Comparison with model-independent and dependent analyses for pion charge radius

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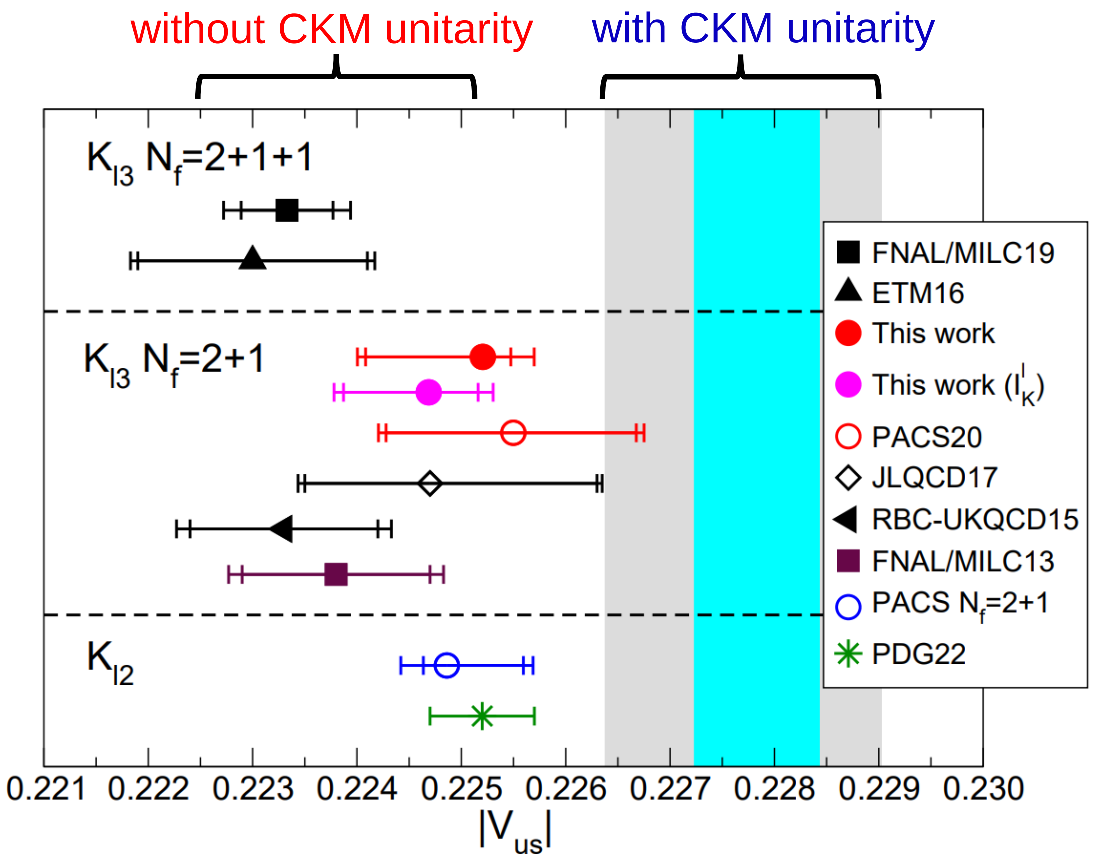
Fermilab

Introduction

■ Motivation

Recently, precise verification of standard model has been actively studied.

Ex) CKM matrix element $|V_{us}|$



arXiv:2212.00255 [hep-lat]

In the K_{l3} decay $\Gamma_{K_{l3}} = C_{K_{l3}}(|V_{us}|f_+(0))^2 I_K^l$

· decay width

known values

· Fermi coupling constant

· Kaon mass and so on

(Experimental value**)=(**Known constant**)** ×

(Matrix element) × **(Hadron matrix element)**

not known values

· Form factor

Obtained using lattice QCD calculations

· CKM matrix element $|V_{us}|$

For precise verification

· Reduce the (systematic+statistic) error

· Multiple groups calculate in different methods

New algorithm development is important.

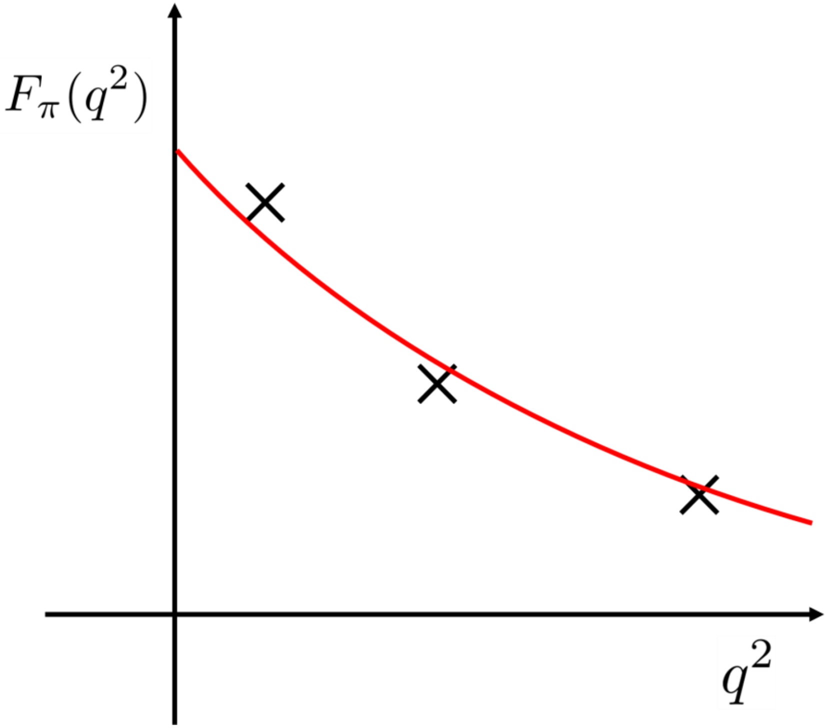
Introduction

$$\Gamma_{K_{l3}} = C_{K_{l3}} (|V_{us}| f_+(0))^2 I_K^l$$

○Traditional method for form factor

The form factor analysis is performed by Making $(q^2, F_\pi(q^2))$ data and fitting it.

fit ansatz
(model-dependent)



$$F_\pi(q^2) = \frac{1}{1+q^2/M_{\rm pole}^2} \quad \text{(monopole)}$$

$$F_\pi(q^2) = 1 + f_1 q^2 + f_2 (q^2)^2$$

$$F_\pi(q^2) = 1 + f_1 q^2 + f_2 (q^2)^2 + f_3 (q^2)^3 \quad \text{(polynomial)}$$

$$F_\pi(q^2) = \sum_{k=0}^{2} a_k z^k \quad \left(z := \frac{\sqrt{4m_\pi^2 + q^2} - \sqrt{4m_\pi^2}}{\sqrt{4m_\pi^2 + q^2} + \sqrt{4m_\pi^2}} \right) \quad \text{(z-expansion)}$$

These fit ansatz $\begin{cases} \text{causes the systematic error} \\ \text{increase the lattice QCD error} \end{cases}$

$$F_\pi(q^2) = \frac{2E_\pi(p) Z_\pi(0)}{(E_\pi(p)+m_\pi) Z_\pi(p)} \frac{\tilde{C}_{\pi V \pi}(t, t_{\rm sink}; p)}{\tilde{C}_{\pi V \pi}(t, t_{\rm sink}; 0)} e^{(E_\pi(p) - m_\pi)t}$$

3-point function: $\tilde{C}_{\pi V \pi}(t, t_{\rm sink}; p) = Z_V \sum_{\vec{x},\vec{y},\vec{z}} \langle 0 | \pi^+(\vec{z}, t_{\rm sink}) V_4(\vec{y}, t) \pi^{+\dagger}(\vec{x}, 0) | 0 \rangle \, e^{ip(x_1 - y_1)}$

Introduction

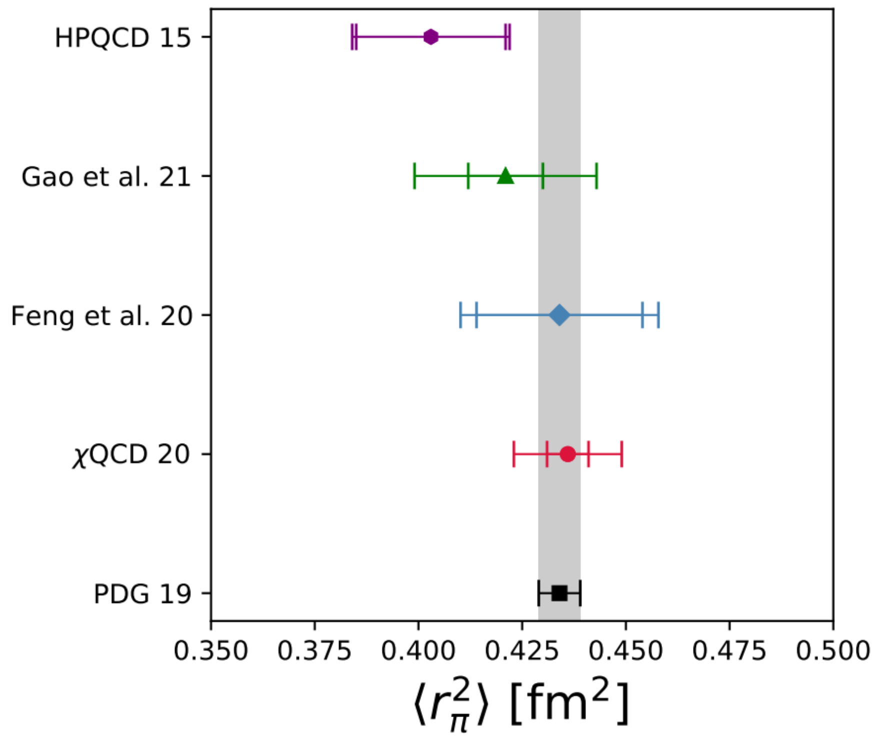
○Pion charge radius

electromagnetic
form factor

$$\langle \pi^+(p_f)|V_\mu|\pi^+(p_i)\rangle = (p_f + p_i)_\mu F_\pi(q^2)$$

charge radius:

$$\langle r_\pi^2 \rangle = -6 \frac{\mathrm{d}}{\mathrm{d}q^2} F_\pi(q^2)\Big|_{q^2=0}$$



Phys.Rev.D104,114515(2021)

○Phys.Lett.B324,85(1994) ○Nucl.Phys.B444,401(1995) ○PoS LATTICE2016,170(2016)

- Proposed method for directly calculating the first-order derivative of the form factor
- Applied to calculate nucleon form factor

improve

○Phys.Rev.D101,051502(R)(2020)

- Applied to calculate pion charge radius
- Proposed method to reduce the high-order contamination.

Improve(Our work last year)

○PoS LATTICE2022,122 (2023)

- Found that large finite volume effect in small M^2_{pole} and volume.
- Proposed method to solve the above problem.

model-independent

This talk:

Efficient way to reduce the finite volume effect of original model-independent method for pion charge radius

Outline

- ~~Introduction~~
- Model-independent method
  - Original model-independent method
- Our improved model-independent method
  - Proposal for our method
  - Introduce new function $\mathrm{G}(q^2)$ to reduce the finite volume effect
- Preliminary result
  - Compare the results of Traditional method, Original method and Our method
- Summary

model-independent method

Phys.Lett.B324,85(1994) ; Nucl.Phys.B444,401(1995) ; PoS LATTICE2016,170(2016)

$$\tilde{C}_{\pi V\pi}(t,t_{\rm sink};p) = Z_V Z_\pi(0) Z_\pi(p) L^2 \frac{(E_\pi(p)+m_\pi)}{2m_\pi 2E_\pi(p)} F_\pi(q^2) e^{-E_\pi(p)t} e^{-m_\pi(t_{\rm sink}-t)}$$

For $a \to 0$ and $V \to \infty$

$$F(\vec{x}) = F(|\vec{x}|)$$

$$\left.\frac{\mathrm{d}\tilde{F}(\vec{p})}{\mathrm{d}|\vec{p}|^2}\right|_{|\vec{p}|^2=0} = \left.\frac{\mathrm{d}}{\mathrm{d}|\vec{p}|^2}\int \mathrm{d}^3x\, F(\vec{x}) e^{-i\vec{p}\cdot\vec{x}}\right|_{|\vec{p}|^2=0} = -\frac{1}{3!}\int \mathrm{d}^3x\, |\vec{x}|^2 F(\vec{x})$$

n-th order momentum-derivative at $|\vec{p}|^2 = 0$ ⟺ $2n$-th order spatial moment ($|\vec{x}|^{2n}$)

For finite V, the higher-order contaminations appear.

1-dimension 3-point function

$$C_{\pi V\pi}(t,t_{\rm sink};r) := Z_V \sum_{\vec{z}} \sum_{y_2,y_3} \sum_{x_2,x_3} \langle 0|\pi^+(\vec{z},t_{\rm sink}) V_4(\vec{y},t) \pi^{+\dagger}(\vec{x},0)|0\rangle$$

$$(r := |x_1 - y_1|\,;\ 0 \le r \le L/2;\ \text{Periodic B.C.})$$

$$C^{(n)}(t) := \sum_r r^{2n} C_{\pi V\pi}(t,t_{\rm sink};r) = \sum_r r^{2n} \frac{1}{L} \sum_p \tilde{C}_{\pi V\pi}(t,t_{\rm sink};p) e^{ipr}$$

$$= \sum_p \Delta(t,t_{\rm sink},p) T_n(p) F_\pi(q^2) \quad \left(\tilde{C}_{\pi V\pi}(t,t_{\rm sink};p) = \Delta(t,t_{\rm sink},p) F_\pi(q^2),\ \ T_n(p) := \frac{1}{L}\sum_r r^{2n} e^{ipr} \right)$$

$$= f_0\beta_{0,n}(t) + f_1\beta_{1,n}(t) + \underbrace{f_2\beta_{2,n}(t) + \cdots}_{\text{higher-order contamination}} \quad \left(F_\pi(q^2) = \sum_{m=0}^{\infty} f_m q^{2m},\ \ \underbrace{\beta_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t,t_{\rm sink},p) T_n(p) q^{2m} \right)$$

$$\langle r_\pi^2 \rangle = -6 \boxed{\left.\frac{\mathrm{d}}{\mathrm{d}q^2} F_\pi(q^2)\right|_{q^2=0}}$$

model-independent method

Phys.Rev.D101,051502(R)(2020)

$$(C^{(0)}(t) := 1, \quad f_0 = 1)$$

$$\underbrace{C^{(n)}(t)}_{\text{moment function}} := \sum_r r^{2n} C_{\pi V \pi}(t, t_{\text{sink}}; r) = \sum_{m=0}^{\infty} f_m \beta_{m,n}(t)$$

$$\underbrace{\beta_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t, t_{\text{sink}}, p) T_n(p) q^{2m}$$

To reduce the higher-order contamination

$$R(t) := \alpha_1 C^{(1)}(t) + \alpha_2 C^{(2)}(t) + h$$

$$= (\alpha_1 \beta_{0,1} + \alpha_2 \beta_{0,2} + h) + (\alpha_1 \beta_{1,1} + \alpha_2 \beta_{1,2}) \underline{f_1} + (\alpha_1 \beta_{2,1} + \alpha_2 \beta_{2,2}) f_2 + \cdots$$

$$\langle r_\pi^2 \rangle = -6 \boxed{\frac{\mathrm{d}}{\mathrm{d}q^2} F_\pi(q^2)\Big|_{q^2=0}}$$

Define parameters α_1, α_2, h to satisfy the following

$$\alpha_1 \beta_{0,1} + \alpha_2 \beta_{0,2} + h = 0 \qquad \alpha_1 \beta_{1,1} + \alpha_2 \beta_{1,2} = 1 \qquad \alpha_1 \beta_{2,1} + \alpha_2 \beta_{2,2} = 0$$

$$R(t) = \underbrace{f_1}_{\text{constant}} + \sum_{m=3}^{\infty} \left(\underbrace{\sum_{k=1}^{2} \alpha_k \beta_{m,k}(t)}_{\text{time-dependent}} \right) f_m$$

$$\langle r_\pi^2 \rangle = -6 \frac{\frac{\mathrm{d}}{\mathrm{d}q^2} F_\pi(q^2)\Big|_{q^2=0}}{f_1} \sim -6 \times R(t)$$

If the high-order contamination terms is small, we get the charge radius

Our improved model-independent method

PoS LATTICE2022,122(2023)

$$R(t) = f_1 + \sum_{m=3}^{\infty}\left(\sum_{k=1}^{2}\alpha_k\beta_{m,k}(t)\right)f_m$$

Original method remains the contamination from high-order at small $M_{\rm pole}^2$ and volume.

➡ Improve the convergence of f_m and reduce the contamination

-- Fact --

Pion form factor is well represented by

$$F_\pi(q^2) = \frac{1}{1+q^2/M_{\rm pole}^2}$$

from phenomenology.

$$\begin{aligned} C^{(n)}(t) &= \sum_p \Delta(t,t_{\rm sink},p)T_n(p)F_\pi(q^2) = \sum_p \Delta(t,t_{\rm sink},p)T_n(p)F_\pi(q^2)\frac{G(q^2)}{G(q^2)} \\ &= \sum_p \Delta(t,t_{\rm sink},p)T_n(p)S(q^2)\frac{1}{G(q^2)} \qquad \left(S(q^2) := F_\pi(q^2)G(q^2)\right) \\ &= \sum_m s_m\tilde{\beta}_{m,n}(t) \qquad \left(S(q^2) = \sum_m s_m q^{2m}, \quad \underbrace{\tilde{\beta}_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t,t_{\rm sink},p)T_n(p)q^{2m}/G(q^2)\right) \end{aligned}$$

Original model-independent method changes to $R(t) = s_1 + \sum_{m=3}^{\infty}\left(\sum_{k=1}^{2}\alpha_k\tilde{\beta}_{m,k}(t)\right)s_m$

➡ Change $F_\pi(q^2)$ to $S(q^2)$ and choose $G(q^2)$ with good convergence s_m

What is the good function $G(q^2)$?

$$R(t) = s_1 + \sum_{m=3}^{\infty}\left(\sum_{k=1}^{2}\alpha_k\tilde{\beta}_{m,k}(t)\right)s_m$$

The function $G(q^2)$ is arbitrary.

Temporarily, we assume that the form factor is monopole form to see the convergence of s_m.

$$F_\pi(q^2) = \frac{1}{1+q^2/M_{\mathrm{pole}}^2} = \sum_{m=0}^{\infty}\left(-\frac{1}{M_{\mathrm{pole}}^2}\right)^m q^{2m}$$

$$G(q^2) = 1 + \sum_{m=1} g_m q^{2m}$$

$$S(q^2) := F_\pi(q^2)G(q^2)$$

$$\begin{aligned} s_0 &= 1 \\ s_m &= \left(-\frac{1}{M_{\mathrm{pole}}^2}\right)s_{m-1} + g_m \quad (m \geq 1) \end{aligned}$$

original method

If $G(q^2) = 1$

$$\left[s_m \sim \mathcal{O}\left(\left(-\frac{1}{M_{\mathrm{pole}}^2}\right)^m\right) \right]$$

Ex.1) polynomial (dim=2)

$$G(q^2) = 1 + g_1 q^2 + g_2 q^4$$

$$s_m = \left(-\frac{1}{M_{\mathrm{pole}}^2}\right)s_{m-1} \quad (m \geq 3)$$

How to determine parameters g_1, g_2

satisfy $s_2 = 0$

- High-order coefficient s_m exactly disappear.

Ex.2) Log function

$$G(q^2) = 1 + g_1' \log\left(1 + g_2' q^2\right) = 1 + \sum_{m=1} \frac{-(-g_2')^m}{m} g_1' q^{2m}$$

$$s_m \sim \mathcal{O}\left(\left(-\frac{1}{M_{\mathrm{pole}}^2}\right)^{m-2} {g_2'}^2\right)$$

- Convergence is Improved from m-th power to (m-2)-th power.
- Convergence can be adjusted by g_2'.

How to determine parameters g_1', g_2' satisfy $s_2 = 0$

We can reduce the influence of high-order coefficient (contamination) with the function $G(q^2)$.

Simulation details

- gauge configuration (Phys. Rev. D 86, 074514 (2012))
  - $L^3 \times T = 32^3 \times 48,\ a^{-1} = 2.194\,\mathrm{GeV} \to a = 0.089\,95\,\mathrm{fm},\ L = 2.9\,\mathrm{fm}$
  - $m_\pi = 0.51\,\mathrm{GeV},\ (\kappa_{ud}, \kappa_s) = (0.1373316, 0.1367526)$
  - $N_f = 2 + 1$, Iwasaki gauge + $\mathcal{O}(a)$-improved Wilson quark
  - $\beta = 1.90,\ c_{\mathrm{SW}} = 1.715,\ t_{\mathrm{sink}} = 22$
- measurement parameter
  - ・80 configurations, 50 traj. each
  - ・16 sources × 3 directions (x, y, z) × 4 random sources = 192 meas. per config.
  - ・bin size:100 traj.

⇨ We obtain the charge radius and compare the traditional, original and our method.

Preliminary result

○Traditional method

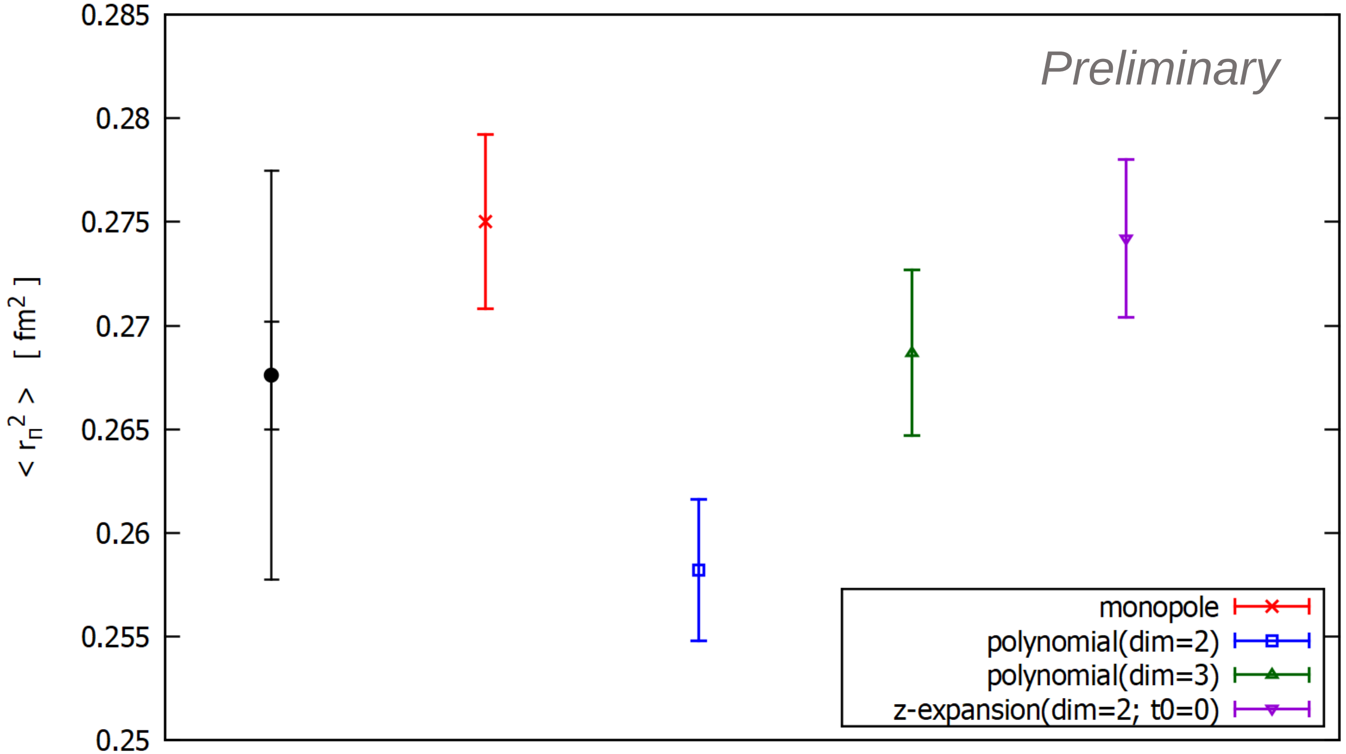
Now, we use 4 forms as fit ansatz.

$$F_\pi(q^2) = \frac{1}{1+q^2/M_{\rm pole}^2} \quad \text{(monopole)}$$

$$F_\pi(q^2) = 1 + f_1 q^2 + f_2 (q^2)^2$$

$$F_\pi(q^2) = 1 + f_1 q^2 + f_2 (q^2)^2 + f_3 (q^2)^3 \quad \text{(polynomial)}$$

$$F_\pi(q^2) = \sum_{k=0}^{2} a_k z^k \quad \left(z := \frac{\sqrt{4m_\pi^2+q^2} - \sqrt{4m_\pi^2}}{\sqrt{4m_\pi^2+q^2} + \sqrt{4m_\pi^2}} \right) \quad \text{(z-expansion)}$$



- Central value

  Weighted mean for 4 forms

- Statistic error

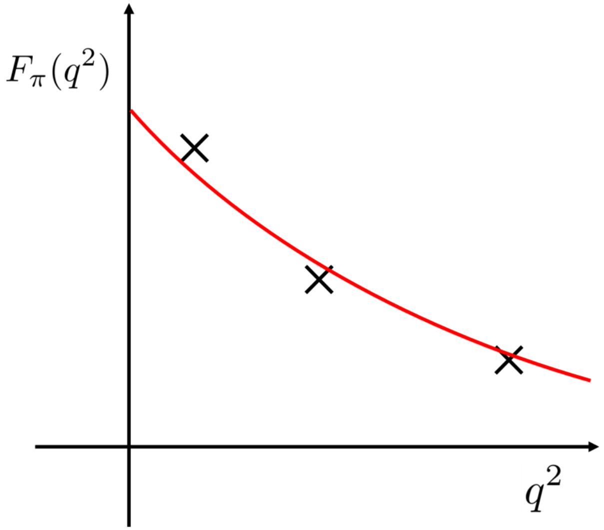
  Jackknife error of the central value

$$\Rightarrow \langle r_\pi^2 \rangle_{\rm Trad.} = 0.2676(26)(95){\rm fm}^2$$

Central(Stat.) (Sys.)

- Systematic error

  Maximum difference between the central value and value on each form

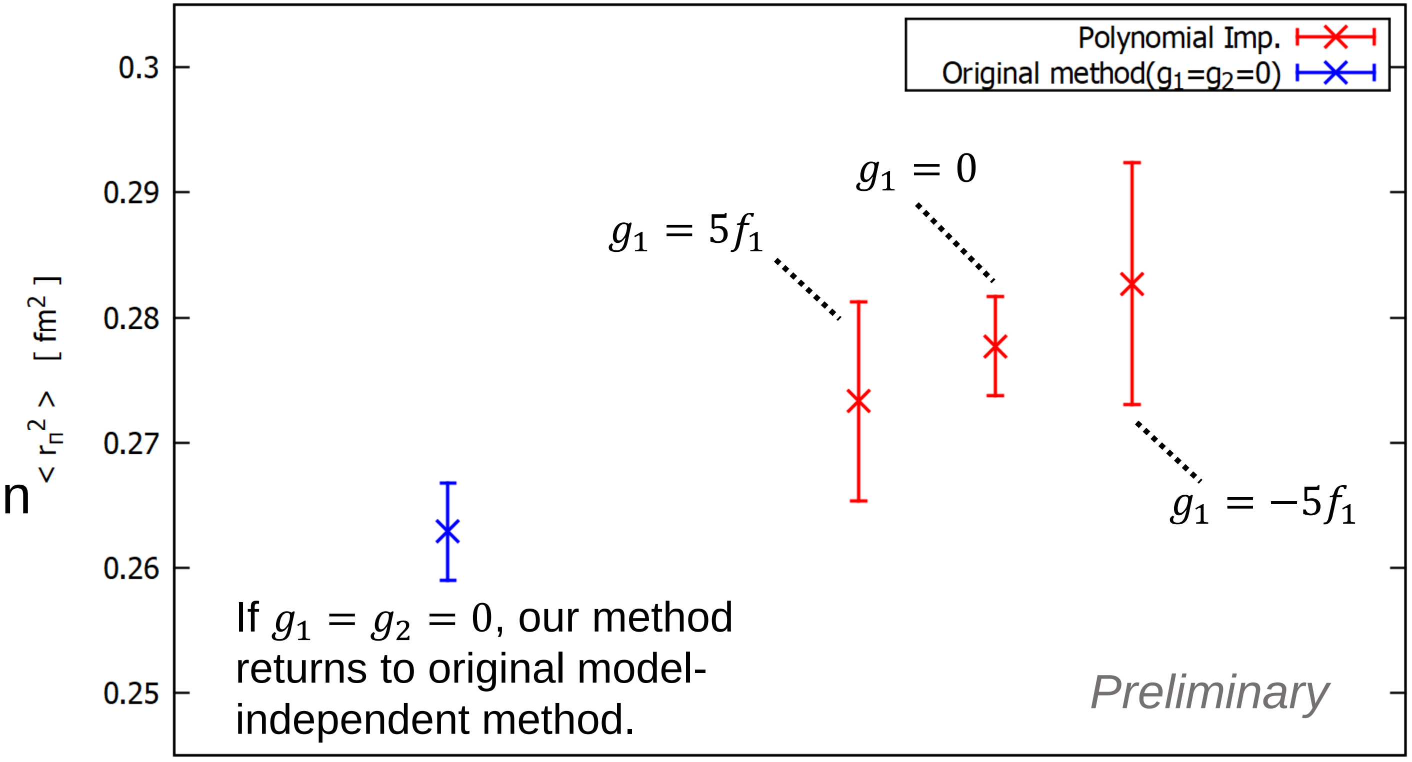


Preliminary result

○Model-independent method

Ex.1) polynomial (dim=2)

$$G(q^2) = 1 + g_1 q^2 + g_2 q^4$$



Now, we consider the sufficiently large g_1 region ($g_1 = -5\underline{f_1}, 0, 5\underline{f_1}$).

Estimated from traditional method

➡ Evaluate the systematic error of g_1

g_2 is determined to satisfy $s_2 = 0$

$$R(t) = s_1 + \sum_{m=3}^{\infty} \left(\sum_{k=1}^{2} \alpha_k \tilde{\beta}_{m,k}(t) \right) s_m \quad \left[R(t) = s_2 + \sum_{m=3}^{\infty} \left(\sum_{k=1}^{2} \alpha'_k \tilde{\beta}_{m,k}(t) \right) s_m \right]$$

- Central value

  Result value at $g_1 = 0$

- Statistic error

  Jackknife error of the central value

- Systematic error

  Maximum difference between the central value and Result value at $g_1 = 5f_1$ or $g_1 = -5f_1$

Central (Stat.)

$$\langle r_\pi^2 \rangle_{\text{Original}} = 0.2629(39)\text{fm}^2$$

➡ $$\langle r_\pi^2 \rangle_{\text{PolynomialImp.}} = 0.2777(39)(50)\text{fm}^2$$

Central (Stat.) (Sys.)

Preliminary result

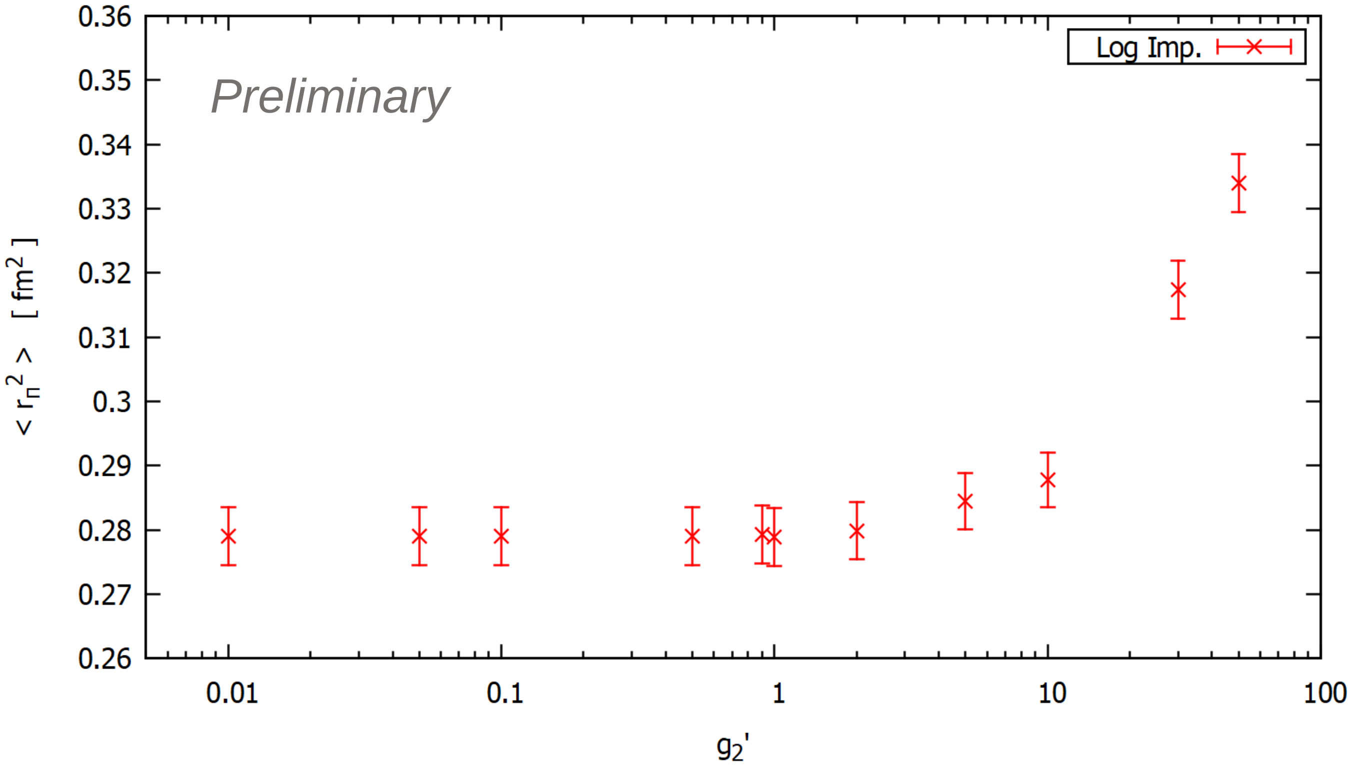
○Model-independent method

Ex.2) Log function

$$G(q^2) = 1 + g'_1 \log\left(1 + g'_2 q^2\right) = 1 + \sum_{n=1} \frac{-(-g'_2)^n}{n} g'_1 q^{2n}$$

Now, we consider the sufficiently large $g_2{}'$ region.

➡ Evaluate the systematic error of $g_2{}'$



$g_1{}'$ is determined to satisfy $s_2 = 0$

$$R(t) = s_1 + \sum_{m=3}^{\infty}\left(\sum_{k=1}^{2}\alpha_k\tilde{\beta}_{m,k}(t)\right)s_m \quad \left[R(t) = s_2 + \sum_{m=3}^{\infty}\left(\sum_{k=1}^{2}\alpha'_k\tilde{\beta}_{m,k}(t)\right)s_m\right]$$

➡ If $g_2{}'$ is sufficiently small, the charge radius is constant.

⬇

We can control the high-order contamination(Systematic error).

- Central value

  Result value at $g_2{}' = 1$

- Statistic error

  Jackknife error of the central value

- Systematic error

  We consider it to be almost zero.

  ($\because \langle r_\pi^2 \rangle = \mathrm{const.}$ at $g'_2 \leq 1$)

➡ $$\langle r_\pi^2 \rangle_{\mathrm{LogImp.}} = 0.2789(45)(0)\mathrm{fm}^2$$

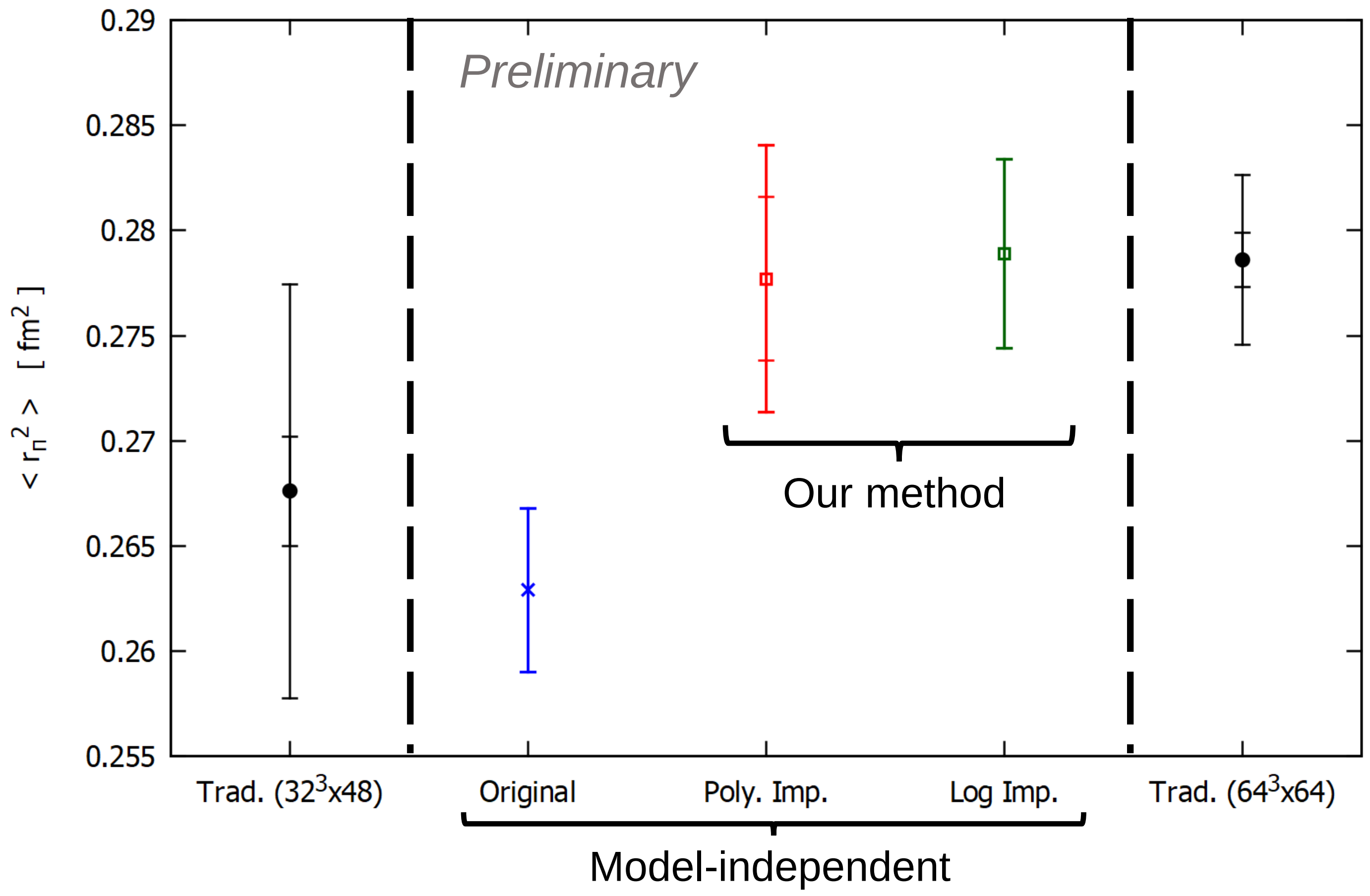
Central(Stat.)(Sys.)

$$s_m \sim \mathcal{O}\left(\left(-\frac{1}{M_{\mathrm{pole}}^2}\right)^{m-2} {g'_2}^2\right)$$

Preliminary result

$$\begin{cases} \langle r_\pi^2 \rangle_{\text{Traditional}} & = 0.2676(26)(95)\text{fm}^2 \\ \color{blue}{\langle r_\pi^2 \rangle_{\text{Original}}} & = 0.2629(39)\text{fm}^2 \\ \color{red}{\langle r_\pi^2 \rangle_{\text{PolynomialImp.}}} & = 0.2777(39)(50)\text{fm}^2 \\ \color{teal}{\langle r_\pi^2 \rangle_{\text{LogImp.}}} & = 0.2789(45)(0)\text{fm}^2 \end{cases}$$

Central(Stat.)(Sys.)



・The model-independent method's error is smaller than the traditional method's error.

➡ Model-independent method does not have the error from fit ansatz.

・There is a difference between the original method and our method.

➡ Our method can suppress the finite volume effect.

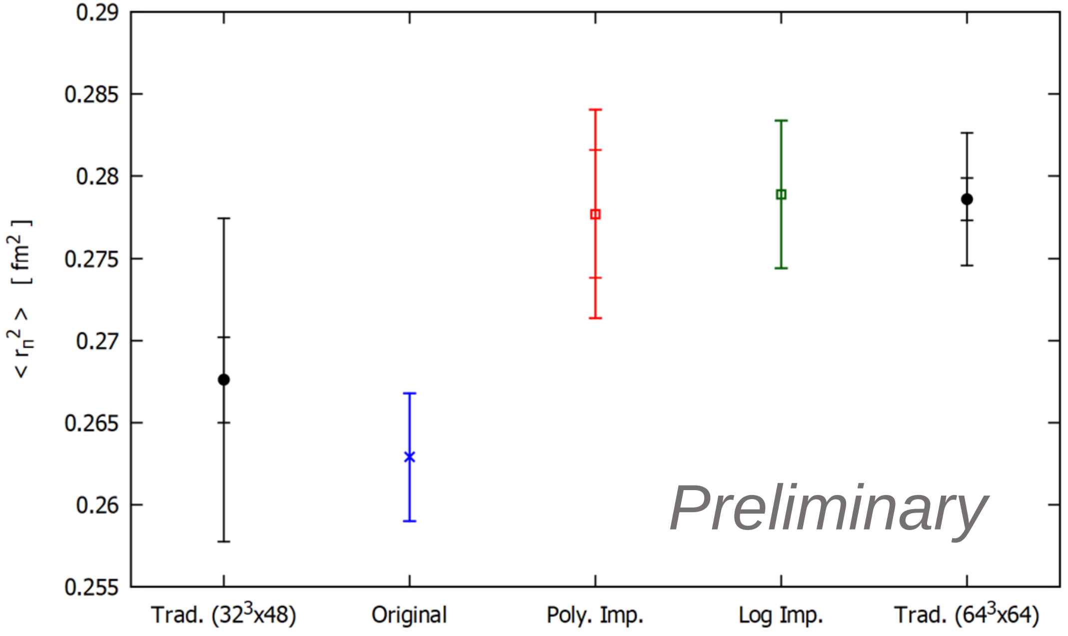
(∵ Our method is consistent with large volume result)

Summary

- We discuss improvement of model independent method to obtain the pion charge radius.
- We propose our method which reduces the high-order contribution included in the original model-independent method.
- We show the arbitrarily of the appropriate function $G(q^2)$ and examples of parameter settings.
- We apply the method to actual lattice QCD data and find that the error (statistics + systematic) was smaller than the traditional method.

Future works

- Analysis by various functions $G(q^2)$
- Analysis at various m_π, a, V
- Analysis at physical point
- Analysis with other particles such as K_{l3}
- Other methods (direct fit, etc.)



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Relationship between momentum-derivative and spatial moment

$$
\begin{aligned}
\left.\frac{\mathrm{d}\tilde{F}(\vec{p})}{\mathrm{d}|\vec{p}|^2}\right|_{|\vec{p}|^2=0} &= \left.\frac{\mathrm{d}}{\mathrm{d}|\vec{p}|^2}\int \mathrm{d}^3x\, F(\vec{x})e^{-i\vec{p}\cdot\vec{x}}\right|_{|\vec{p}|^2=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}|\vec{p}|^2}\int \mathrm{d}|\vec{x}|\,\mathrm{d}(\cos\theta)\,\mathrm{d}\phi\,|\vec{x}|^2 F(|\vec{x}|)e^{-i|\vec{p}||\vec{x}|\cos\theta}\right|_{|\vec{p}|^2=0} \qquad (\textbf{if}\ \ F(\vec{x}) := F(|\vec{x}|)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}|\vec{p}|^2}\int \mathrm{d}|\vec{x}|\,\mathrm{d}\phi\,|\vec{x}|^2 F(|\vec{x}|)\frac{1}{-i|\vec{p}||\vec{x}|}\left[e^{-i|\vec{p}||\vec{x}|\cos\theta}\right]_{\cos\theta=-1}^{1}\right|_{|\vec{p}|^2=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}|\vec{p}|^2}\int \mathrm{d}|\vec{x}|\,\mathrm{d}\phi\,|\vec{x}|^2 F(|\vec{x}|)\times 2\frac{\sin(|\vec{p}||\vec{x}|)}{|\vec{p}||\vec{x}|}\right|_{|\vec{p}|^2=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}|\vec{p}|^2}\int \mathrm{d}|\vec{x}|\,\mathrm{d}(\cos\theta)\,\mathrm{d}\phi\,|\vec{x}|^2 F(\vec{x})\frac{1}{|\vec{p}||\vec{x}|}\sum_{n=0}^{\infty}(-1)^n\frac{(|\vec{p}||\vec{x}|)^{2n+1}}{(2n+1)!}\right|_{|\vec{p}|^2=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}|\vec{p}|^2}\int \mathrm{d}^3x\, F(\vec{x})\sum_{n=0}^{\infty}(-1)^n\frac{(|\vec{p}||\vec{x}|)^{2n}}{(2n+1)!}\right|_{|\vec{p}|^2=0} \\
&= \left.\int \mathrm{d}^3x\, F(\vec{x})\sum_{n=0}^{\infty}(-1)^n\frac{n|\vec{p}|^{2(n-1)}|\vec{x}|^{2n}}{(2n+1)!}\right|_{|\vec{p}|^2=0} \\
&= -\frac{1}{3!}\int \mathrm{d}^3x\,|\vec{x}|^2 F(\vec{x})
\end{aligned}
$$

Why consider a small M_{pole}^2 ?

Monopole form : $$F(q^2) = \frac{1}{1 + \frac{\langle r_\pi^2 \rangle}{6} q^2}$$

To consider on a lattice, Nondimensionalizing the form factor using lattice spacing a

$$F(q^2) = \frac{1}{1 + \frac{\langle r_\pi^2 \rangle}{6a^2} (aq)^2}$$

$$\xrightarrow{a \to 0} \quad M_{\mathrm{pole}}^2 = \frac{6a^2}{\langle r_\pi^2 \rangle} \to 0$$

s_m relationship

$$F(x) = \frac{1}{1+Ax} = \sum_m f_m x^m, \quad G(x) = 1 + g_1 x + g_2 x^2, \quad S(x) = F(x)G(x) = \sum_m s_m x^m$$

For $m \geq 2$

$$\begin{aligned} s_m = \frac{1}{m!}\frac{d^m S}{dx^m}\Big|_{x=0} &= \frac{1}{m!}\left\{\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} F^{(m-k)} G^{(k)}\right\}\Bigg|_{x=0} \\ &= \frac{1}{m!}\sum_{k=0}^{m} \frac{m!}{k!(m-k)!}(m-k)!(-A)^{m-k} G^{(k)} \qquad (\because F^{(m)} = m!(-A)^m) \\ &= \sum_{k=0}^{m} \frac{(-A)^{m-k}}{k!} G^{(k)} = (-A)^m + (-A)^{m-1} g_1 + (-A)^{m-2} g_2 \end{aligned}$$

For $m \geq 3$

$$s_m = (-A) \times \left\{(-A)^{m-1} + (-A)^{m-2} g_1 + (-A)^{m-3} g_2\right\} = (-A) s_{m-1}$$

Other methods

$$\underbrace{C^{(n)}(t)}_{\text{moment function}} := \sum_r r^{2n} C_{\pi V \pi}(t, t_{\rm sink}; r) = \sum_{m=0}^{\infty} f_m \beta_{m,n}(t)$$

- Direct fit (moment 3-point function)

$$C^{(n)}(t) = \sum_p \Delta(t, t_{\rm sink}, p) T_n(p) F_\pi(q^2) = \sum_p \Delta(t, t_{\rm sink}, p) T_n(p) F_\pi(q^2) \frac{G(q^2)}{G(q^2)}$$

Input data (LQCD data)

$$= \sum_p \Delta(t, t_{\rm sink}, p) T_n(p) S(q^2) \frac{1}{G(q^2)} \quad \left(S(q^2) := F_\pi(q^2) G(q^2), \;\; G(q^2) := 1 + g_1 q^2 + g_2 q^4\right)$$

$$= \sum_m s_m \tilde{\beta}_{m,n}(t) \quad \left(S(q^2) = \sum_m s_m q^{2m}, \quad \underbrace{\tilde{\beta}_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t, t_{\rm sink}, p) T_n(p) q^{2m} / G(q^2)\right)$$

Parameters Known function ⇒ curve fitting

- Direct fit (3-point function)

$$C_{\pi V \pi}(t, t_{\rm sink}; r) := Z_V \sum_{\vec{z}} \sum_{y_2, y_3} \sum_{x_2, x_3} \langle 0 | \pi^+(\vec{z}, t_{\rm sink}) V_4(\vec{y}, t) \pi^{+\dagger}(\vec{x}, 0) | 0 \rangle$$

$$= \sum_p \Delta(t, t_{\rm sink}, p) T_r(p) F_\pi(q^2) \quad \left(T_r(p) := \frac{1}{L} e^{ipr}\right)$$

$$= \sum_m s_m \tilde{\beta}'_{m,n}(t) \quad \left(\tilde{\beta}'_{m,n}(t) := \sum_p \Delta(t, t_{\rm sink}, p) T_r(p) q^{2m} / G(q^2)\right)$$

Other methods

・Model-dependent Direct fit (moment 3-point function)

$$C^{(n)}(t) = \sum_p \Delta(t, t_{\mathrm{sink}}, p) T_n(p) F_\pi(q^2)$$

Input data
(LQCD data)

$$= \sum_p \Delta(t, t_{\mathrm{sink}}, p) T_n(p) \frac{1}{1 + q^2/M_{\mathrm{pole}}^2}$$

Parameters

curve fitting

・ Model-dependent Direct fit (3-point function)

$$C_{\pi V \pi}(t, t_{\mathrm{sink}}; r) = \sum_p \Delta(t, t_{\mathrm{sink}}, p) T_r(p) F_\pi(q^2)$$

$$= \sum_p \Delta(t, t_{\mathrm{sink}}, p) T_r(p) \frac{1}{1 + q^2/M_{\mathrm{pole}}^2}$$

How to find the parameter g_1, g_2

$$(C^{(0)}(t) := 1, \quad f_0 = s_0 = 1)$$

$$\underline{C^{(n)}(t)}_{\text{moment function}} := \sum_r r^{2n} C_{\pi V \pi}(t, t_{\text{sink}}; r) = \sum_{m=0}^{\infty} s_m \tilde{\beta}_{m,n}(t)$$

$$\underline{\tilde{\beta}_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t, t_{\text{sink}}, p) T_n(p) q^{2m} / G(q^2)$$

- Exhaustive search

$$\begin{aligned} R(t) &:= \alpha_1 C^{(1)}(t) + \alpha_2 C^{(2)}(t) + h \\ &= (\alpha_1 \tilde{\beta}_{0,1} + \alpha_2 \tilde{\beta}_{0,2} + h) + (\alpha_1 \tilde{\beta}_{1,1} + \alpha_2 \tilde{\beta}_{1,2}) s_1 + (\alpha_1 \tilde{\beta}_{2,1} + \alpha_2 \tilde{\beta}_{2,2}) s_2 + \cdots \end{aligned}$$

Define parameters α_1, α_2, h to satisfy the following

$$\alpha_1 \tilde{\beta}_{0,1} + \alpha_2 \tilde{\beta}_{0,2} + h = 0 \qquad \alpha_1 \tilde{\beta}_{1,1} + \alpha_2 \tilde{\beta}_{1,2} = 0 \qquad \alpha_1 \tilde{\beta}_{2,1} + \alpha_2 \tilde{\beta}_{2,2} = 1$$

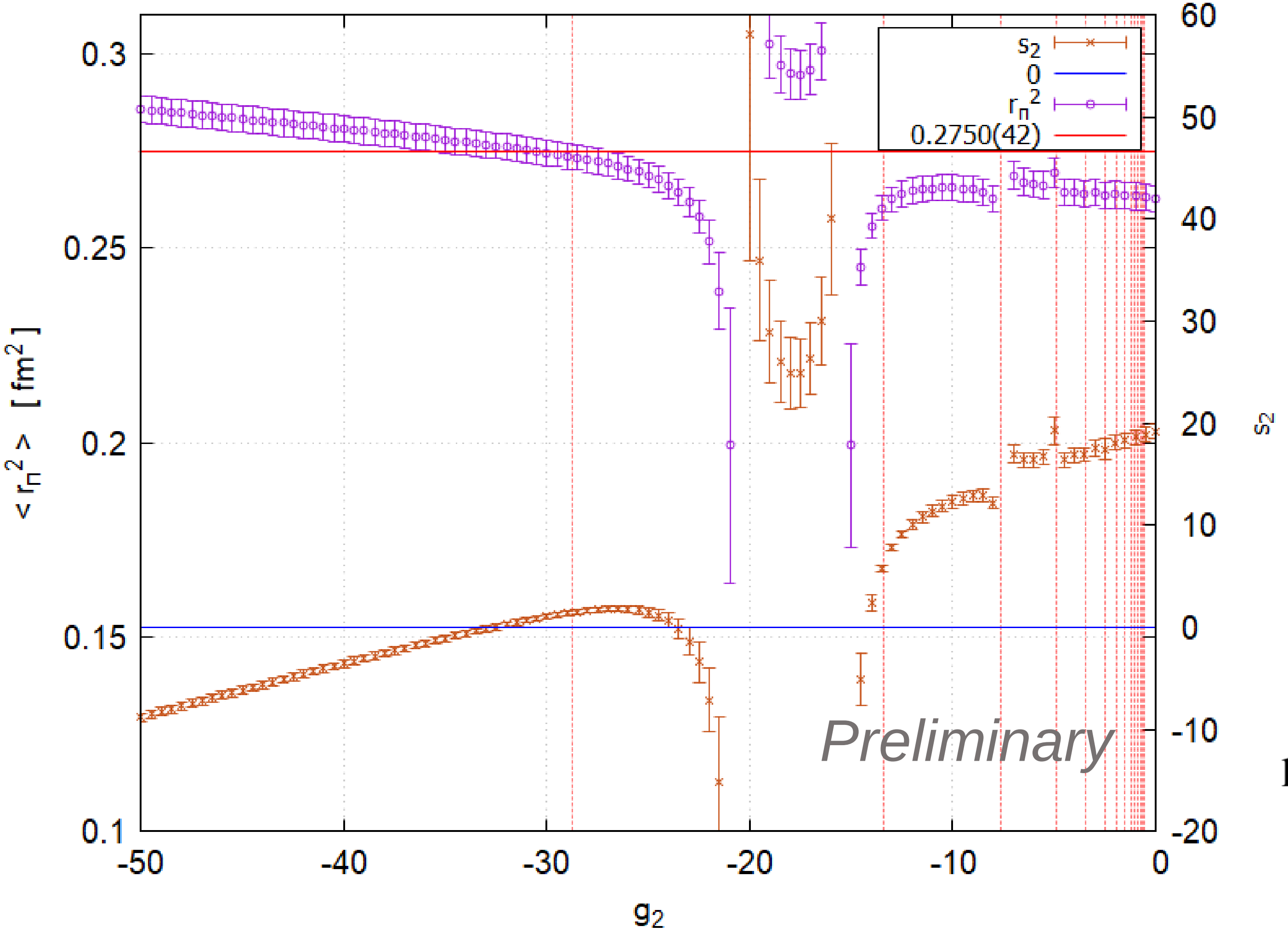
$$\Rightarrow \quad R(t) = s_2 + \sum_{m=3}^{\infty} \left(\sum_{k=1}^{2} \alpha_k \tilde{\beta}_{m,k}(t) \right) s_m$$

How to…

Fix g_1 ,g_2 → Obtain $R(t)$ (s_2) → Change g_2 (or g_1) → Plot s_2- g_2(or g_1) graph and find the crossing point ($s_2 = 0$)

How to find the parameter g_1, g_2

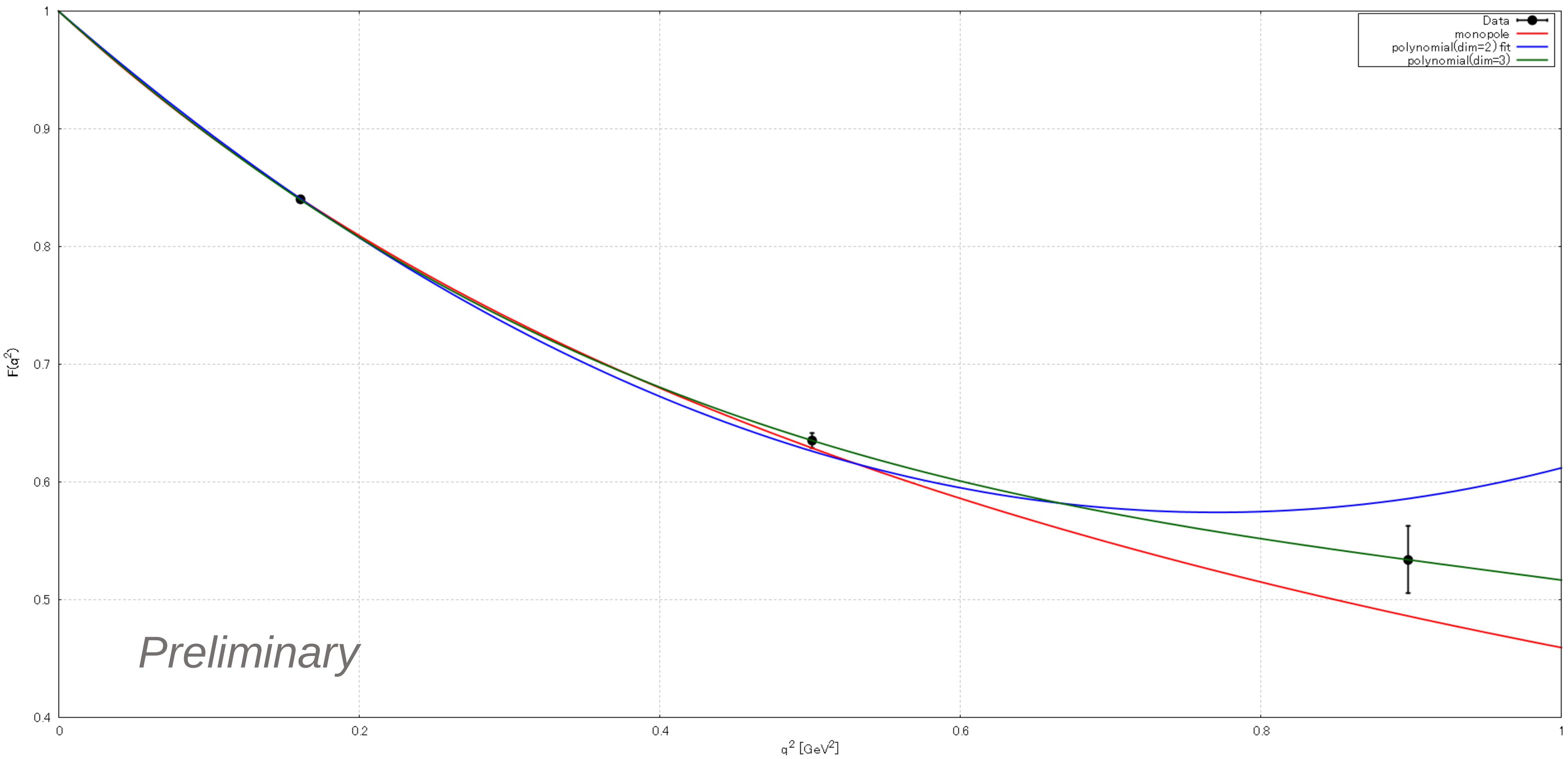
・Method 2 : Exhaustive search



$$\begin{cases} g_1 = 0 \\ g_2 = -32.5 \end{cases}$$

$$\underbrace{\tilde{\beta}_{m,n}(t)}_{\text{known function}} := \sum_p \Delta(t, t_{\text{sink}}, p) T_n(p) q^{2m} / G(q^2)$$

Traditional method and Form factor



Model-independent method

$$\langle r_\pi^2 \rangle = -6 \left. \frac{\mathrm{d}}{\mathrm{d}q^2} F_\pi(q^2) \right|_{q^2=0} \sim -6 \times R(t)$$

$$\langle r_\pi^2 \rangle = -6 \left. \frac{\mathrm{d}}{\mathrm{d}q^2} F_\pi(q^2) \right|_{q^2=0} \sim -6 \times (R(t) - g1)$$