Chiral Transition via Strong Coupling expansion

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Strong-interaction matter under extreme conditions

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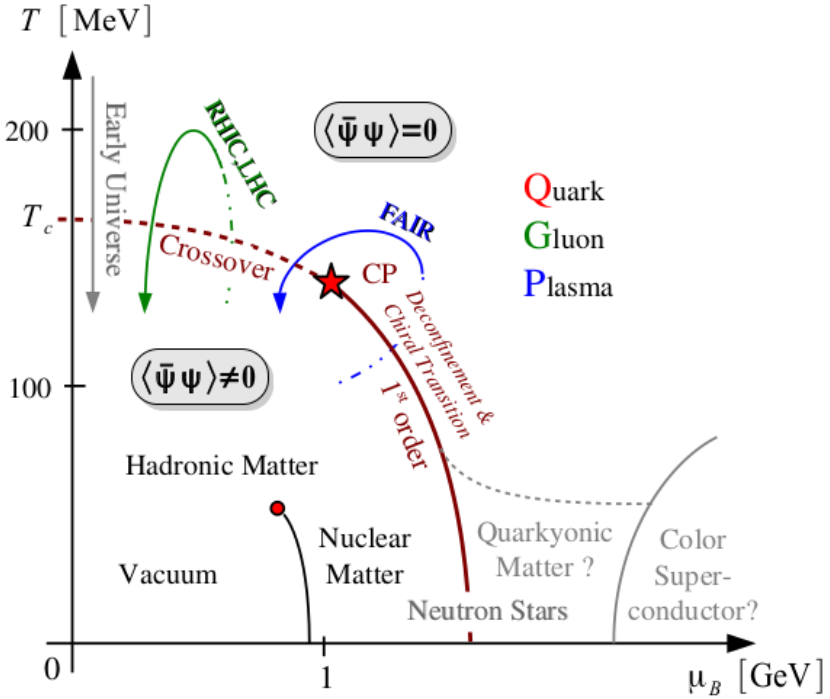
Outline

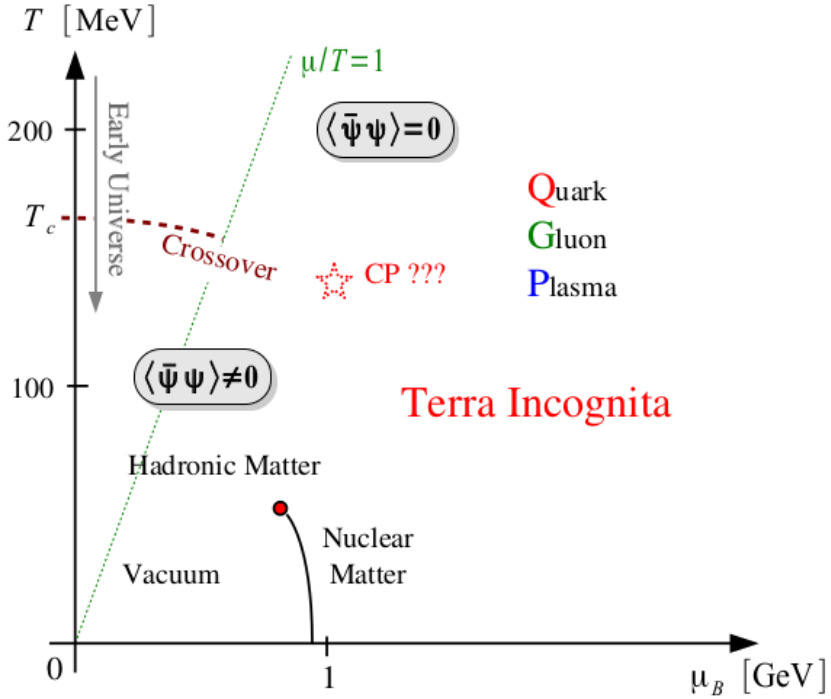
Motivation

Conjectured phase diagram (W. Unger)



- Crossover transition: At low μ_B and high T
- CEP followed by 1^{st} order phase transition: As μ_B increases
- Nuclear phase transition: At low T

Known phase diagram (W. Unger)



- LQCD: PD in $\mu_B/T < 1$ region
- Experiments and other theories: nuclear phase transition
- Unknown: location of CEP, existence of 1^{st} order phase transition

Motivation

- Sign problem: severe in LQCD, methods circumventing it is limited to $\mu_B/T < 1$ (Forcrand 2010)
- Alternative approach: study LQCD using strong coupling expansion
  - change order of integration $\rightarrow$ dual variable formulation
  - strong coupling limit ($\beta \rightarrow 0$): well established (Rossi and Wolff 1984), full $\mu_B - T$ phase diagram determined
  - leading $\mathcal{O}(\beta)$: up to $\beta \leq 1$, sign problem still mild enough, recently used to determine the nuclear liquid-gas transition (Kim, P, and Unger 2023)
  - $\mathcal{O}(\beta^n)$: dual formulation results in **tensor network** (promising approach) (Gagliardi and Unger 2020)
- In this talk, map dual formulation of LQCD to a **vertex model**:
  - suitable for Monte Carlo (in contrast to tensor-network methods)
  - aim: check the validity range in β by comparing different orders in β
- Goal: study the β dependence of the chiral critical end point

Strong Coupling expansion

- Partition function of LQCD:

$$Z = \int_G DU_{n,\mu} d\bar{\chi}(n) d\chi(n) \exp\left(-S_F[\bar{\chi}, \chi, U]\right) \exp\left(-S_G[U]\right)$$

here S_F is the staggered fermionic action:

$$S_F[\bar{\chi}, \chi, U] = \sum_n \left(-\sum_\mu \eta_\mu(n) \left(\bar{\chi}_n U_{\hat{\mu}}(n) \chi_{n+\hat{\mu}} - \bar{\chi}_{n+\hat{\mu}} U^\dagger_{\hat{\mu}}(n) \chi_n \right) + 2 m_q \bar{\chi}\chi \right)$$

and S_G is the Wilson gauge action:

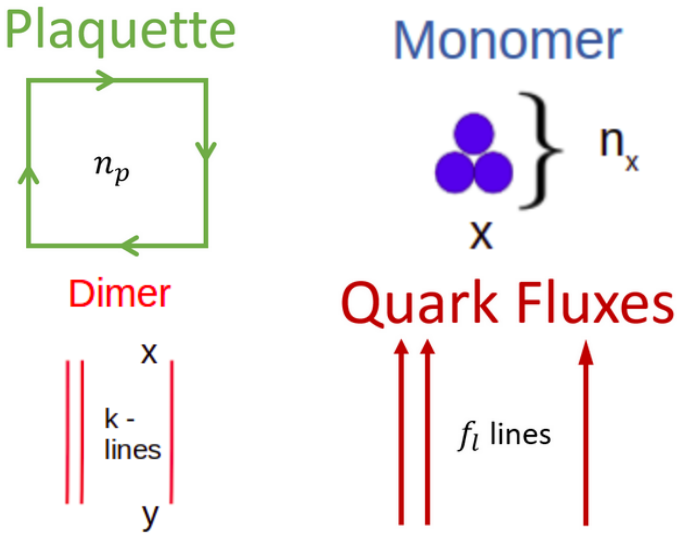
$$S_G[U] = -\frac{\beta}{2N_c} \sum_P \mathrm{Re} \left(\mathrm{tr}\left[U_p\right] + \mathrm{tr}\left[U_p^\dagger\right] \right)$$

- At strong coupling, β is small, allowing for a Taylor expansion of S_G in β:

$$\exp\left(-S_G[U]\right) = \sum_{n_p, \bar{n_p}=0}^{\infty} \frac{1}{n_p!} \frac{1}{\bar{n}_p!} \left(\frac{\beta}{2N}\right)^{n_p + \bar{n}_p} \left(\sum_p \mathrm{tr}\, U_p \right)^{n_p} \left(\sum_P \mathrm{tr}\, U_p^\dagger \right)^{\bar{n}_p}$$

Dual degrees of freedom

- Exponents n_p and $\bar{n}_p$: plaquette and antiplaquette occupation number, are gluon fluxes living on plaquettes
- Taylor expand in m_q with exponents n_x: monomer occupation number, are color singlet mesonic states living at a site
- Expansion of massless Dirac operator in forward (d_l) and backward $(\bar{d}_l)$ direction:
  - min$(d_l,\bar{d}_l)$=k_l, dimers which are meson hoppings on a link
  - unpaired quark (f_l) or antiquark $(\bar{f}_l)$ - $f_l = d_l - \bar{d}_l$, oriented quark fluxes on a link
- Partition function (in terms of dual variables):



$$Z\left(\beta,\hat{m}_q\right) = \sum_{\substack{\{n_p,\bar{n}_p\}\\ \{d_l,\bar{d}_l,n_x\}}} \prod_p \frac{\tilde{\beta}^{n_p+\bar{n}_p}}{n_p!\bar{n}_p!} \prod_l \frac{1}{d_l!\bar{d}_l!} \prod_x \frac{(2\hat{m}_q)_x^n}{n_x!} \mathcal{G}_{n_p,\bar{n}_p,d_l,\bar{d}_l,n_x}$$

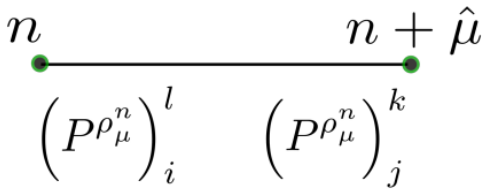
$\mathcal{G}$: non-local Gauge+Grassmann integral over whole lattice

Computing $\mathcal{G}$

(Gagliardi and Unger 2020)

- Obtain the integral in terms of generalized Weingarten functions and integer partitions
- Get a different parametrization of integral using irreducible matrix elements of symmetric group
- Integral decouples into decoupling operators P^{ρ} (ρ=DOI):

$$I = \sum_{\rho} (P^{\rho})^{l}_{i} (P^{\rho})^{j}_{k}$$

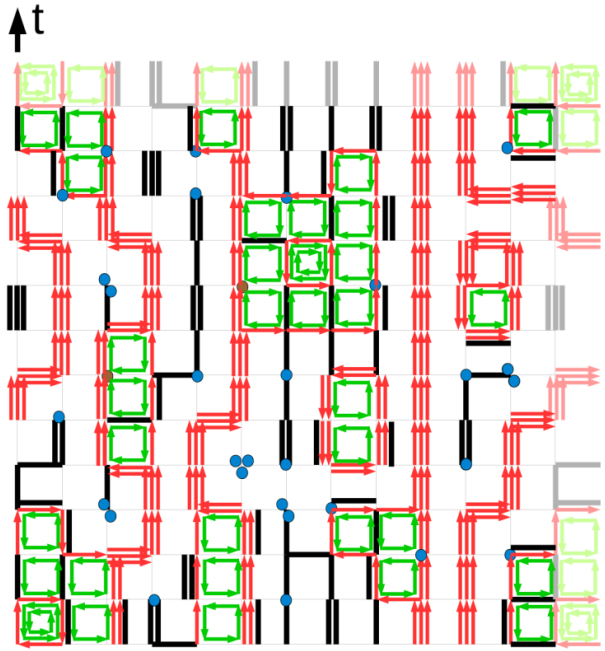


Fixing DOI lets us contract color indices independently at each site

- Contraction of color indices depends on $n_p, \bar{n}_p, d_l, \bar{d}_l$
- Two types of color indices:
  - from expansion of hopping term (fermionic) -
    - determined by $d_l, \bar{d}_l$
    - contracted with Grassmann fields in fermionic matrices
  - from expansion of Wilson term (gluonic) -
    - determined by $n_p, \bar{n}_p$
    - contracted according to the plaquettes, contraction happens between different links sharing a common site

Constraints

(Gagliardi and Unger 2020)



- Grassmann constraint: exactly N_c fermion and anti-fermion fields present at a site

$$n_x + \sum_{\pm\mu} \left(k_{x,\mu} + \frac{|f_{x,\mu}|}{2} \right) = N_c \qquad \sum_{\pm\mu} f_{x,\mu} = 0$$

- Gauge constraint: Sum of gauge flux and fermion flux on each link should be an integer multiple of N_c

$$f_{x,\mu} + \sum_{\nu>\mu} \left[\delta n_{\mu,\nu}(x) - \delta n_{\mu,\nu}(x-\nu) \right] = N_c \mathbb{Z}$$

(Gagliardi and Unger 2020)

- The integrations and the contraction of the color indices results in a tensor:

$$T_n^{\rho} = \mathrm{Tr}[\prod_{\pm\mu} P^{\rho_\mu^n}] \in R$$

- The partition function depends on local variables:

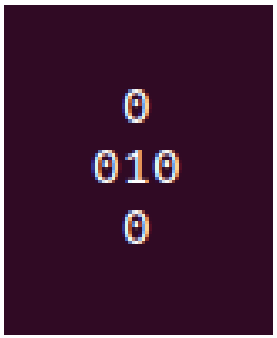
$$\mathcal{Z}\left(\beta, \mu_q, \hat{m}_q\right) = \sum_{\substack{\{n_p, \bar{n}_p\} \\ \{k_\ell, f_\ell, m_n\}}} \sigma_f \sum_{\left\{\rho_\mu^n\right\}} \prod_p \frac{\left(\frac{\beta}{2N}\right)^{n_p+\bar{n}_p}}{n_p! \bar{n}_p!} \prod_{\ell=(n,\mu)} \frac{e^{\mu_q \delta_{\mu,0} f_{n,\mu}}}{k_\ell!\left(k_\ell + |f_\ell|\right)!}$$
$$\prod_n \frac{\left(2\hat{m}_q\right)^{m_n}}{m_n!} T_n^{\rho_{-D}^n \cdots \rho_{+D}^n}\left(\mathcal{D}_n\right)$$

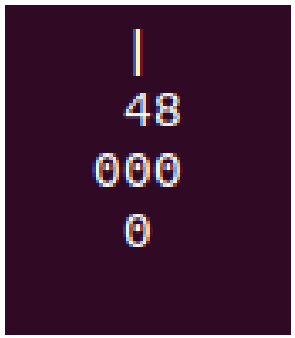
- Sign σ_f: includes the staggered phases and the global geometric sign due to quark fluxes

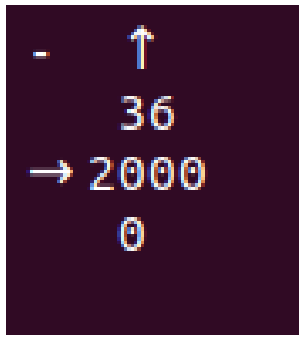
Computing Vertices

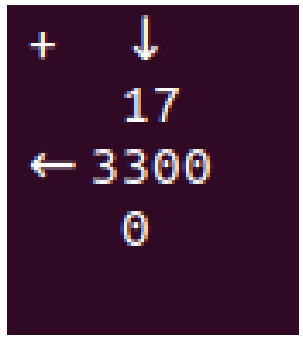
(Gagliardi and Unger 2020)

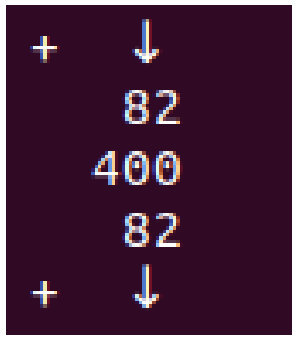
- From now on: restrict to U(3) up to $O(\beta^2)$, which has $\sigma_f = 1$
- Weight in front of tensor: subsued in to tensor to form vertices
- Calculation of vertices has been automatized:
  - Input - order in β, gauge group $(SU(N), U(N))$, space-time dimension $D+1$
  - Output - list of vertices with corresponding weight
  - Vertices can be parsed back into dual degrees of freedom, example: U(1) gauge group with $O(\beta^2)$ in $1+1$ dimension





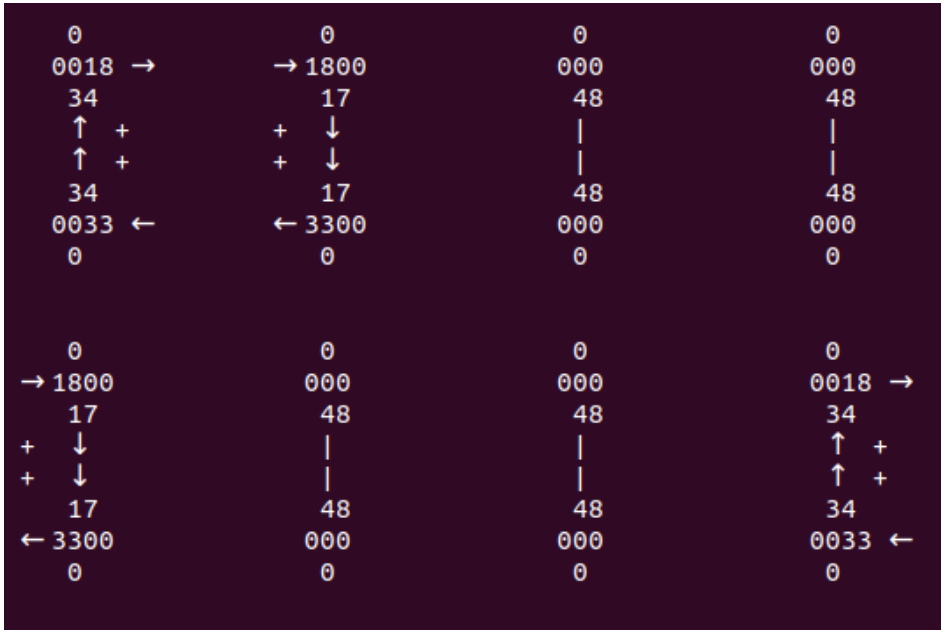






Monte Carlo for Vertex Model

Sample configuration with the vertices:



- Sampling: via **Metropolis algorithm** with vertex weight

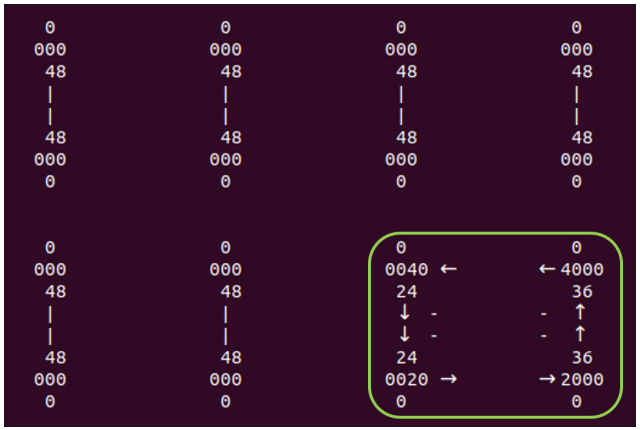
$$w(v) = (2m_q)^{N_m} (\gamma)^{0.5(N_{D_t} + N_{F_{t,+}} + N_{F_{t,-}})} (\tilde{\beta})^{0.25(N_p + \bar{N}_p)} (e^{\mu/\gamma^2})^{0.5(N_{F_{t,+}} - N_{F_{t,-}})}$$

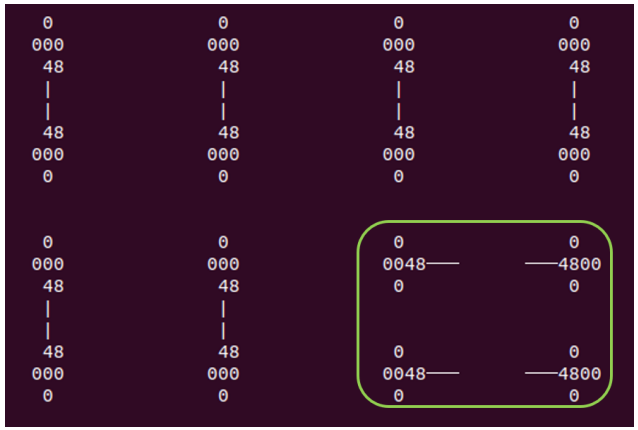
- Acceptance probability:

$$\min\left(1, \frac{\prod_i w(v_i^{\text{new}})}{\prod_i w(v_i^{\text{old}})}\right)$$

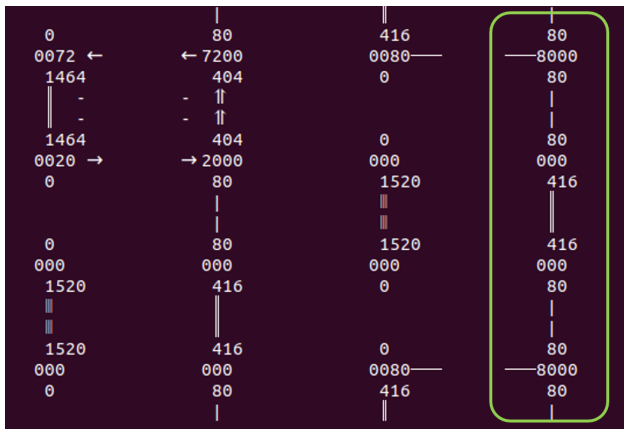
Updates

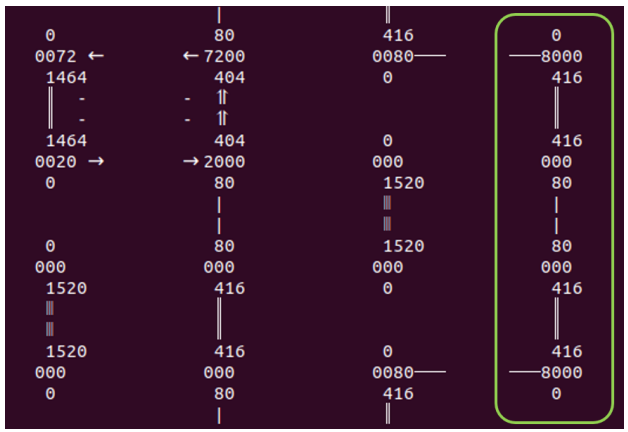
- All possible configurations are attainable for $U(3)$ by combining two types of updates:
  - Plaquette update - choose random plaquette, update all vertices attached to it with new vertices compatible with surrounding vertices





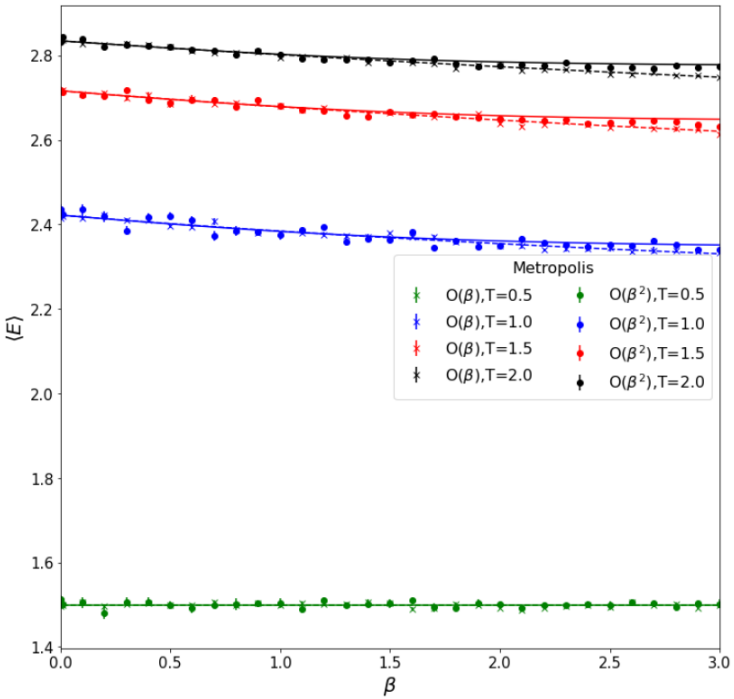
  - Line update - choose random line (spatial or temporal), update all vertices in the line with new vertices compatible with surrounding vertices

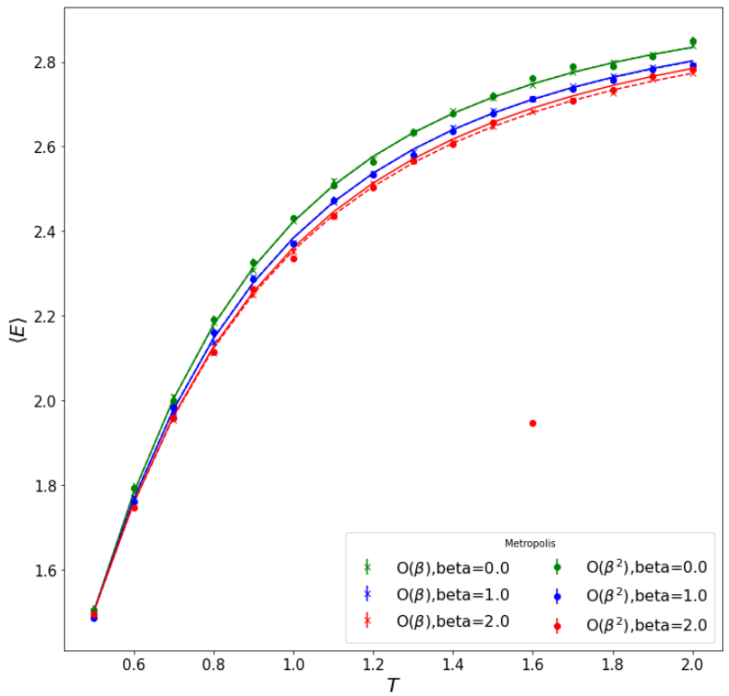




Energy

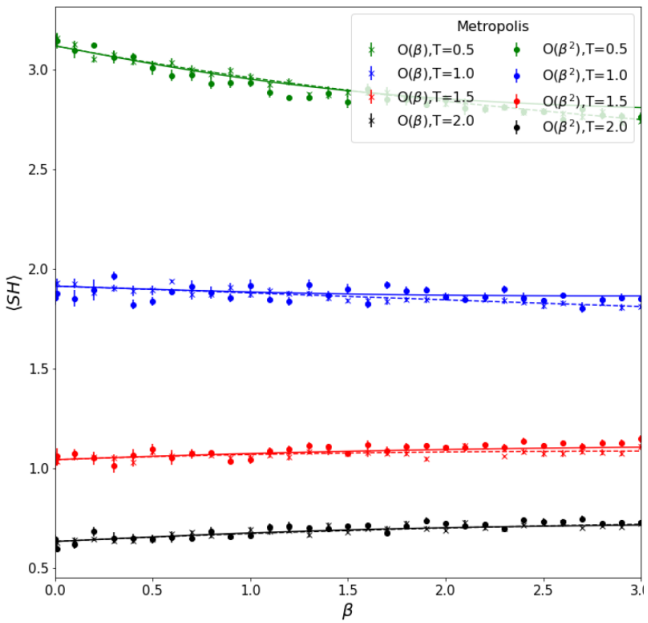
- To verify the vertex model updates, we compare our results with **exact enumeration** results in 2×2 lattice
- Energy: $N_{D_t} + N_{F_{t,+}} + N_{F_{t,-}}$
- Exact enumeration: dashed lines - $O(\beta)$, solid lines - $O(\beta^2)$
- Results agree for all β and T for $O(\beta)$ and $O(\beta^2)$ corrections

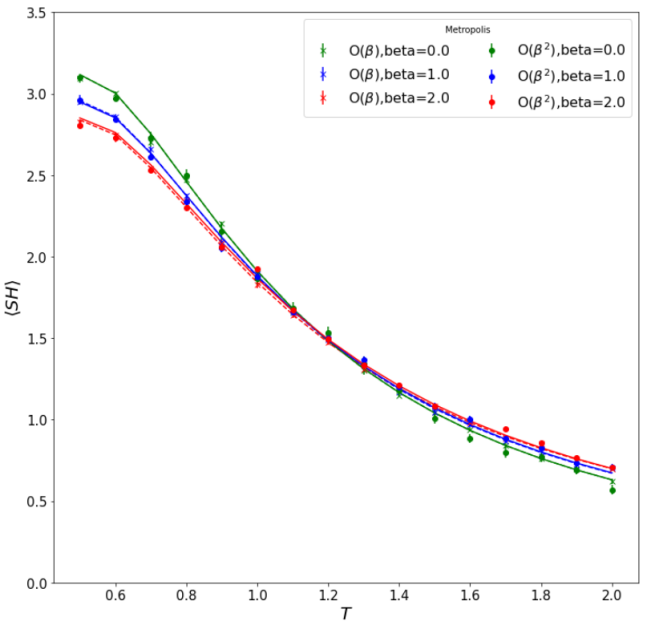


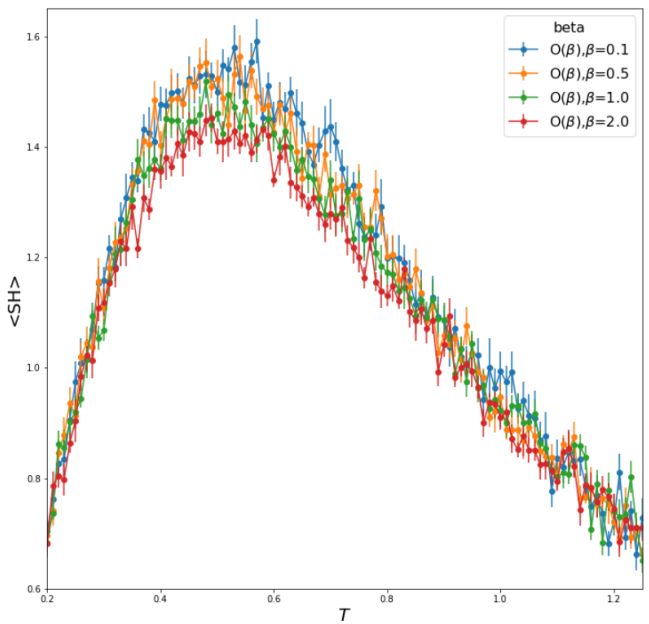


Specific heat

- $C_v = \langle E^2 \rangle - \langle E \rangle^2$
- peak at the transition temperature, let us determine the crossover temperature (higher statistics, larger volumes required)

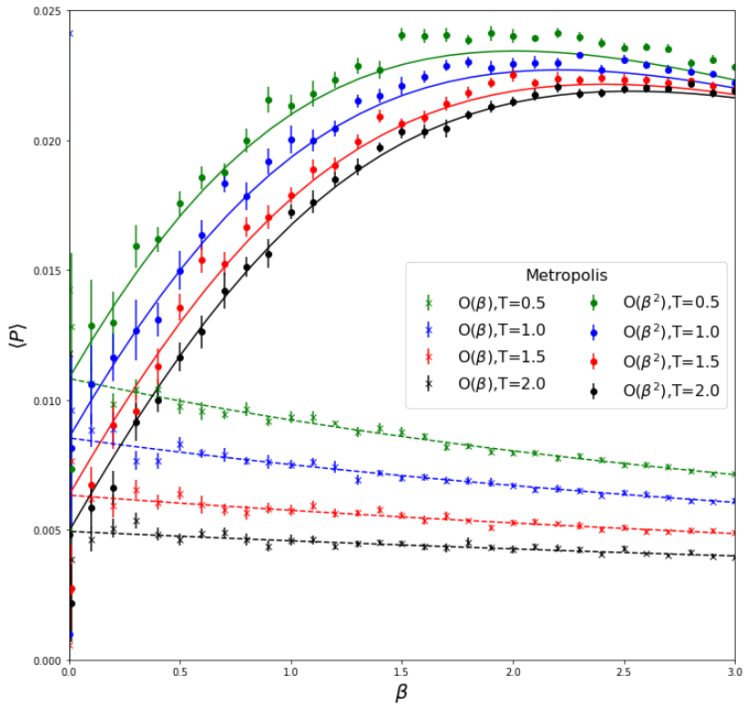


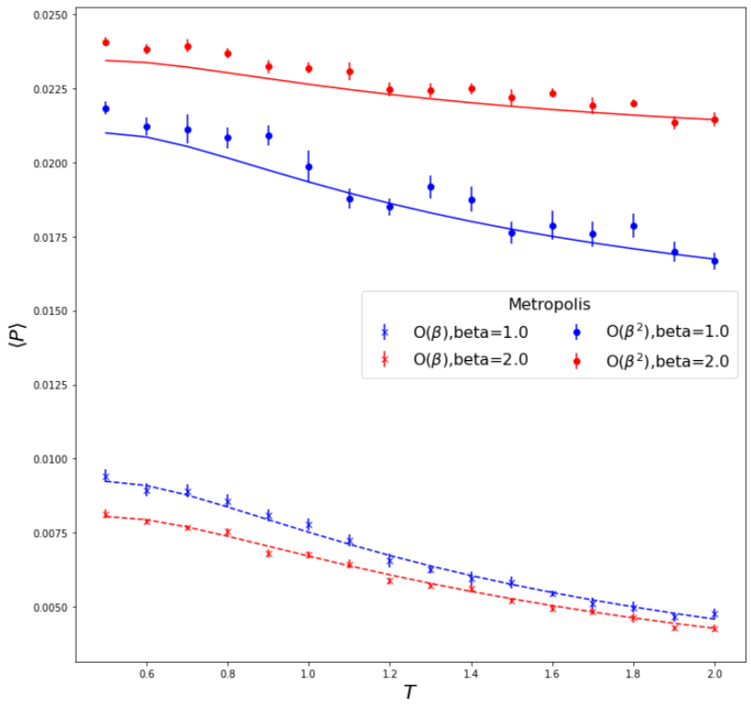




Average plaquette

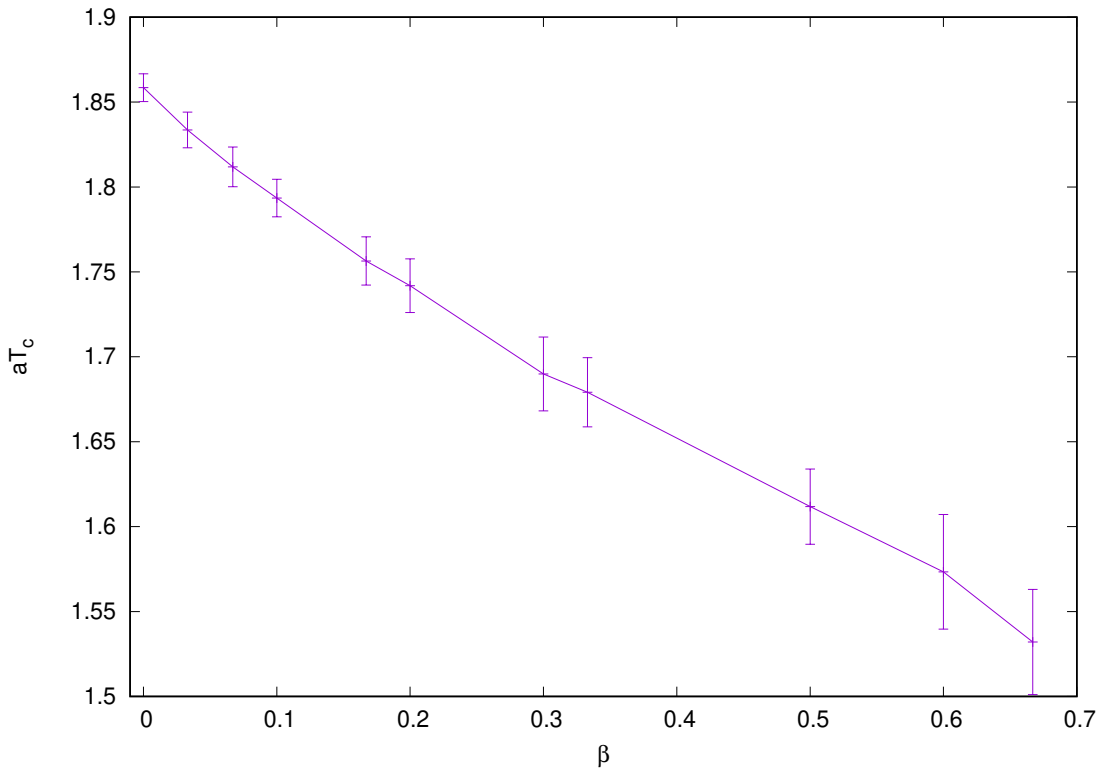
- Average plaquette: $\langle P \rangle = \frac{n_p + \bar{n}_p}{2\beta V}$
- $O(\beta)$ and $O(\beta^2)$ have different results, but match at $\beta = 0$





Dependence on β of aT_c

- Prelimnary result with $O(\beta)$ corrections, $O(\beta^2)$ results not yet satisfactory
- aT_c decreases with increasing β: expected because the lattice spacing decreases with increasing β



Conclusion and Outlook

- Implemented vertex model from tensor network representation, allows for metropolis sampling for $U(N)$ gauge groups
- Agrees with results from exact enumeration
- Peak of susceptibility gives us the chiral transition temperature, higher statistics and larger volumes will let us find the transition temperature
- Extend the model to $SU(N)$ gauge group
- Perform simulations with non-zero m_q and μ_B
- Include sampling via worm algorithm for faster sampling

THANK YOU

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