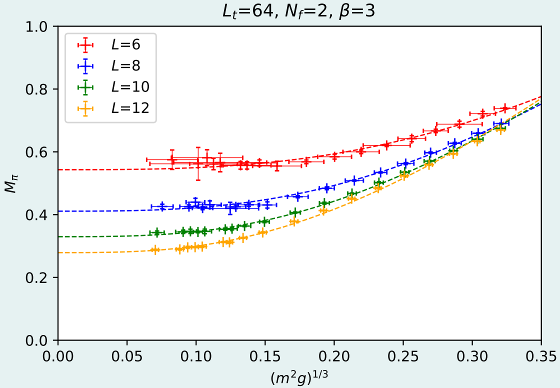
Simulations

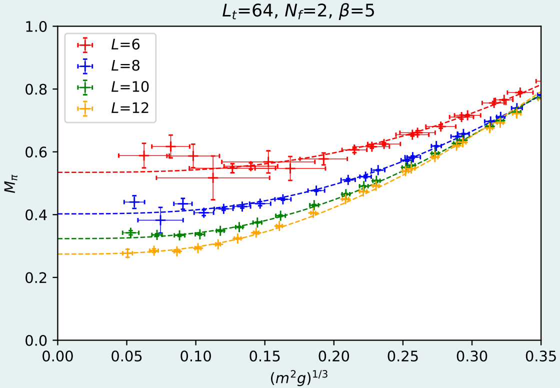
- We perform simulations in the δ-regime to determine F_π by measuring M_π^R for several space extensions L, with 10^4 configurations.
- For $N_f = 2$ we simulate Wilson fermions with the HMC algorithm. The degenerate fermion mass is computed with the PCAC relation.
- For $N_f \geq 2$ we simulate overlap-hypercube fermions [12,13] with quenched re-weighted configurations.

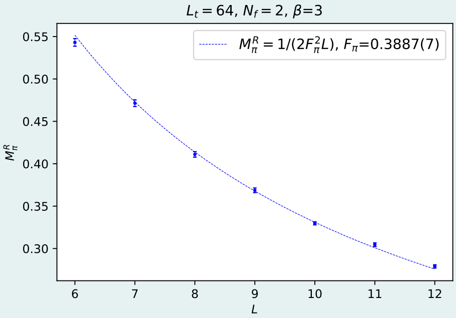
[12] H. Neuberger, *Phys. Lett. B* **417** (1998).

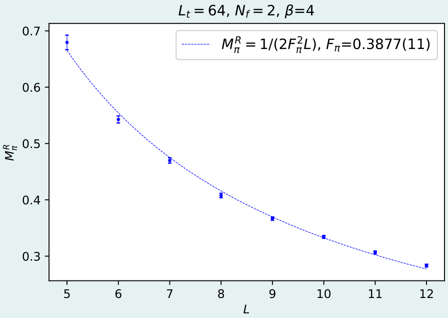
[13] W. Bietenholz and I. Hip, *Nucl. Phys. B* **570** (2000).

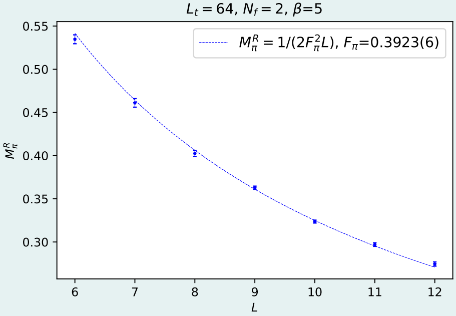
Wilson fermions results







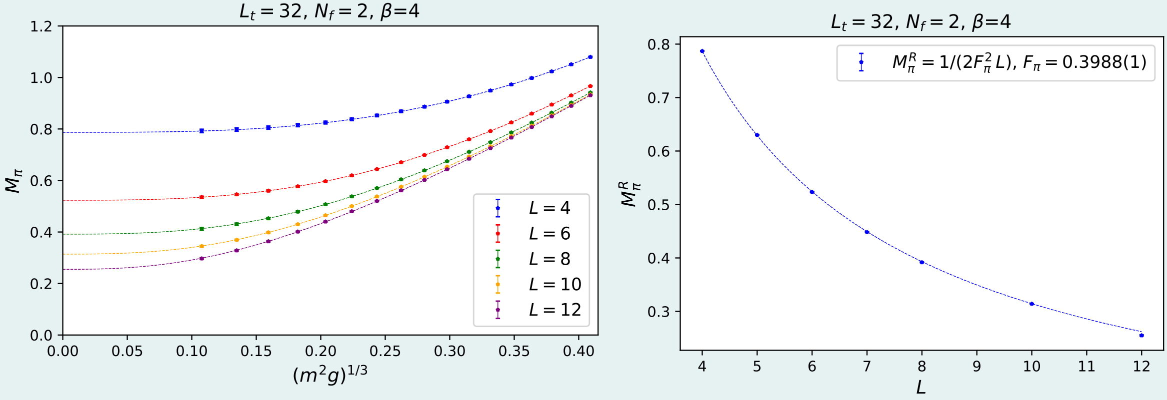




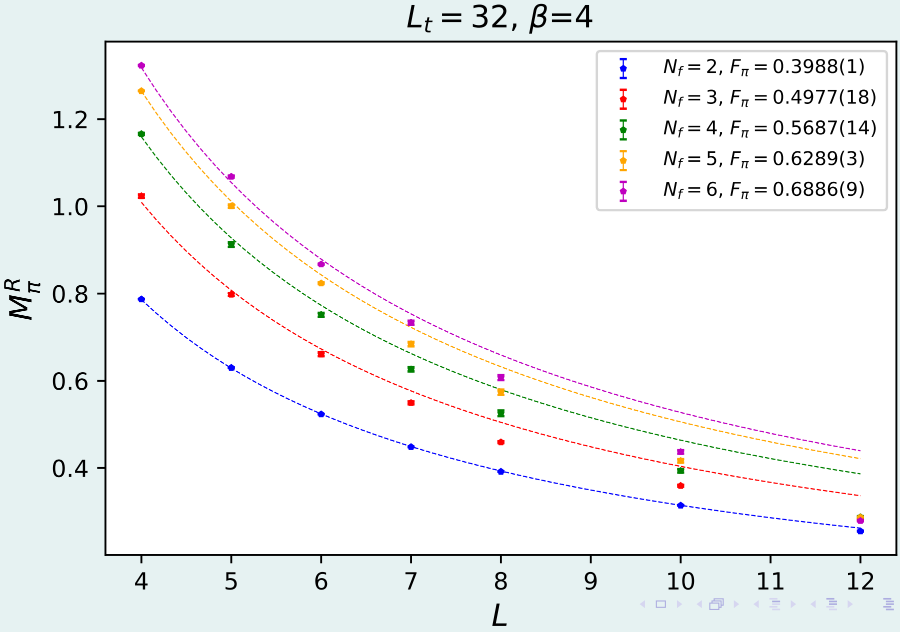
β	3	4	5
F_π	0.3887(7)	0.3923(6)	0.3877(11)

$$M_\pi^R = \frac{N_f - 1}{2\, F_\pi^2 L}, \qquad \frac{1}{\sqrt{2\pi}} \simeq 0.3989$$

Overlap-hypercube fermions results



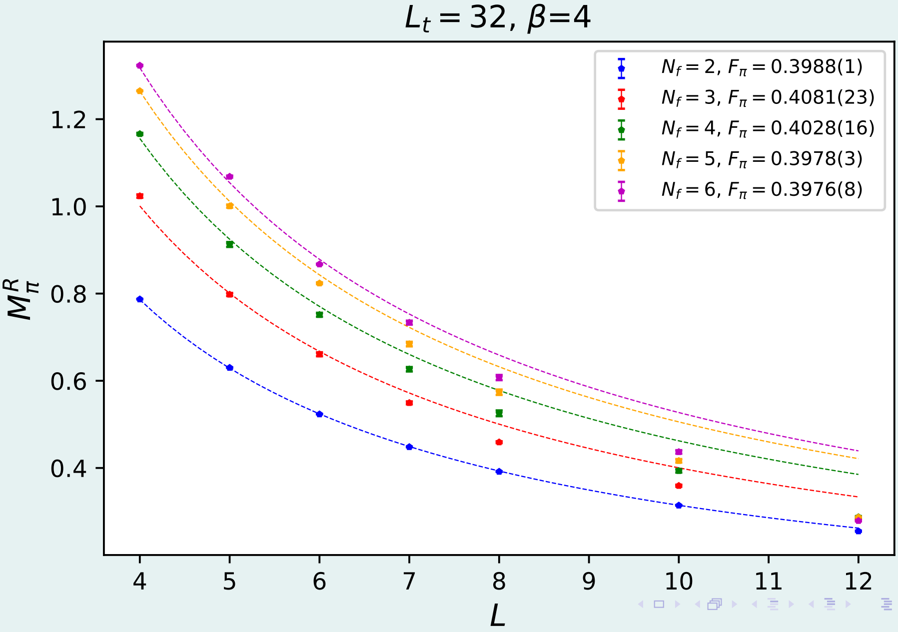
$$M_\pi^R = \frac{N_f - 1}{2\, F_\pi^2 L}$$



$$M_\pi^R = \frac{N_f - 1}{2\,F_\pi^2 L}$$

$\downarrow$

$$M_\pi^R = \frac{N_f - 1}{N_f\,F_\pi^2 L}$$



Conclusions

- For $N_f = 2$ the result of F_π is compatible with $1/\sqrt{2\pi}$, for simulations with Wilson fermions and with overlap fermions in the δ-regime. This same value is found through alternative ways: GMOR relation and W-V formula.
- The results suggest that F_π is flavor independent if we insert $N_\pi = (N_f - 1)/N_f$.
- Considering the previous point, we find $F_\eta = F_\pi$.
- We attract attention to F_π in the Schwinger model, which has been almost overlooked in the literature.