Three ways of calculating mass spectra for the 2-flavor Schwinger model in the Hamiltonian formalism

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Simulation in the Hamiltonian formalism

Schwinger model (QED$_{1+1d}$)

- the simplest nontrivial gauge theory sharing some features with QCD
  —> good testing ground

Nf=1

- chiral condensate, $q\bar{q}$ potential, mass spectrum, $\cdots$

[Chakraborty et al. (2022)]
[Honda et al. (2022)]
[Banuls et al. (2013)]

Nf=2 —> "hadron" as analogy with QCD

We develop three methods to compute the hadron mass spectrum.

(1) correlation-function scheme: conventional method

(2) one-point-function scheme: makes good use of the boundary effects

(3) dispersion-relation scheme: generates excited states directly

“Hadron” in the 2-flavor Schwinger model

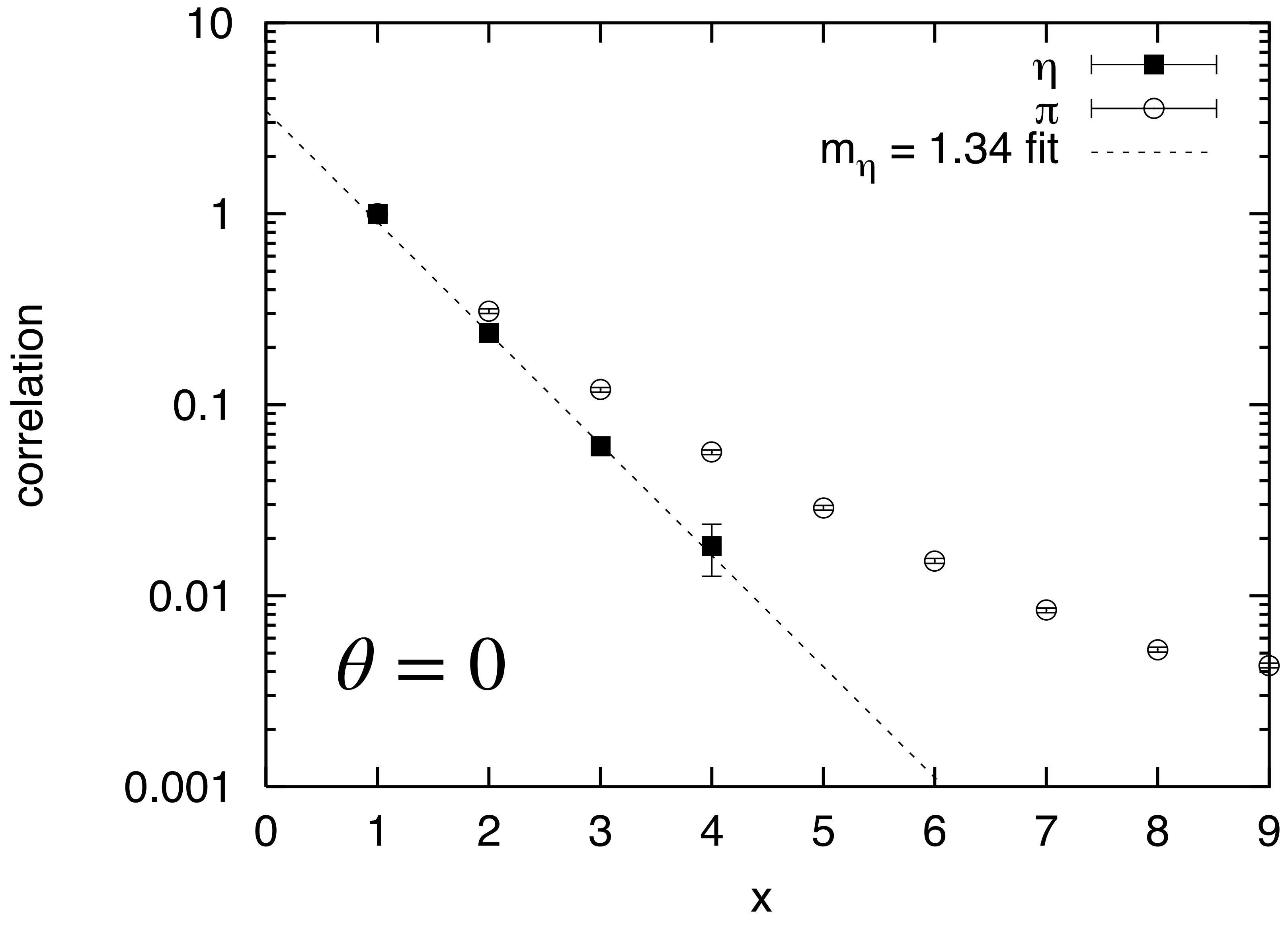
“hadron”: composite particles

$$\pi = -i\left(\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2\right):\ \ J^{PG} = 1^{-+}$$

$$\eta = -i\left(\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2\right):\ \ J^{PG} = 0^{--}$$

$$\sigma = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2:\ \ J^{PG} = 0^{++}$$

Monte Carlo study



[Fukaya & Onogi (2003)]

quantum number to distinguish the hadrons

- isospin J: SU(2) symmetry acting on the flavor doublet
- parity P
- G-parity $G = Ce^{i\pi J_y}$ (generalization of C)

toy model of QCD with up and down quarks

Calculation strategy

- Hamiltonian on the lattice with open boundary condition

$$H=\frac{g^2a}{2}\sum_{n=0}^{N-2}\left(L_n+\frac{\theta}{2\pi}\right)^2+\sum_{f=1}^{N_f}\left[\frac{-i}{2a}\sum_{n=0}^{N-2}\left(\chi_{f,n}^{\dagger}U_n\chi_{f,n+1}-\chi_{f,n+1}^{\dagger}U_n^{\dagger}\chi_{f,n}\right)+m_{\mathrm{lat}}\sum_{n=0}^{N-1}(-1)^n\chi_{f,n}^{\dagger}\chi_{f,n}\right]$$

- solving Gauss law condition
- gauge fixing $U_n = 1$

[Kogut & Susskind (1975)]

[Dempsey et al. (2022)]

- Jordan-Wigner transformation for Nf=2

$$\chi_{1,n}=\sigma_{1,n}^{-}\prod_{j=0}^{n-1}(-\sigma_{2,j}^{z}\sigma_{1,j}^{z}),\qquad \chi_{2,n}=\sigma_{2,n}^{-}(-i\sigma_{1,n}^{z})\prod_{j=0}^{n-1}(-\sigma_{2,j}^{z}\sigma_{1,j}^{z})$$

—> spin Hamiltonian with a finite-dimensional Hilbert space

Density matrix renormalization group (DMRG)

[White (1992)] [Schollwock (2005)]

variational method to find the eigenstates of H using matrix product state (MPS)

- cost function: energy $E = \langle \Psi | H | \Psi \rangle$

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}\left[A_0(s_0)\, A_1(s_1)\ \cdots\right] |\, s_0\, s_1\ \cdots\rangle$$

- $A_i(s_i)$: $D_{i-1} \times D_i$ matrix (D_i : bond dimension)
- introduce a cutoff ε to controls the accuracy
  singular values smaller than ε are neglected in SVD
  —> small ε needs large D_i
- k-th excited state $|\Psi_k\rangle$ —> cost function: $\langle \Psi_k | H | \Psi_k \rangle + W \sum_{k'=0}^{k-1} \left| \langle \Psi_{k'} | \Psi_k \rangle \right|^2$

The C++ library of ITensor is used in this work. [Fishman et al. (2022)]

Simulation results

1. Correlation-function scheme
2. One-point-function scheme
3. Dispersion-relation scheme

Simulation results

1. **Correlation-function scheme**
2. One-point-function scheme
3. Dispersion-relation scheme

(1) Correlation-function scheme

- spatial correlation function: $C_\pi(r) = \langle \pi(x)\pi(y) \rangle$
- effective mass: $M_{\pi,\text{eff}}(r) = -\frac{d}{dr}\log C_\pi(r), \quad r = |x-y|$

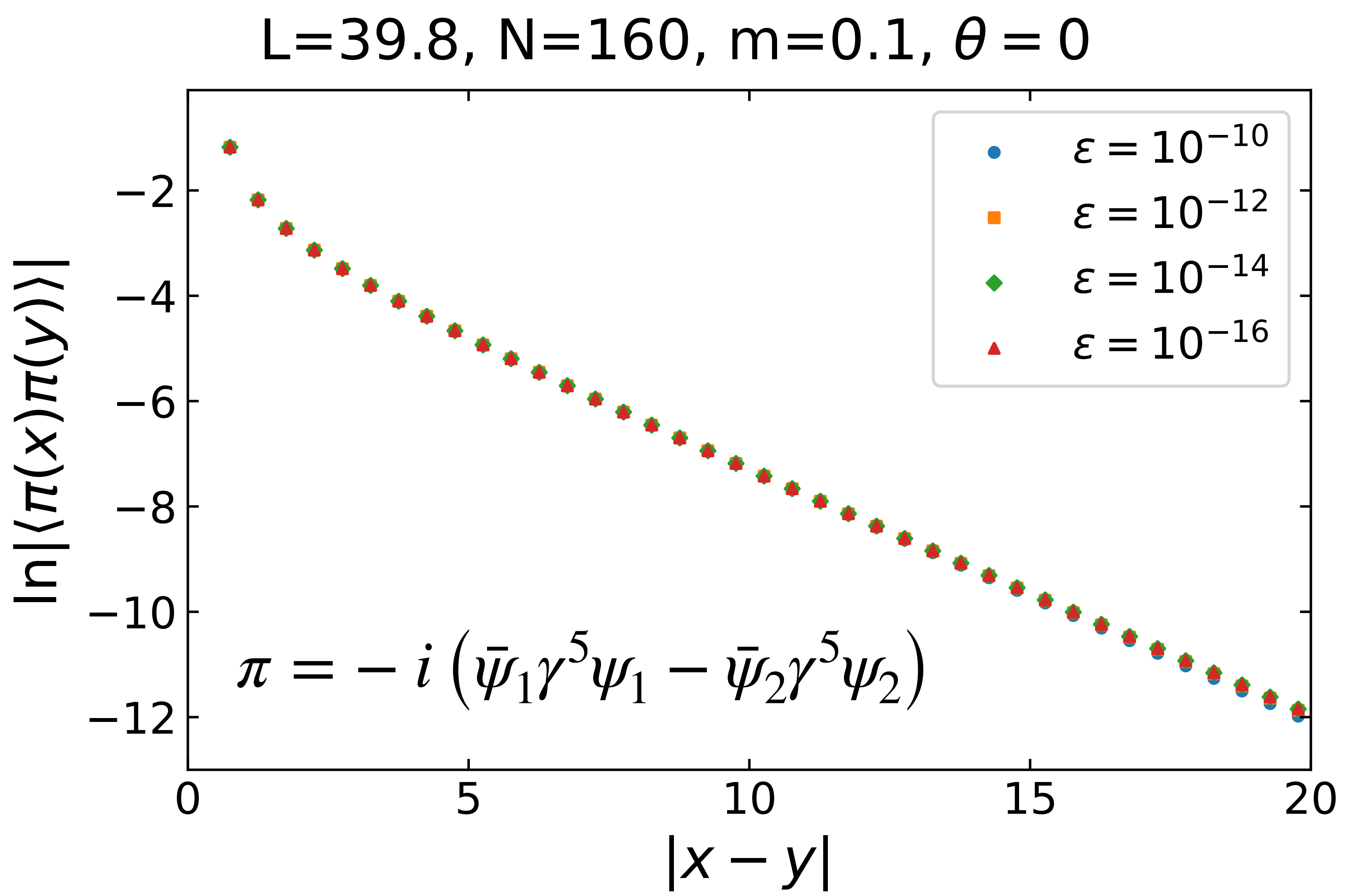
plateau value = pion mass?

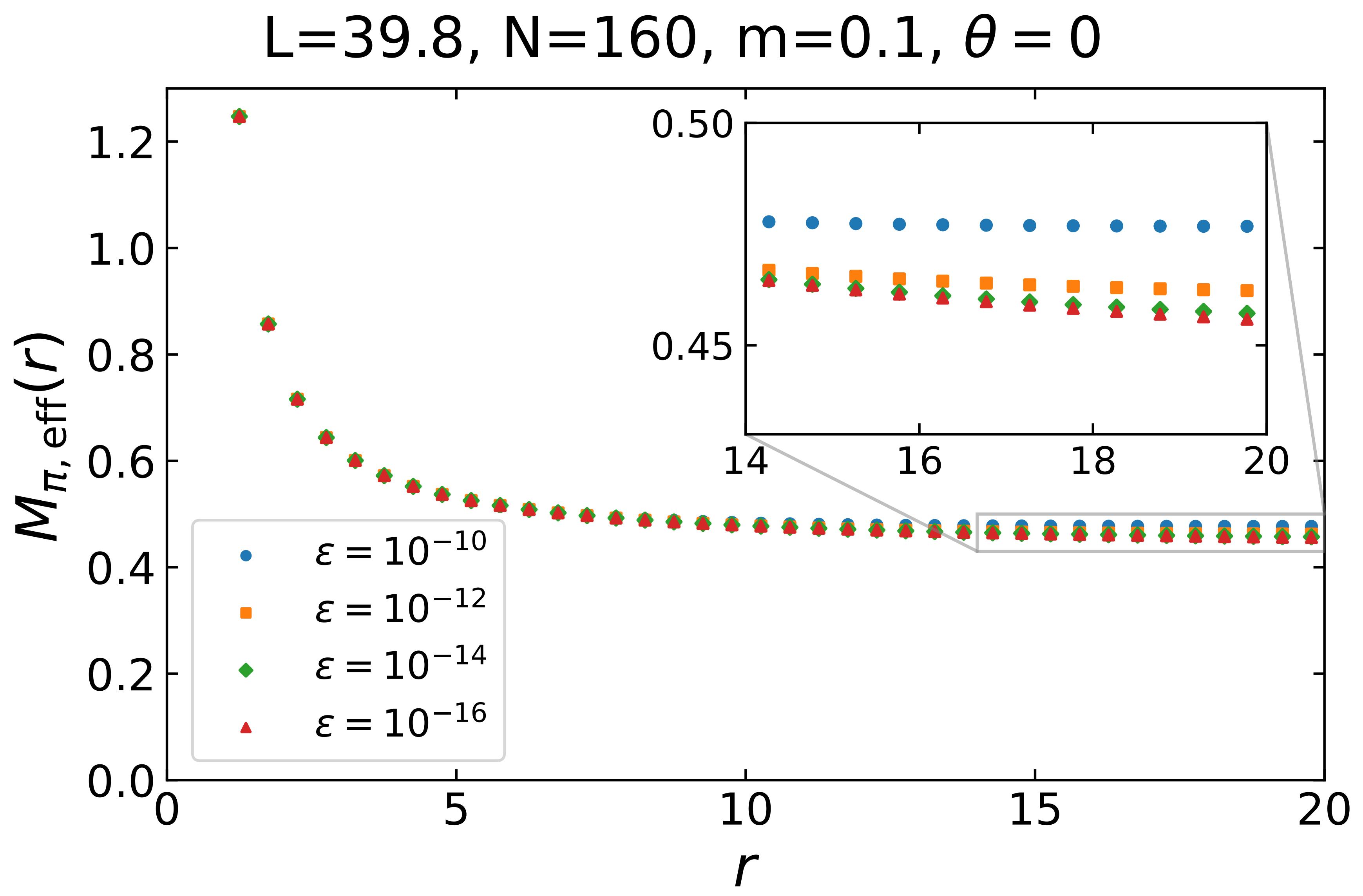
⚠️ plateau behavior gets modification in precise calc.

$\varepsilon = 10^{-10}$ ($D_i \sim 400$) : $M_{\pi,\text{eff}}(r)$ is almost flat

$\varepsilon = 10^{-16}$ ($D_i \sim 2800$) : $M_{\pi,\text{eff}}(r)$ depends on r

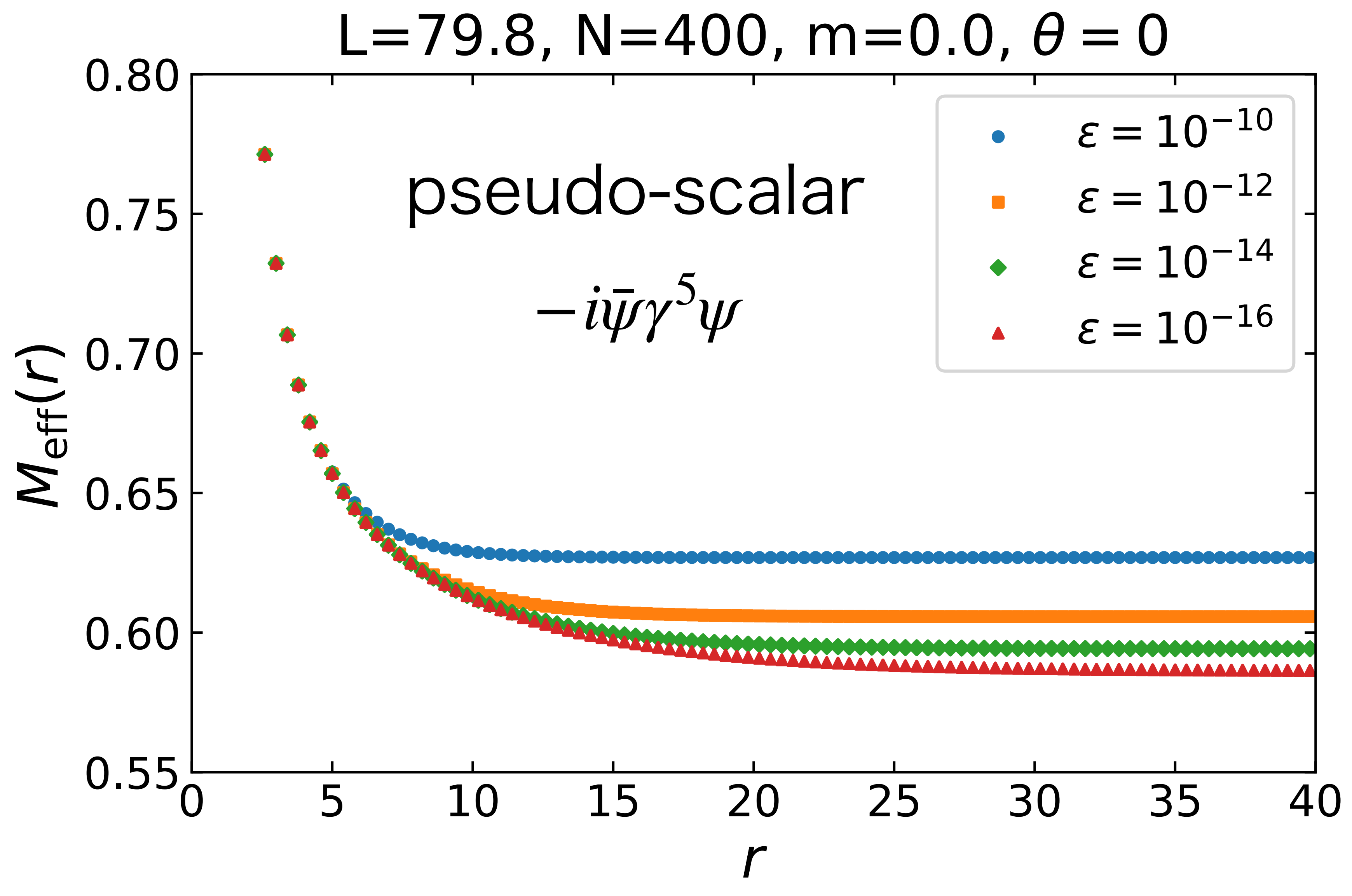
- What's happened?



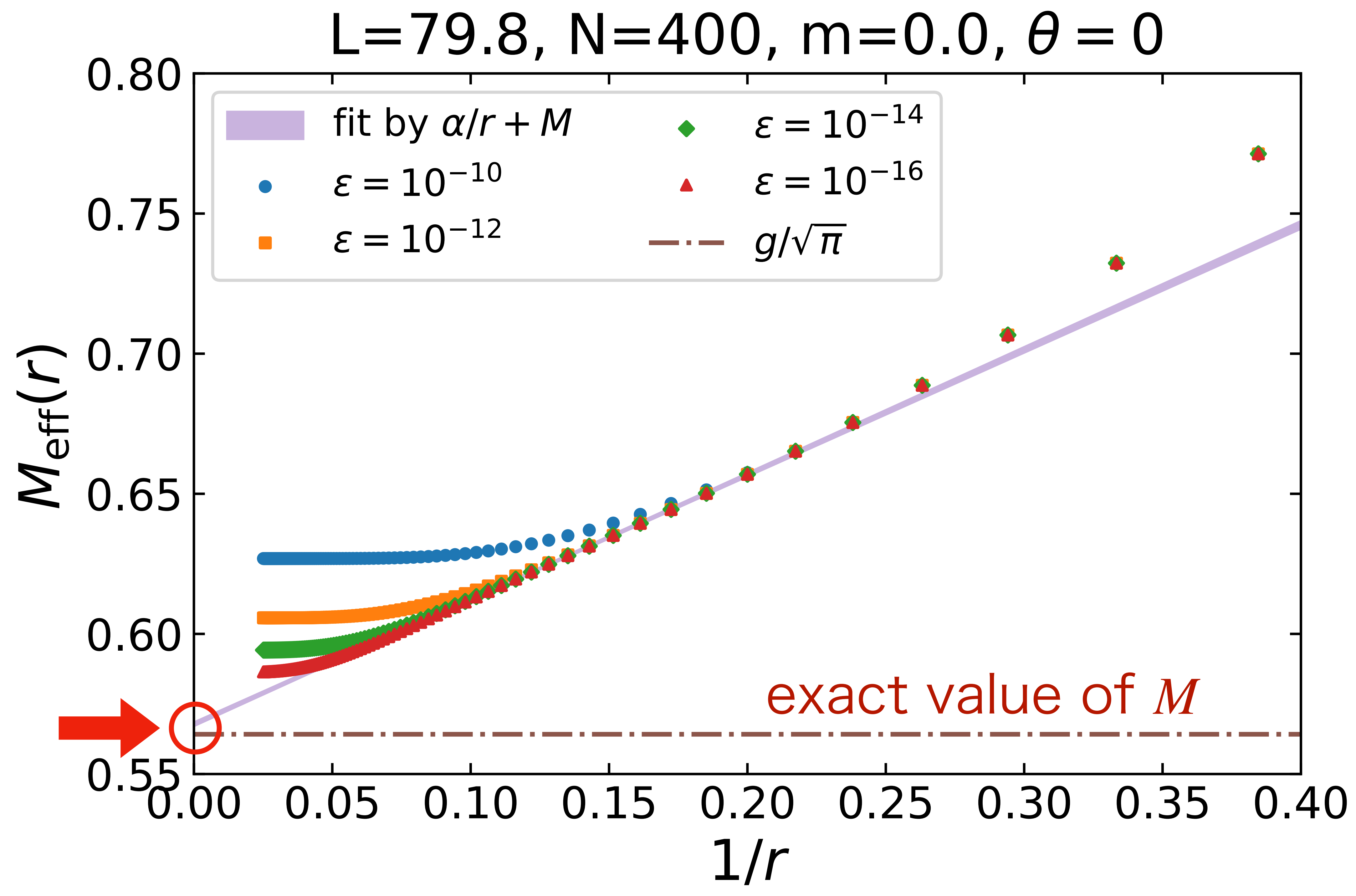


Feasibility test in Nf=1 case

- (1+1)d free particle with mass M : $\langle\phi(x,t)\phi(y,t)\rangle \sim \frac{1}{\sqrt{Mr}} e^{-Mr}$ ⟶ $M_{\text{eff}}(r) \sim \frac{\alpha}{r} + M$
- massless Nf=1 Schwinger model (exactly solvable)



plot against $\frac{1}{r}$

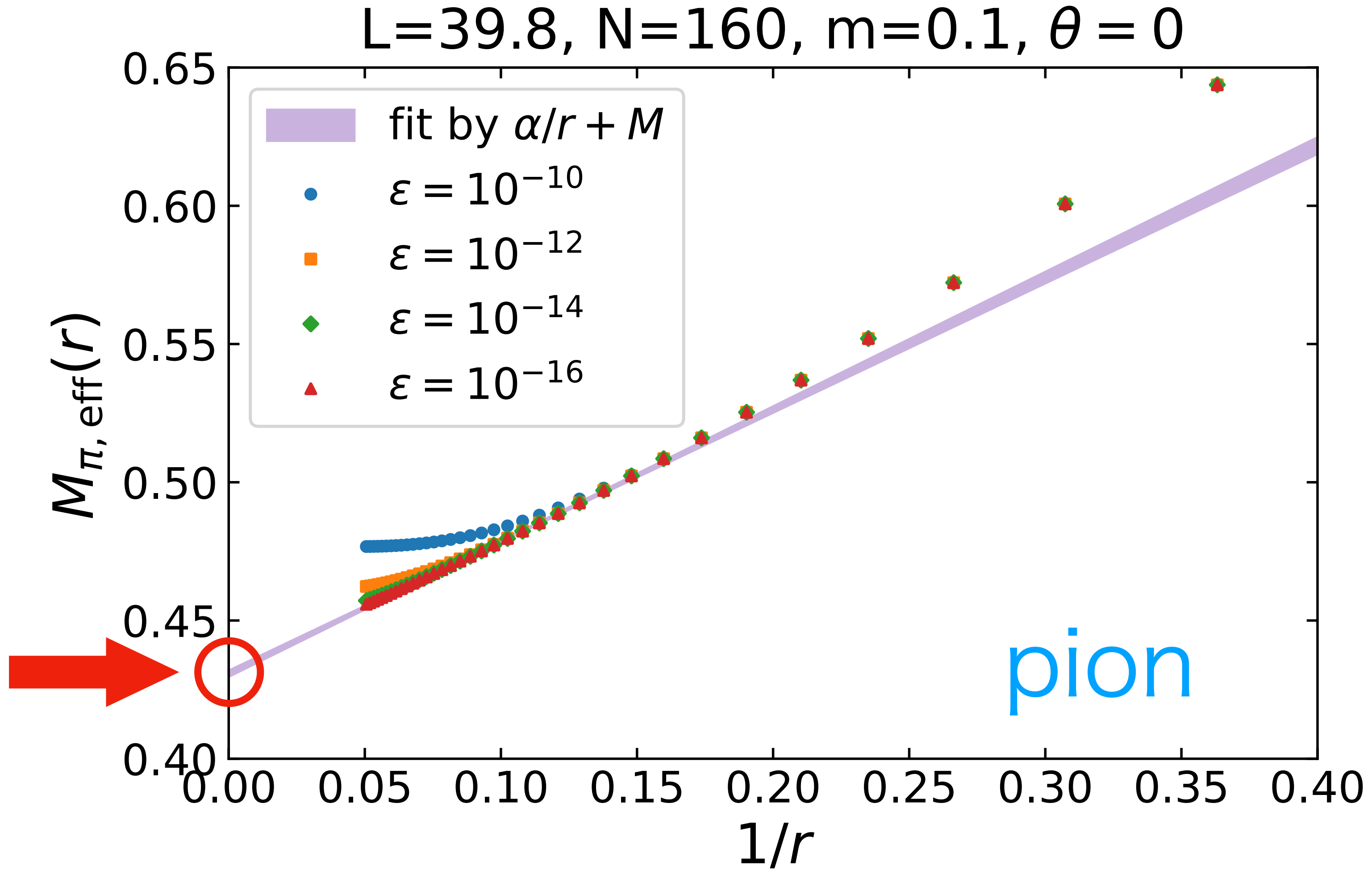


- difficult to reproduce $1/r$ term by DMRG
- we need $1/r \to 0$ extrapolation

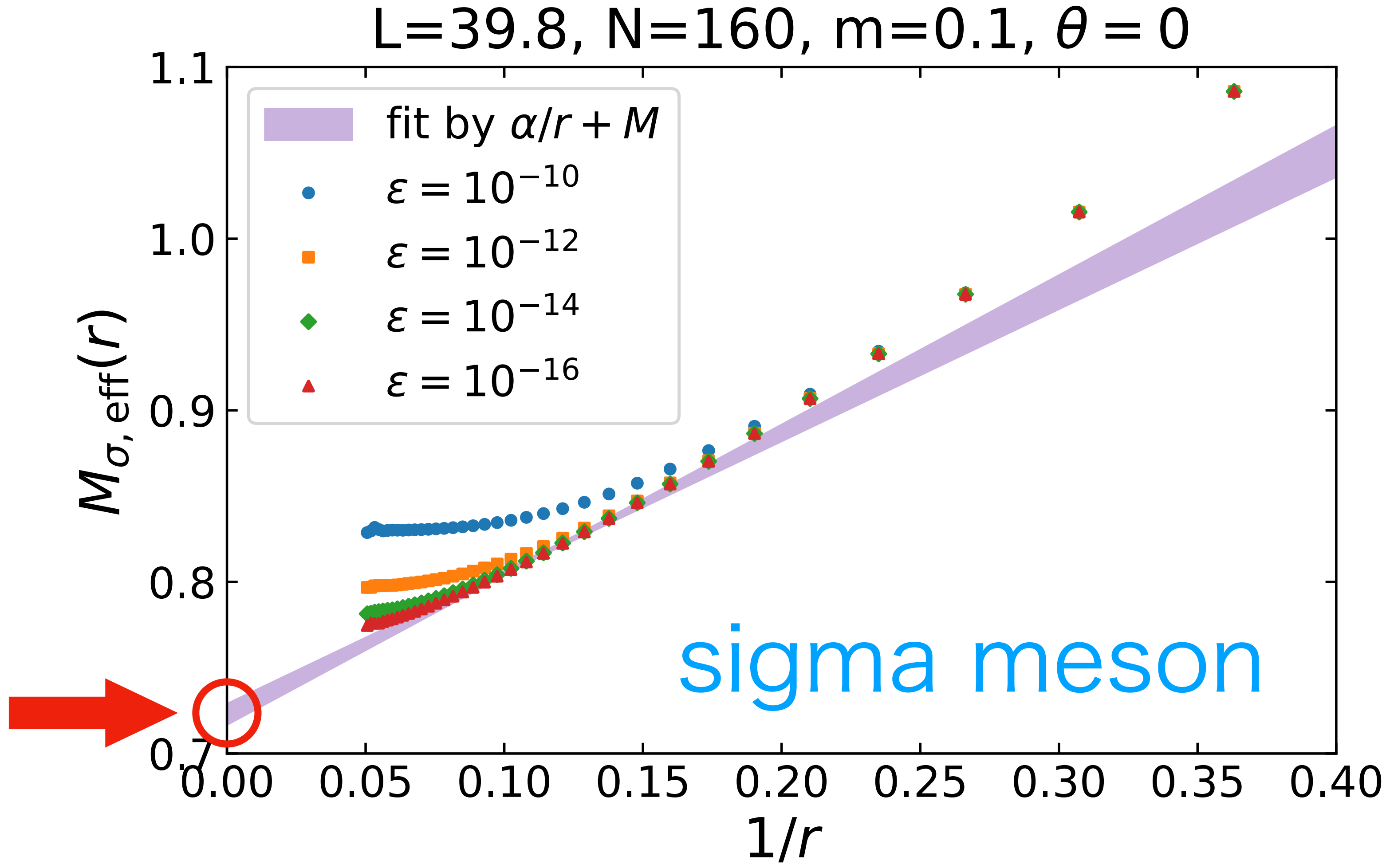
Result of Nf=2 case

extrapolate the effective masses to $1/r \to 0$ for $\varepsilon = 10^{-16}$

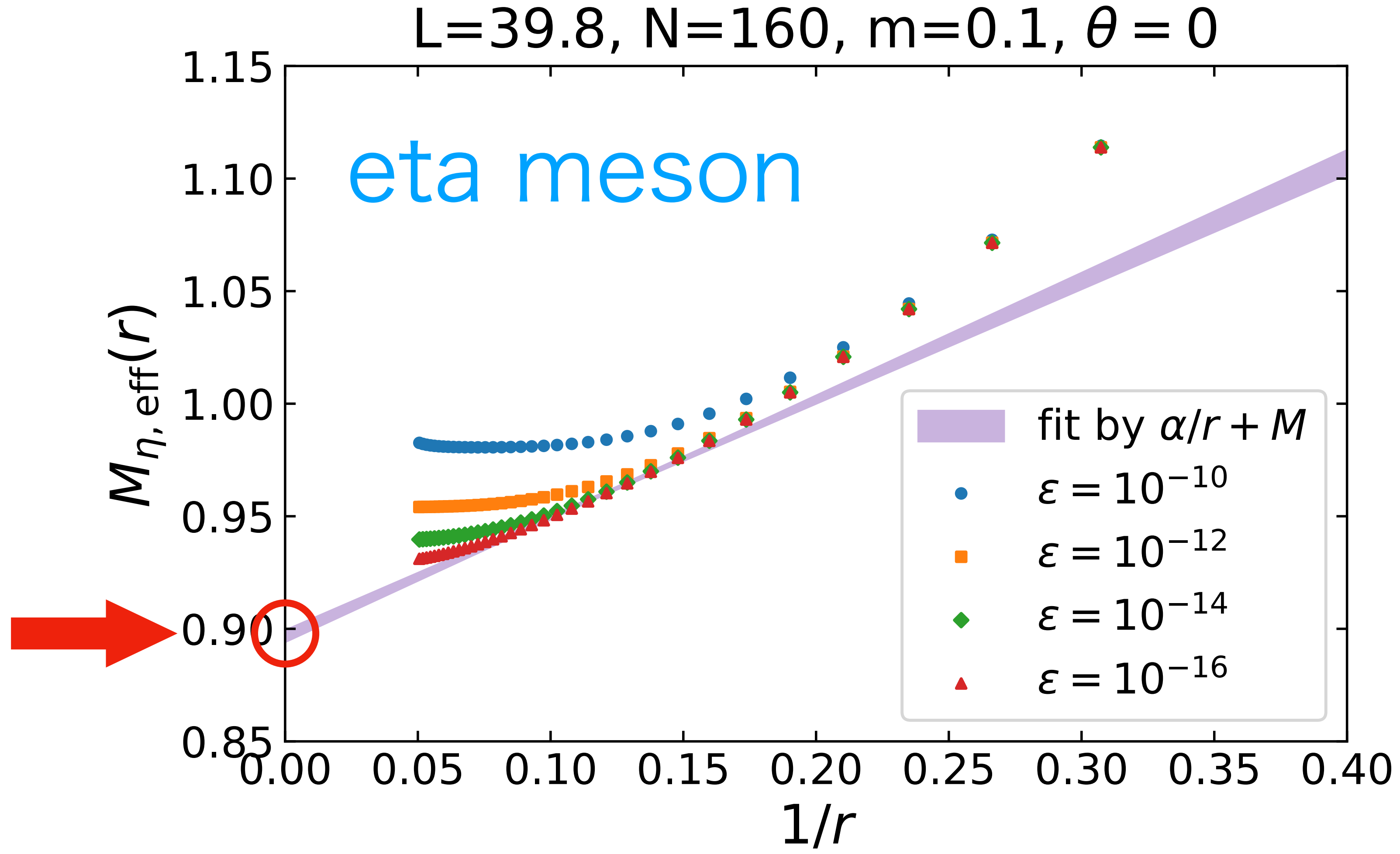
$$\pi = -i\left(\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2\right)$$



$$\sigma = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$$



$$\eta = -i\left(\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2\right)$$



	pion	sigma	eta
M	0.431(1)	0.722(6)	0.899(2)
α	0.477(9)	0.83(5)	0.51(2)

Simulation results

1. Correlation-function scheme
2. **One-point-function scheme**
3. Dispersion-relation scheme

(2) one-point-fn. scheme (eta & sigma)

- At $\theta = 0$, the open boundary is a source of iso-singlet states. (analogous to wall source)
- one-point function $\langle \mathcal{O}(x) \rangle \sim e^{-Mx+C}$

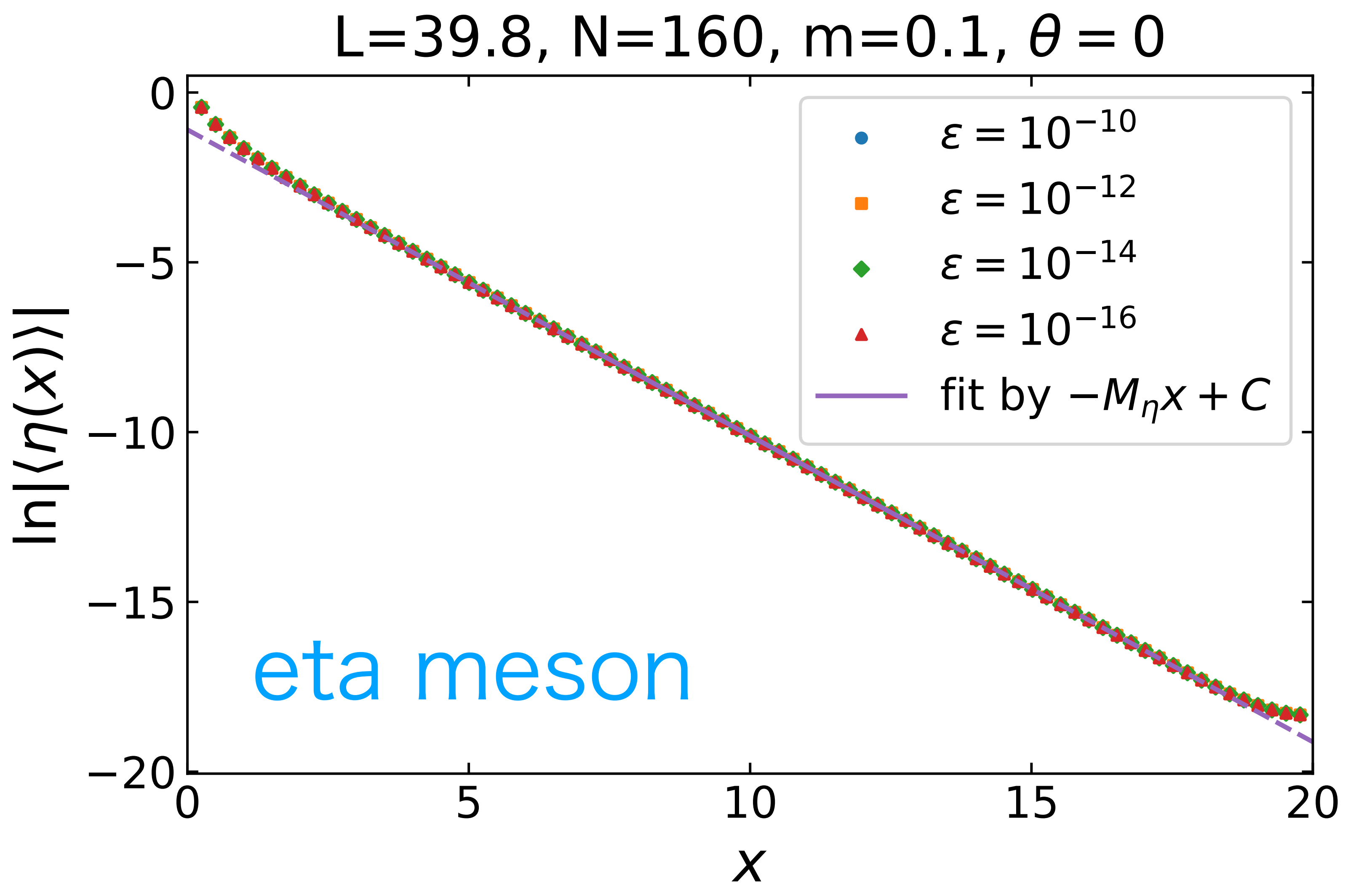
  x : distance from the boundary
- ε-dependence is not observed

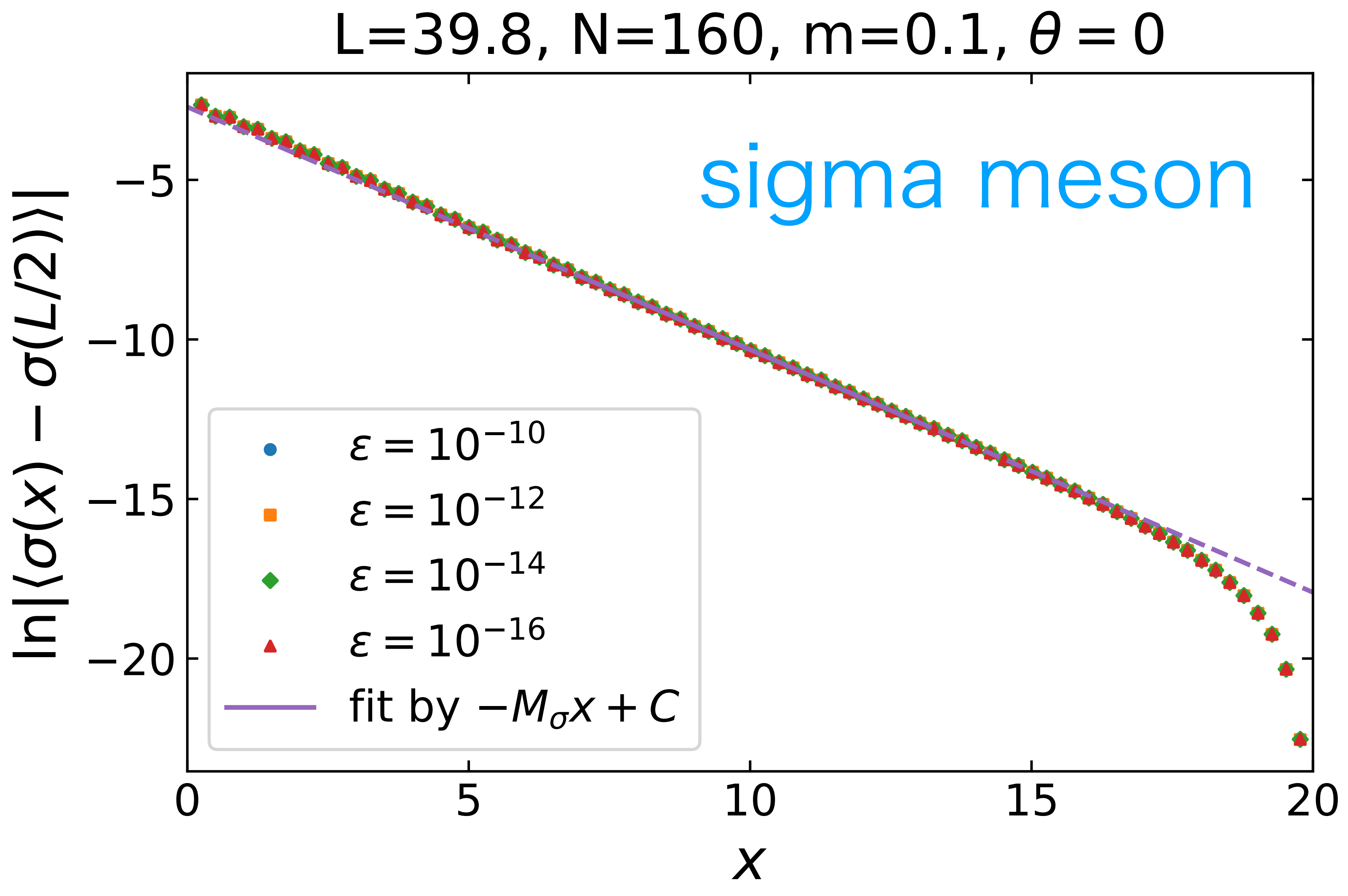
  —> systematic error

  from truncating D_{eff} is sufficiently small

fitting results

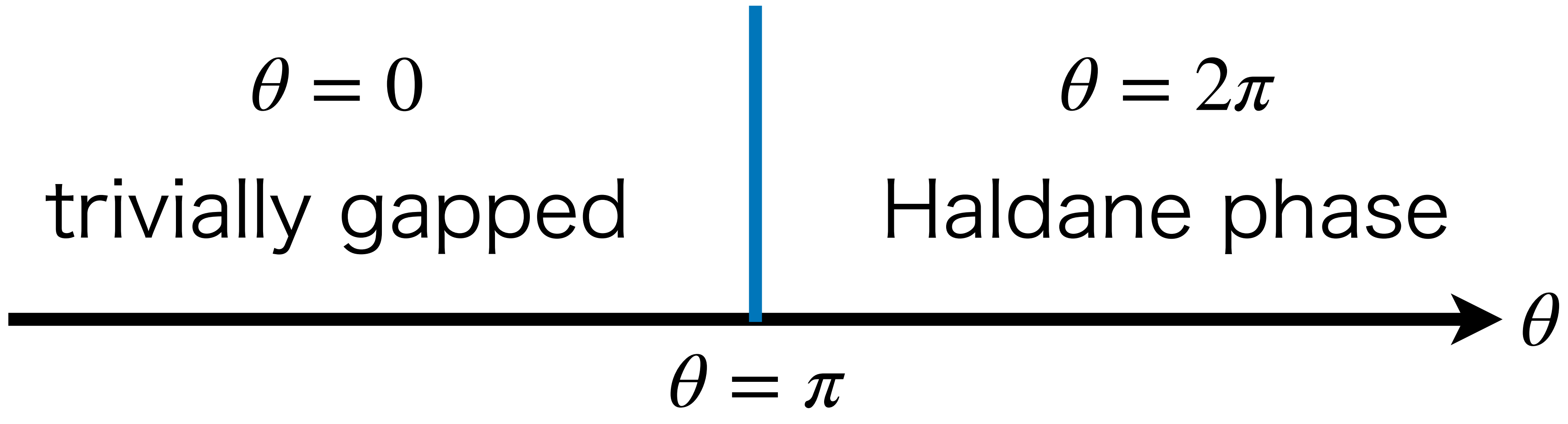
- eta: $M = 0.9014(1),\ C = -1.096(1)$
- sigma: $M = 0.761(2),\ C = -2.71(2)$





(2) pion: tricky case

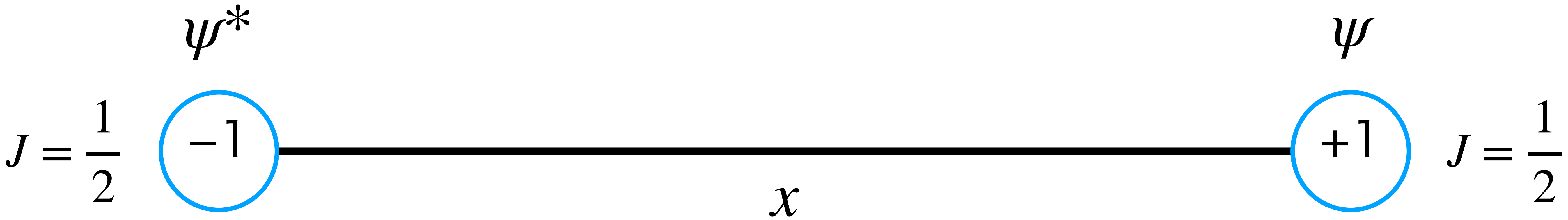
⚠️ $\langle \pi(x) \rangle = 0$ at $\theta = 0$ (trivially gapped phase)



SPT aspect

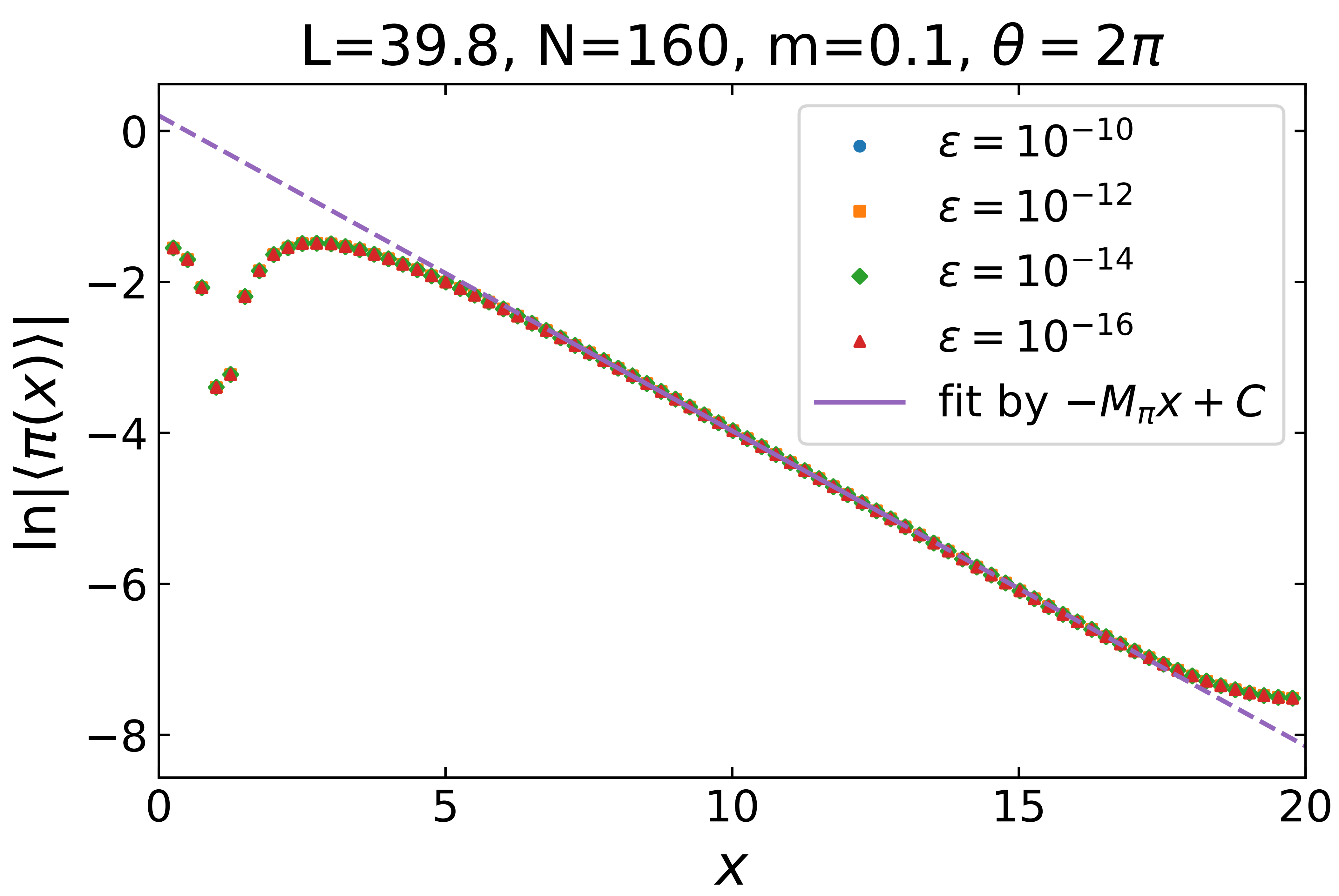
setting $\theta = 2\pi$ —> introducing a background electric field

- Dirac fermions with charge ± 1 are induced at both ends
- isospin 1/2 at the boundary —> a source of iso-triplet mesons



(2) one-point-fn. scheme (pion)

- generate the ground state at $\theta = 2\pi$
- compute $|\langle \pi(x) \rangle| \sim e^{-Mx+C}$
- fitting results:
  $M = 0.4175(9), \quad C = 0.203(9)$
- ε-dependence is not observed



	pion	sigma	eta
M	0.4175(9)	0.761(2)	0.9014(1)

Simulation results

1. Correlation-function scheme

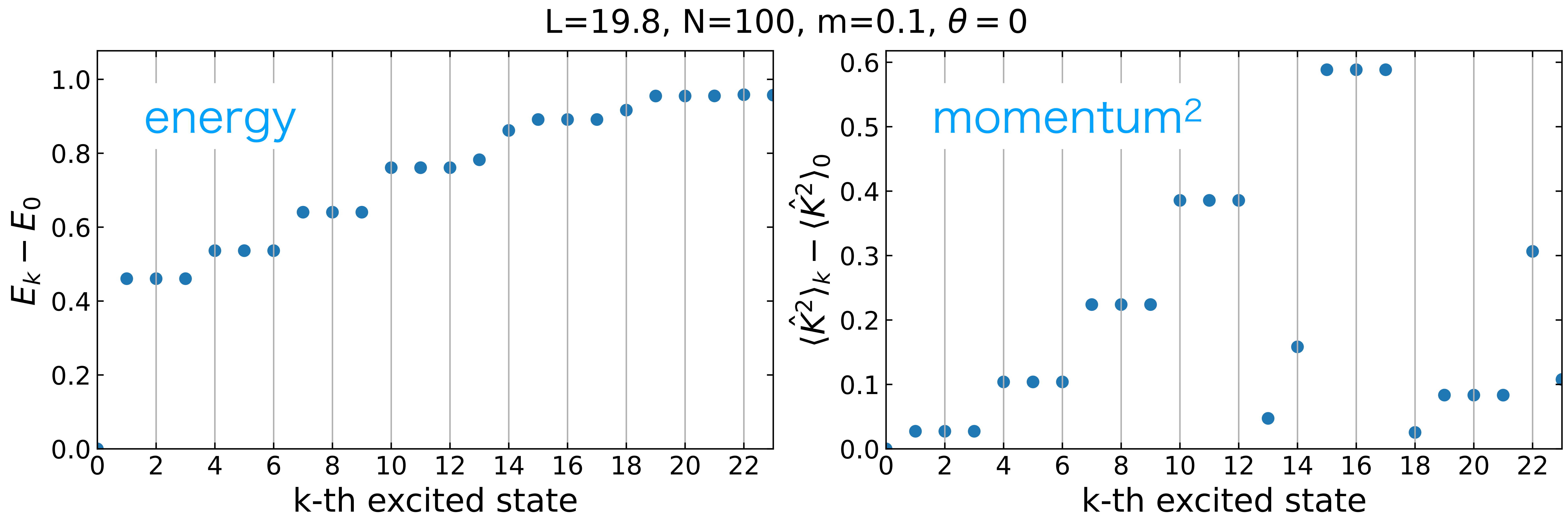
2. One-point-function scheme

3. **Dispersion-relation scheme**

(3) Dispersion-relation scheme

- Energy gap: $\Delta E_k = E_k - E_0$ Momentum square: $\Delta K_k^2 = \langle K^2 \rangle_k - \langle K^2 \rangle_0$
- triplets —> pion? singlets —> sigma or eta meson?

identify the states by measuring quantum numbers: $\mathbf{J}^2,\ J_z,\ G = Ce^{i\pi J_y}$



Quantum numbers

- triplets: $\mathbf{J}^2 = 2,\ \ J_z = (0, \pm 1),\ \ G > 0$

  —> pion ($J^{PG} = 1^{-+}$)

- singlets: $\mathbf{J}^2 = 0,\ \ J_z = 0,$

  $G > 0\ (k = 13,14,22)$ —> sigma meson ($J^{PG} = 0^{++}$)

  $G < 0\ (k = 18,23)$ —> eta meson ($J^{PG} = 0^{--}$)

singlets

k	$\mathbf{J}^2$	J_z	G
0	0.00000003	-0.00000000	0.27984227
13	0.00000003	0.00000000	0.27865844
14	0.00000003	0.00000000	0.27508176
18	0.00000028	0.00000006	-0.27390909
22	0.00001537	0.00000115	0.26678987
23	0.00003607	-0.00000482	-0.27664779

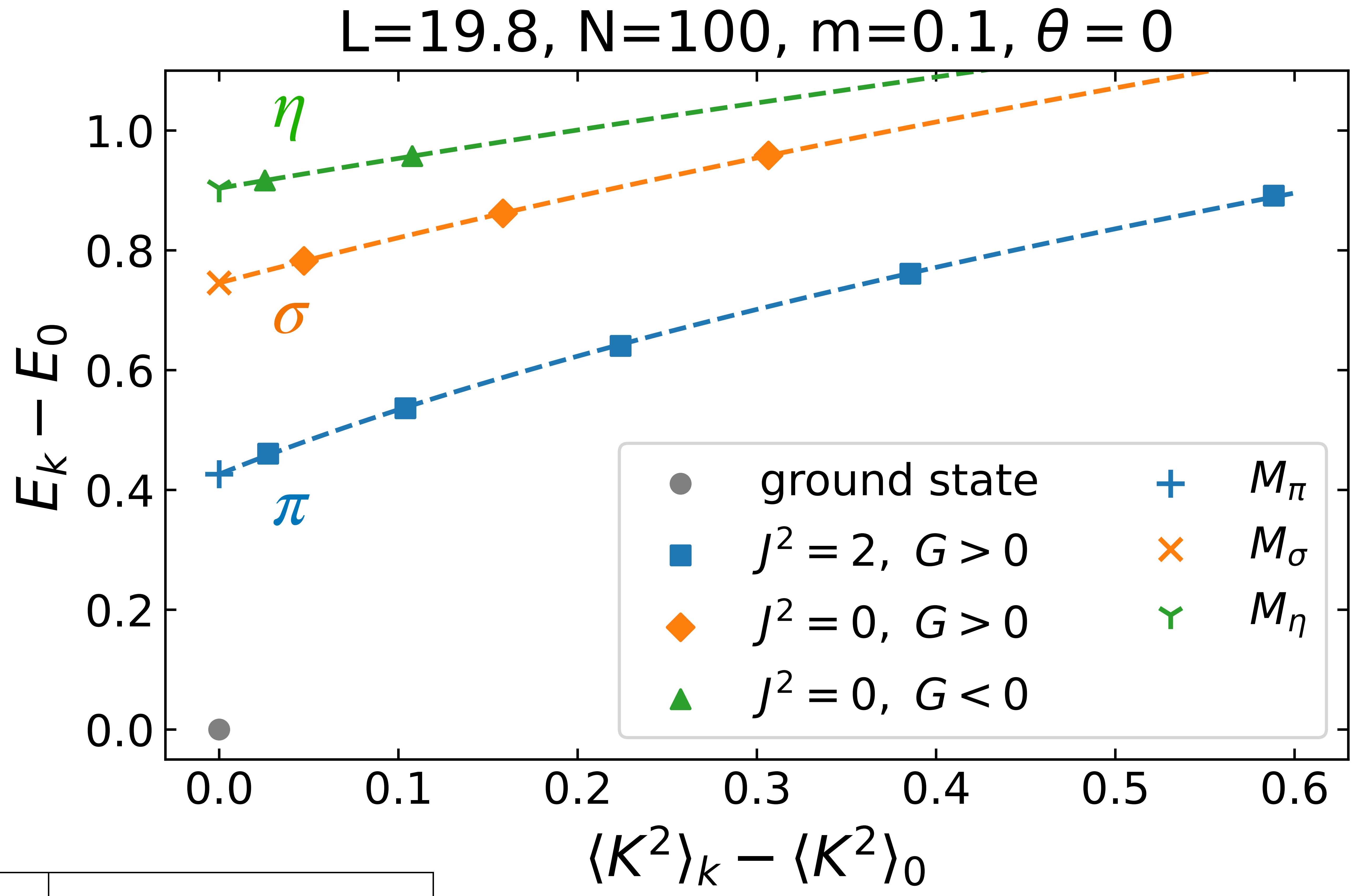
triplets

k	$\mathbf{J}^2$	J_z	G
1	2.00000004	0.99999997	0.27872443
2	2.00000012	-0.00000000	0.27872416
3	2.00000004	-0.99999996	0.27872443
4	2.00000007	0.99999999	0.27736066
5	2.00000006	0.00000000	0.27736104
6	2.00000009	-0.99999998	0.27736066
7	2.00000010	1.00000000	0.27536687
8	2.00000002	0.00000000	0.27536702
9	2.00000007	-0.99999998	0.27536687
10	2.00000007	0.99999998	0.27356274
11	2.00000005	0.00000001	0.27356277
12	2.00000007	-0.99999999	0.27356274
15	1.99999942	0.99999966	0.27173470
16	2.00000052	0.00000000	0.27173482
17	2.00000015	-1.00000003	0.27173470
19	2.00009067	1.00004377	0.27717104
20	2.00002578	-0.00000004	0.27717020
21	2.00003465	-1.00001622	0.27717104

Result of dispersion relation

- plot ΔE_k against ΔK_k^2 for each meson

- fit the data by $\Delta E = \sqrt{b^2 \Delta K^2 + M^2}$



	pion	sigma	eta
M	0.426(2)	0.7456(5)	0.9037
b	1.017(4)	1.087(2)	0.9622

Summary

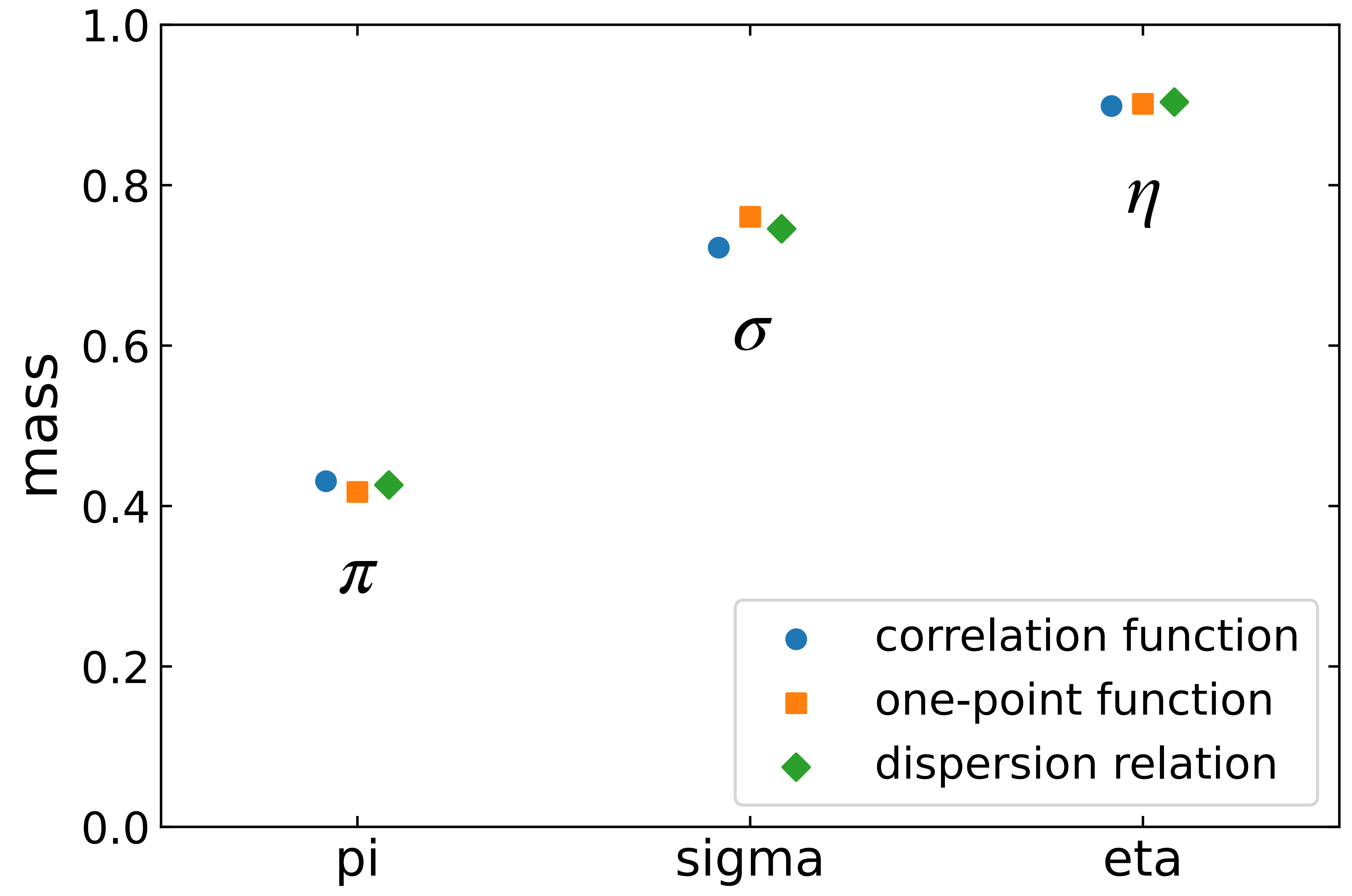
- The three results are consistent with each other and look promising.
- consistent with the analytic predictions

  ✔ $M_\pi < M_\sigma < M_\eta$ —> U(1) problem

  ✔ $M_\eta = \mu + O(m) \quad (\mu = g\sqrt{2/\pi} \sim 0.8,\ m = 0.1)$

  ✔ $M_\sigma / M_\pi = \sqrt{3}$ within 5% deviation

  [Coleman (1976)]   [Dashen et al. (1975)]



	correlation func.	one-point func.	dispersion
M_σ/M_π	1.68(2)	1.821(6)	1.75(1)

Discussion

(1)correlation-function scheme

👍 applicability to other models

😥 sensitive to the bond dimension (DMRG) —> 😊 quantum computation

(2)one-point-function scheme

👍 need to increase neither the bond dimension nor the system size L

😥 only the lowest state having given quantum numbers

(3)dispersion-relation scheme

👍 obtain various states heuristically / directly see wave functions (s/p-wave)

😥 computational cost to generate excited states

Thank you for listening.

Future prospect

go to $\theta \neq 0$?

- meson operators are mixed nontrivially

  —> We need to compute the correlation matrix $\begin{pmatrix} \langle S, S \rangle & \langle S, PS \rangle \\ \langle PS, S \rangle & \langle PS, PS \rangle \end{pmatrix}$

- C and P are explicitly broken by $\theta \neq 0$

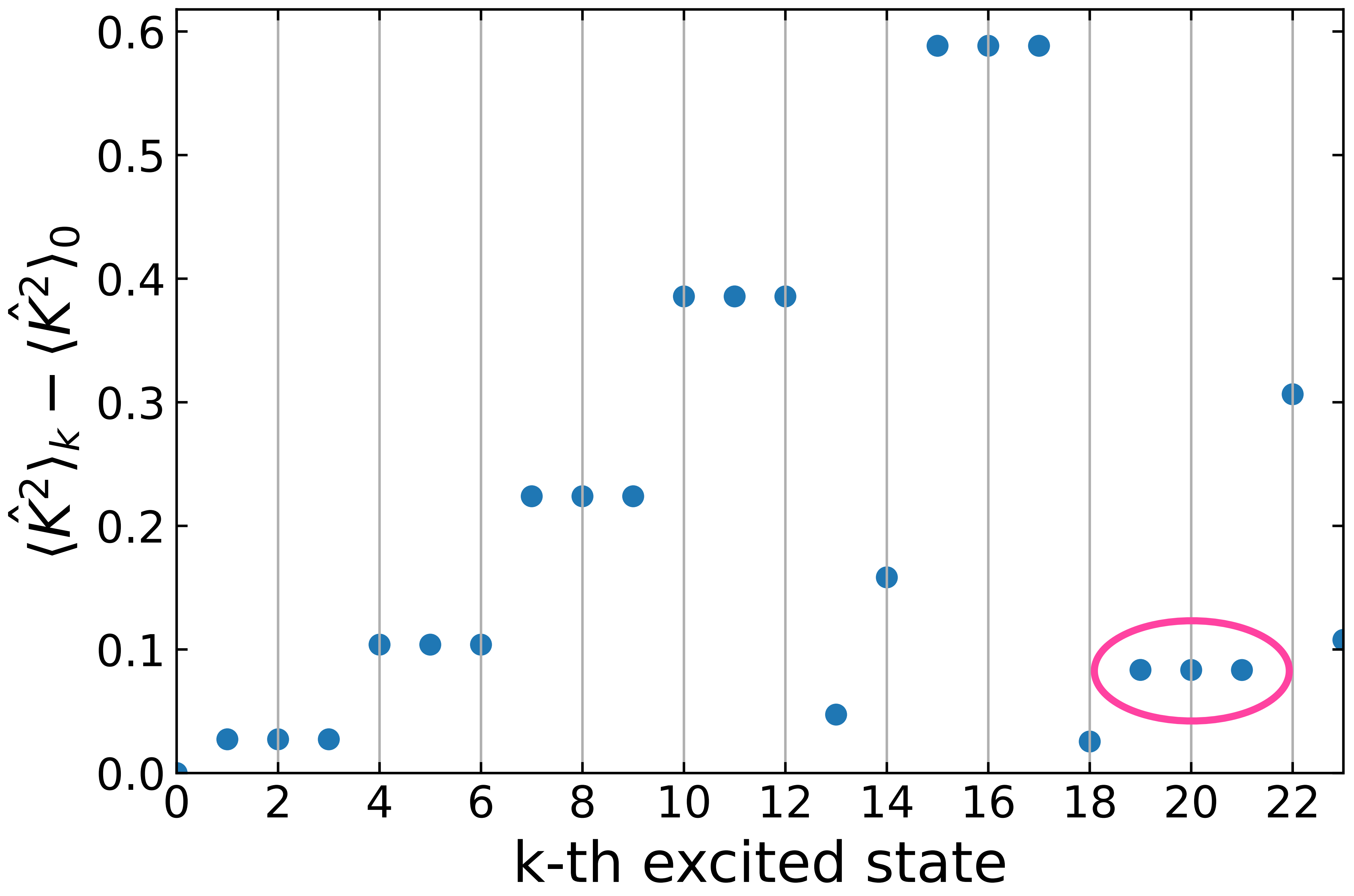
- approach the almost gapless phase $\theta = \pi$

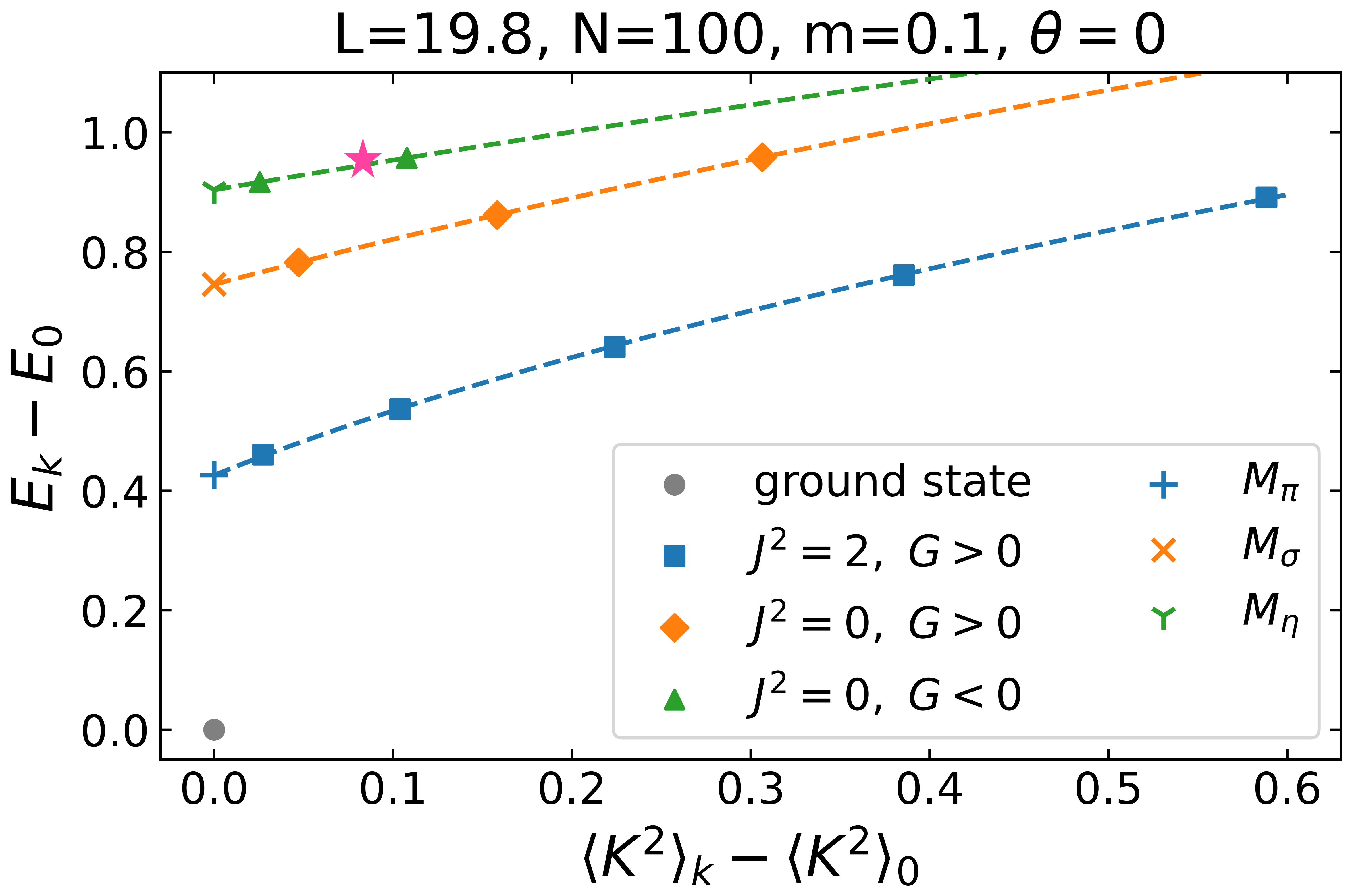
  —> increase the bond dimension

Result of dispersion relation

★ triplet $k = 19{,}20{,}21$ seems to be a two-pion scattering state [Harada et al. (1994)]

the same quantum numbers as the pion $1 \times 1 = 0 + 1 + 2$

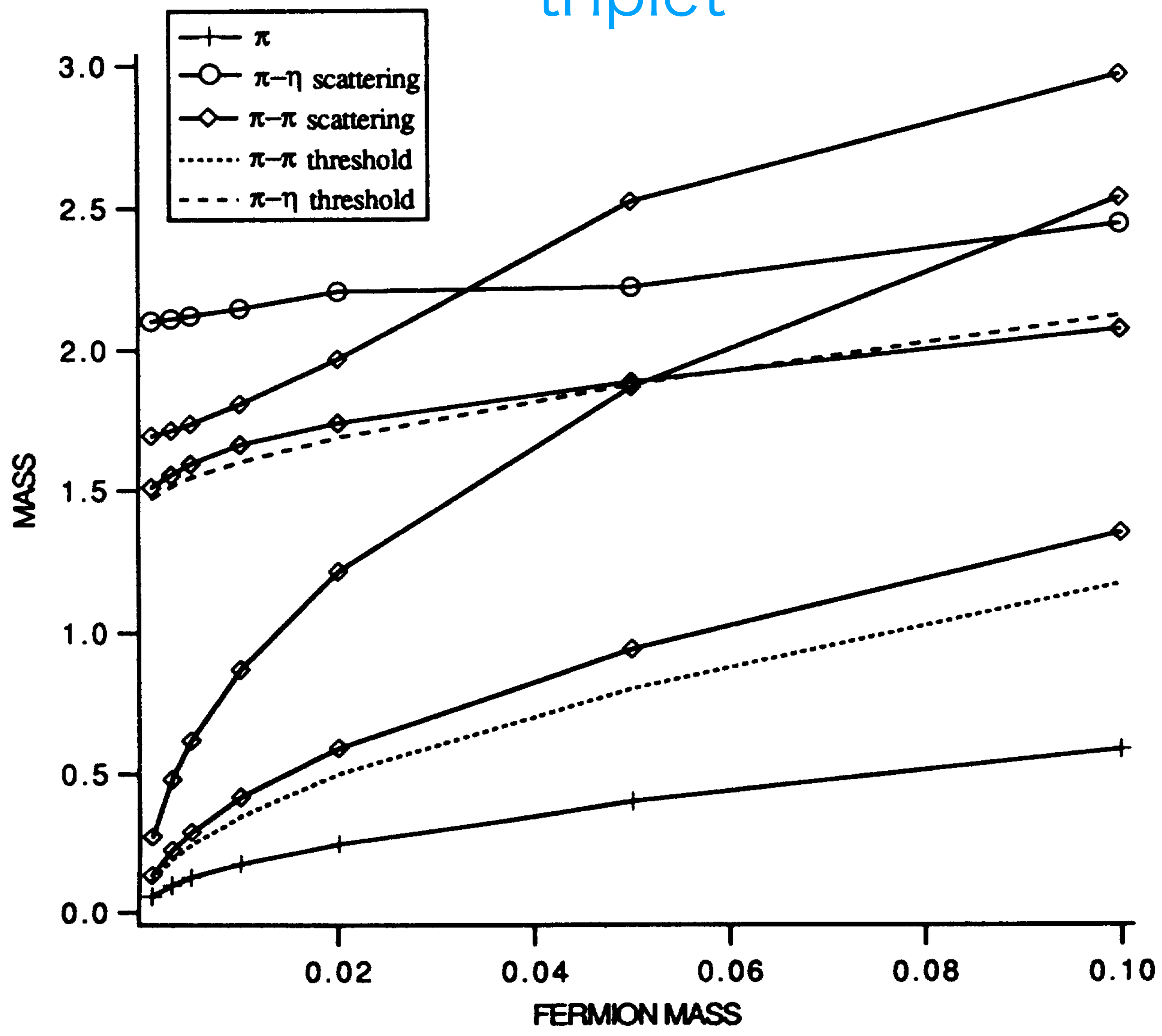




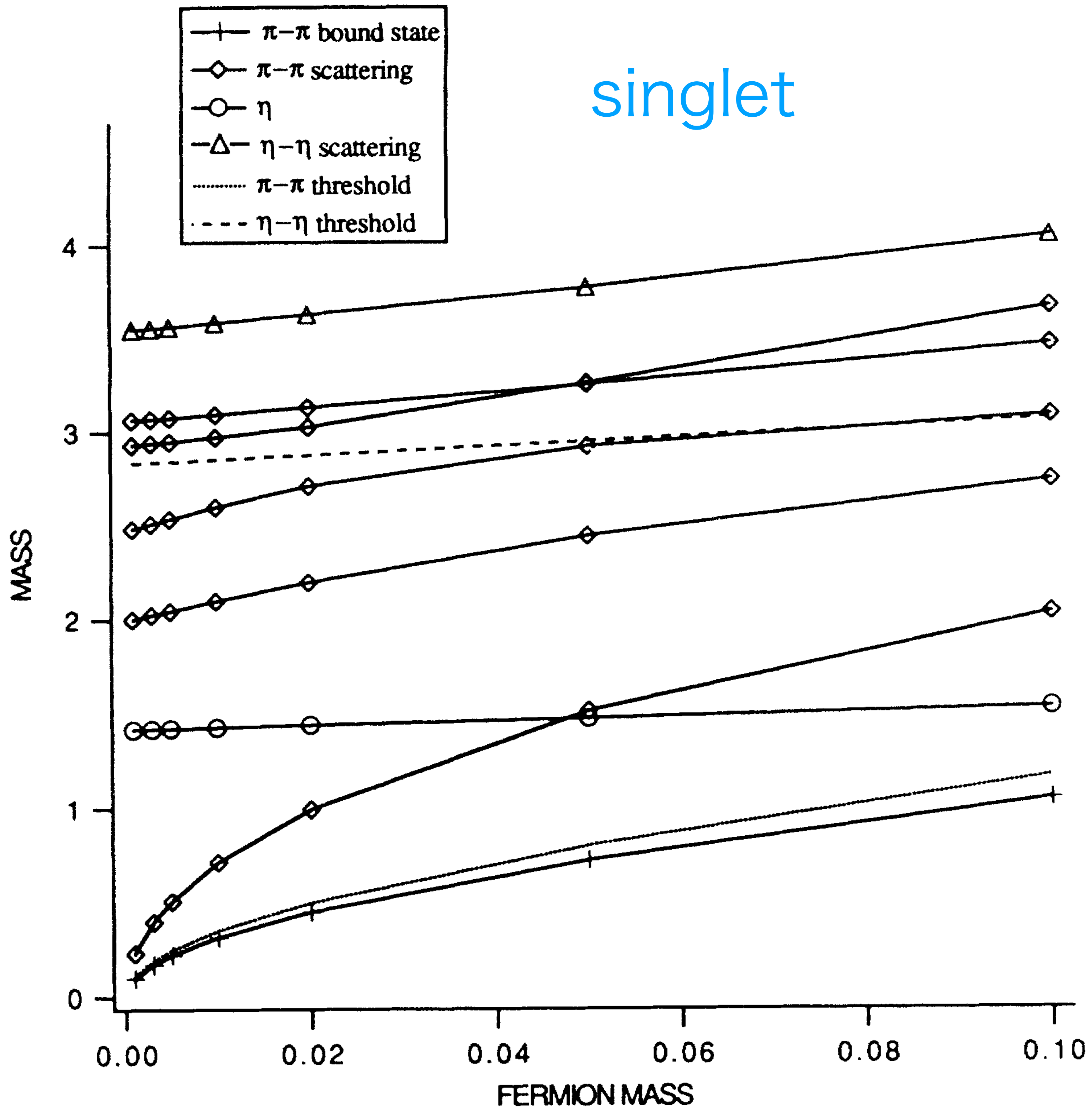
light-front Tamm-Dancoff approximation

- [Harada et al. (1994)]
- "pi-pi scattering state" slightly above the eta meson
- sigma meson ~ "pi-pi bound state"

triplet



singlet



Low energy mass spectrum

- massive 2-flavor Schwinger model is not exactly solvable
- Bosonization technique can be used in the strong coupling region $g \gg m > 0$

$$H = N_m \left[\frac{1}{2}\Pi_+^2 + \frac{1}{2}(\partial_1 \phi_+)^2 + \frac{\mu^2}{2}\phi_+^2 + \frac{1}{2}\Pi_-^2 + \frac{1}{2}(\partial_1 \phi_-)^2 - 2cm^2 \cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2}\right) \cos\left(\sqrt{2\pi}\phi_-\right) \right]$$

[Coleman (1976)]

- ϕ_+ —> eta meson: $M_\eta = \mu + O(m), \ \ \mu^2 = 2g^2/\pi$
- soliton/anti-soliton of ϕ_- and their bound states

  described by sine-Gordon model

  —> pion: $M_\pi \propto |m\mu^{1/2}\cos(\theta/2)|^{2/3}$, sigma meson: $M_\sigma = \sqrt{3}M_\pi$ —> $M_\pi < M_\sigma < M_\eta$

[Dashen et al. (1975)]

Hamiltonian formalism

- Hamiltonian is written only by fermionic operators

$$H = \frac{g^2 a}{2}\sum_{n=0}^{N-2}\left[\sum_{f=1}^{N_f}\sum_{k=0}^{n}\chi_{f,k}^{\dagger}\chi_{f,k} + \frac{N_f}{2}\left(\frac{(-1)^n - 1}{2} - n\right) + \frac{\theta}{2\pi}\right]^2 + \sum_{f=1}^{N_f}\left[\frac{-i}{2a}\sum_{n=0}^{N-2}\left(\chi_{f,n}^{\dagger}\chi_{f,n+1} - \chi_{f,n+1}^{\dagger}\chi_{f,n}\right) + m_{\mathrm{lat}}\sum_{n=0}^{N-1}(-1)^n\chi_{f,n}^{\dagger}\chi_{f,n}\right]$$

- Jordan-Wigner transformation: fermion operator —> spin operator

$$\chi_{1,n} = \sigma_{1,n}^{-}\prod_{j=0}^{n-1}(-\sigma_{2,j}^{z}\sigma_{1,j}^{z}) \qquad \chi_{2,n} = \sigma_{2,n}^{-}(-i\sigma_{1,n}^{z})\prod_{j=0}^{n-1}(-\sigma_{2,j}^{z}\sigma_{1,j}^{z})$$

$$\sigma_{f,n}^{\pm} = \frac{1}{2}(\sigma_{f,n}^{x} \pm i\sigma_{f,n}^{y}) \qquad \left[\sigma_{f,n}^{a}, \sigma_{f',n'}^{b}\right] = 2i\,\delta_{ff'}\,\delta_{nn'}\,\epsilon^{abc}\,\sigma_{f,n}^{c}$$

- useful to apply the tensor network method or quantum computation

Hamiltonian formalism

- spin Hamiltonian: $H = H_{\text{gauge}} + H_{\text{kin}} + H_{\text{mass}}$

$$H_{\text{gauge}} = \frac{g^2 a}{8} \sum_{n=0}^{N-2} \left[\sum_{f=1}^{N_f} \sum_{k=0}^{n} \sigma_{f,k}^z + N_f \frac{(-1)^n + 1}{2} + \frac{\theta}{\pi} \right]^2$$

$$H_{\text{kin}} = \frac{-i}{2a} \sum_{n=0}^{N-2} \left(\sigma_{1,n}^+ \sigma_{2,n}^z \sigma_{1,n+1}^- - \sigma_{1,n}^- \sigma_{2,n}^z \sigma_{1,n+1}^+ + \sigma_{2,n}^+ \sigma_{1,n+1}^z \sigma_{2,n+1}^- - \sigma_{2,n}^- \sigma_{1,n+1}^z \sigma_{2,n+1}^+ \right)$$

$$H_{\text{mass}} = \frac{m_{\text{lat}}}{2} \sum_{f=1}^{N_f} \sum_{n=0}^{N-1} (-1)^n \sigma_{f,n}^z + \frac{m_{\text{lat}}}{2} N_f \frac{1 - (-1)^N}{2}$$

- We compute eigenstates of this Hamiltonian by the tensor network method

DMRG for excited states

- The k-th excited state $|\Psi_k\rangle$ is the lowest energy eigenstate under the orthogonality condition $\langle\Psi_{k'}|\Psi_k\rangle = 0$ for $k' = 0,1,\cdots,k-1$.

- obtained by DMRG for the Hamiltonian with the projection term

$$H_k = H + W\sum_{k'=0}^{k-1} |\Psi_{k'}\rangle\langle\Psi_{k'}|$$ $W > 0$: weight parameter

—> cost function: $\langle\Psi_k|H|\Psi_k\rangle + W\sum_{k'=0}^{k-1} \left|\langle\Psi_{k'}|\Psi_k\rangle\right|^2$

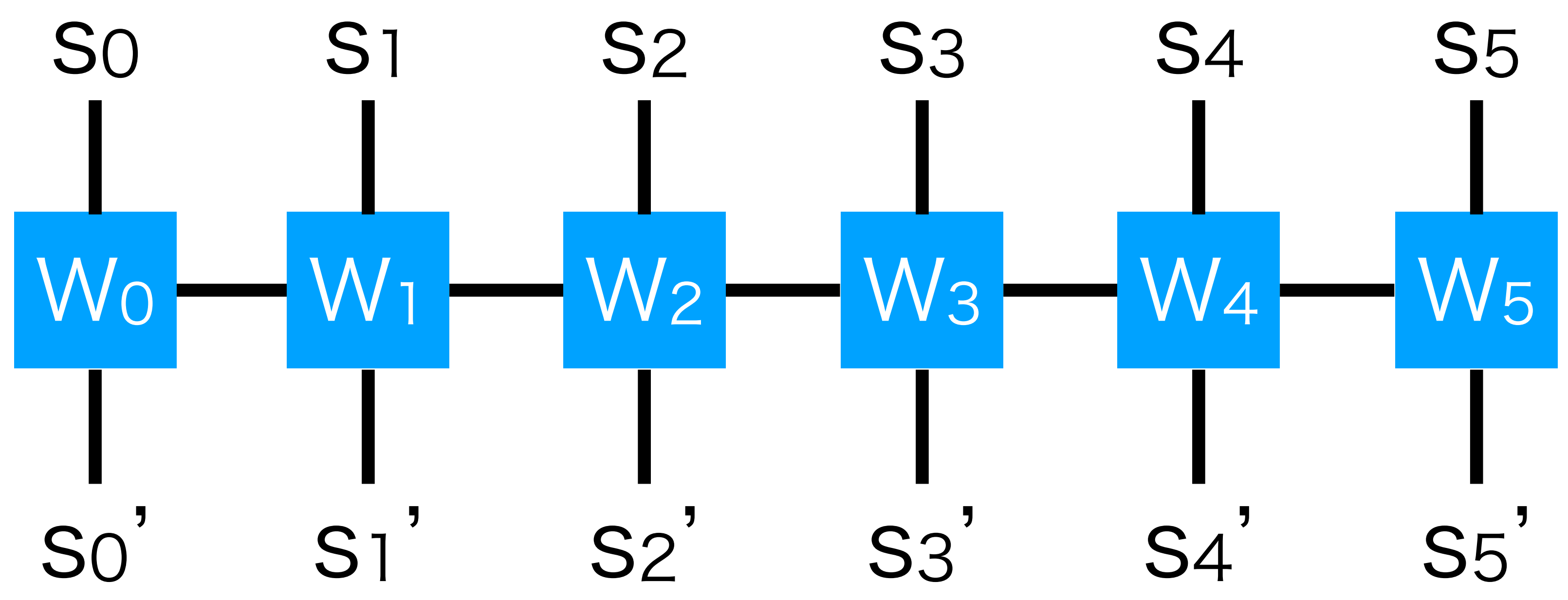
- We can generate the excited state step by step from the bottom.

The C++ library of ITensor is used in this work. [Fishman (2022)]

Measurement

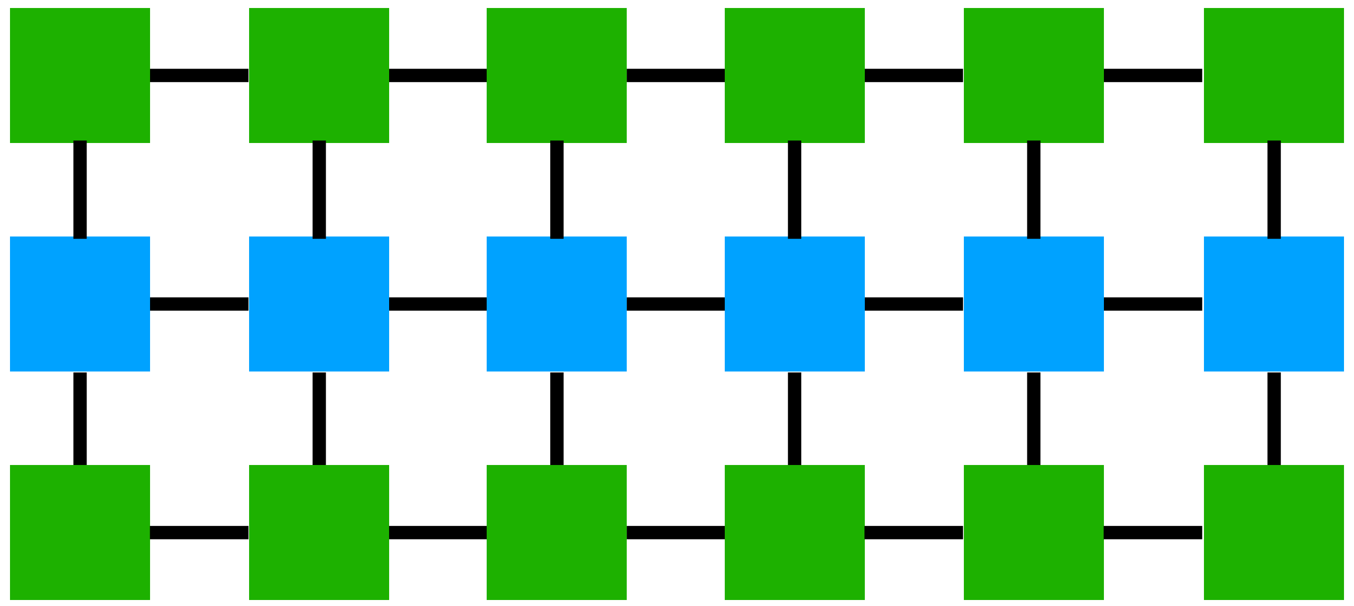
matrix product operator (MPO)

$$\mathcal{O} = \sum_{\{s_i'\}} \sum_{\{s_i\}} \mathrm{Tr}\left[W_0(s_0', s_0) W_1(s_1', s_1) \cdots\right] | s_0' s_1' \cdots \rangle \langle s_0 s_1 \cdots |$$

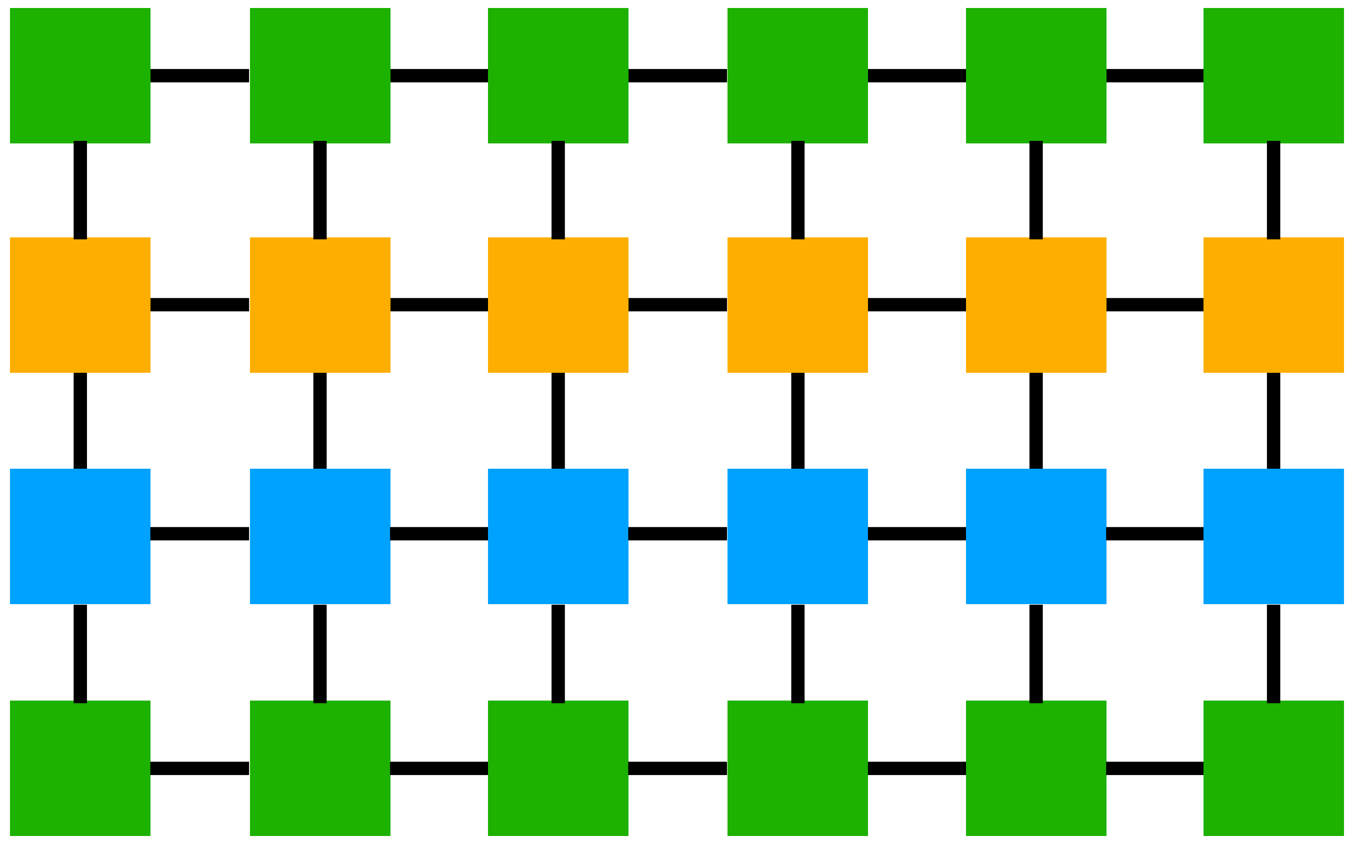


- The expectation value is given by contracting the indices of MPS and MPO.

$\langle \Psi | \mathcal{O} | \Psi \rangle$



$\langle \Psi | \mathcal{O}_1 \mathcal{O}_2 | \Psi \rangle$



Dispersion-relation scheme

In the Hamiltonian formalism, it is straightforward to treat the excited states.

—> We can obtain the dispersion relation $E = \sqrt{K^2 + M^2}$ directly.

- generated the excited states up to the level $k = 23$ by DMRG.

- compute the energy E and the total momentum

$$K = \sum_{f=1}^{N_f} \int dx\, \psi_f^\dagger i\partial_x \psi_f \to \frac{i}{4a} \sum_{f=1}^{N_f} \sum_{n=1}^{N-2} (\chi_{f,n-1}^\dagger \chi_{f,n+1} - \chi_{f,n+1}^\dagger \chi_{f,n-1})$$

  - $[H, K] \neq 0$ due to the absence of translational invariance, but it is still useful as an approximated operator.

Quantum numbers

- isospin operators: conserved charge of SU(2) isospin symmetry

$$J_a = \frac{1}{2}\int dx \sum_{f,f'} \psi_f^\dagger (\sigma^a)_{f,f'} \psi_{f'} \qquad a \in \{x, y, z\}$$

- lattice version

$$J_z = \frac{1}{2}\sum_{n=0}^{N-1}\left(\chi_{1,n}^\dagger \chi_{1,n} - \chi_{2,n}^\dagger \chi_{2,n}\right), \quad J_+ = \sum_{n=0}^{N-1} \chi_{1,n}^\dagger \chi_{2,n} = (J_-)^\dagger, \quad \mathbf{J}^2 = \frac{1}{2}(J_+J_- + J_+J_-) + J_z^2$$

- They exactly commute with the lattice Hamiltonian.

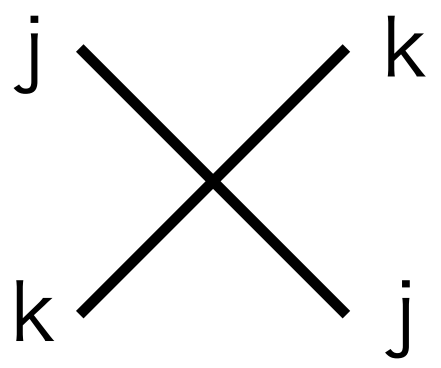
$$[H, J_z] = [H, J_\pm] = [H, \mathbf{J}^2] = 0$$

Quantum numbers

- charge conjugation: exchange particles/anti-particles
  = exchange even/odd sites and flip each spin
  = 1-site translation and σ^x operator

$$C := \prod_{f=1}^{N_f} \left(\prod_{n=0}^{N-1} \sigma^x_{f,n} \right) \left(\prod_{n=0}^{N-2} (\mathrm{SWAP})_{f;N-2-n,N-1-n} \right)$$

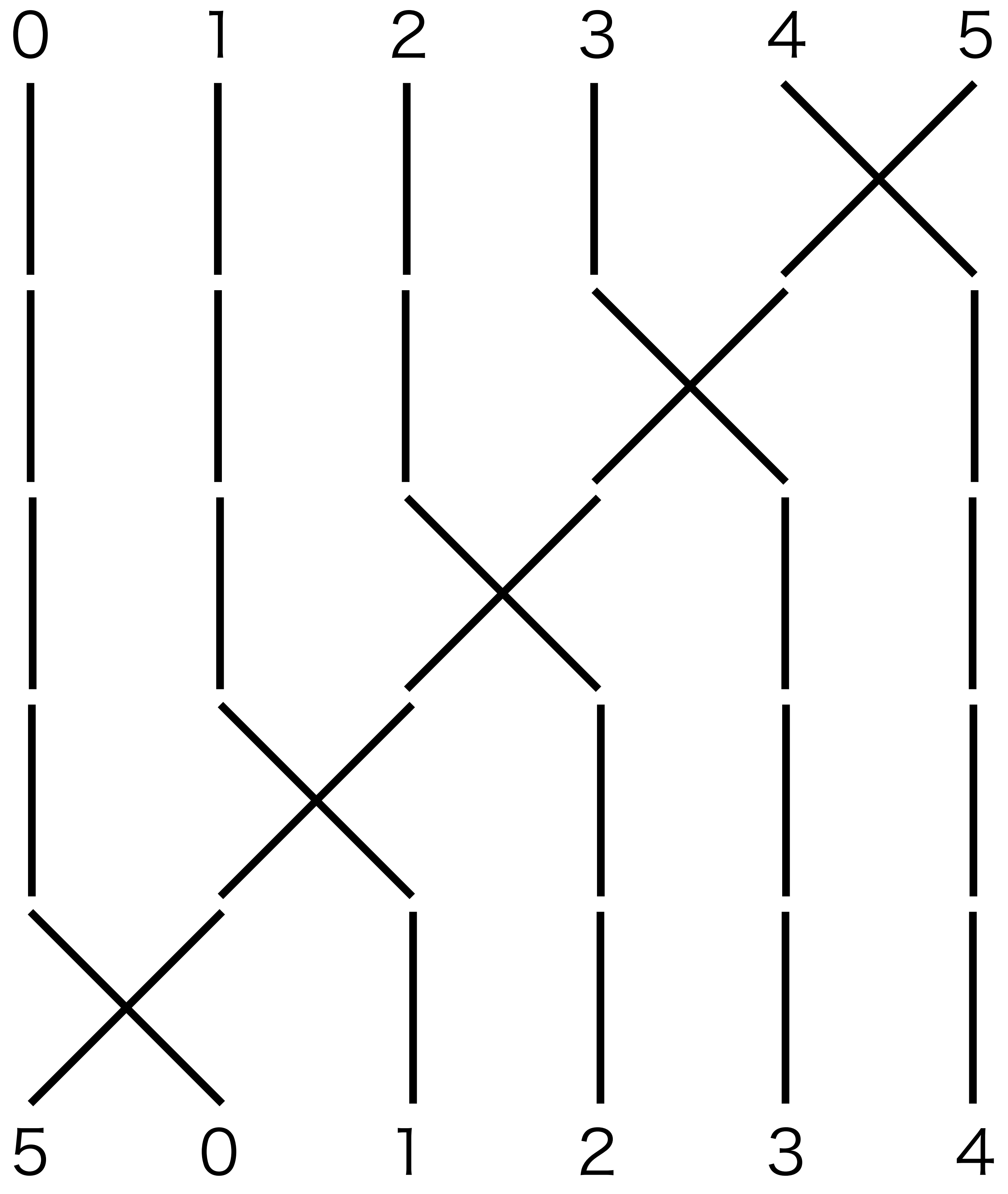
$$(\mathrm{SWAP})_{f;j,k} = \frac{1}{2} \left(\mathbf{1}_{f,j} \mathbf{1}_{f,k} + \sum_a \sigma^a_{f,j} \sigma^a_{f,k} \right) \longrightarrow$$



$[H, C] \neq 0$ due to the boundary

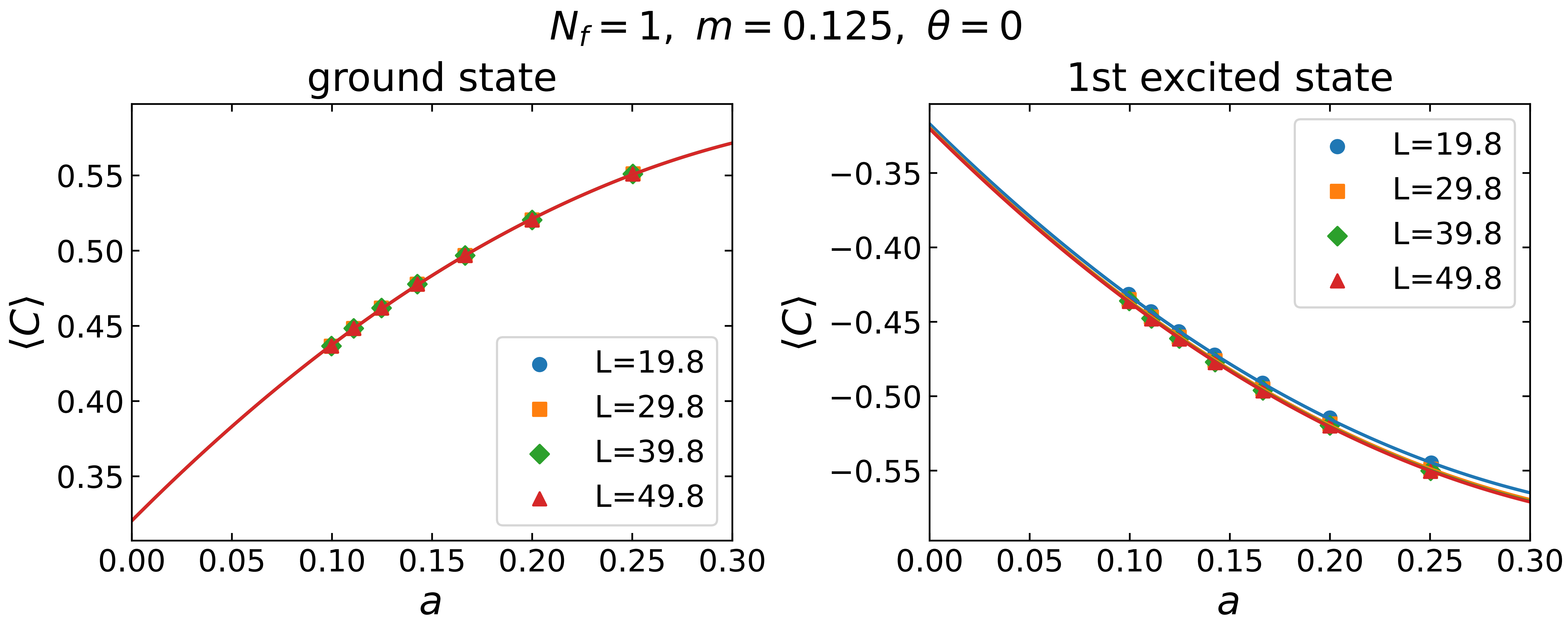
- G-parity: $G = C \exp(i\pi J_y)$ acting on the whole multiplet

1-site translation



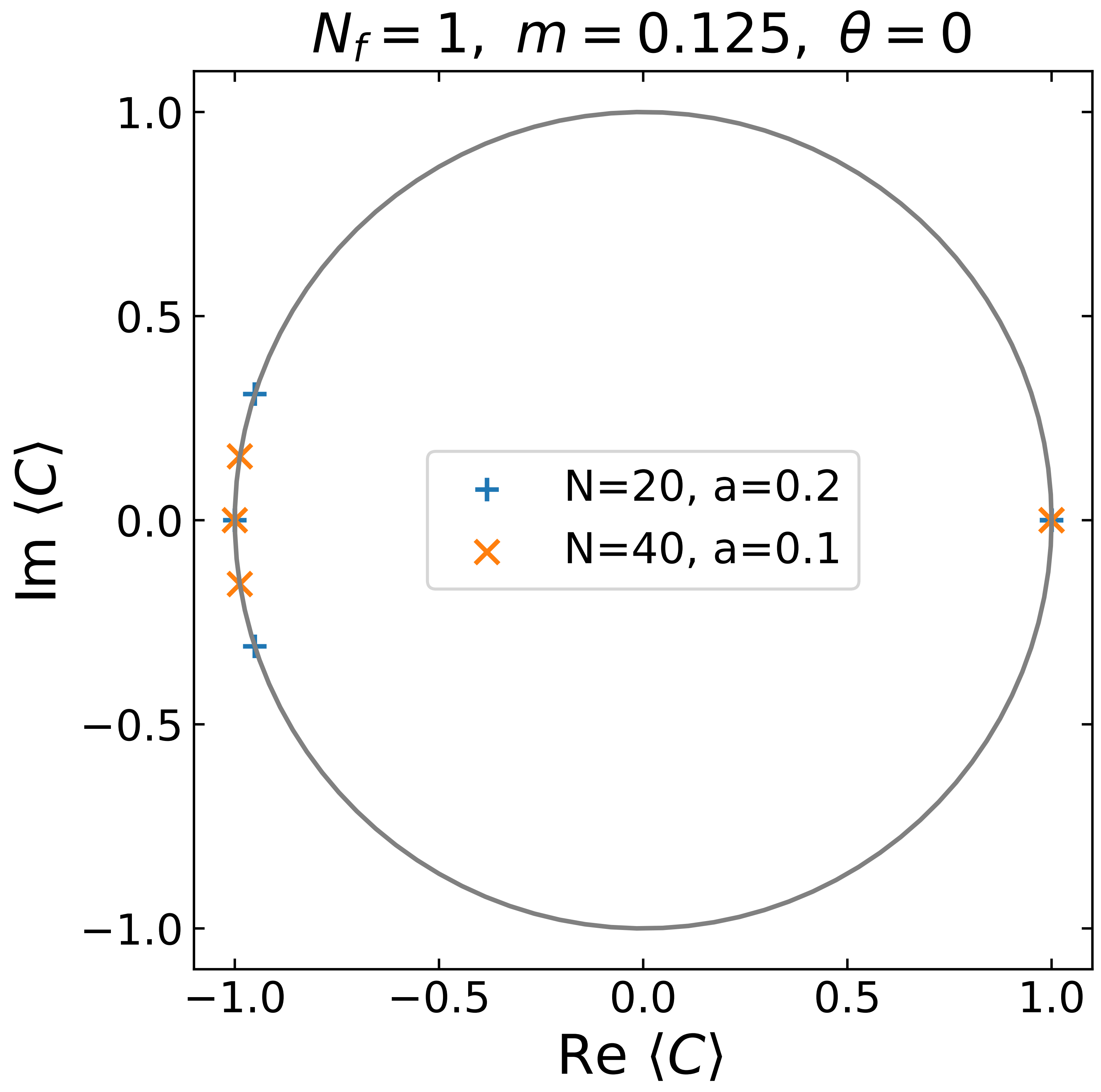
Charge conjugation

continuum limit of $\langle C \rangle$ for various L



1-flavor Schwinger model

boundary effect on $\langle C \rangle$



free fermion with p.b.c

Parity

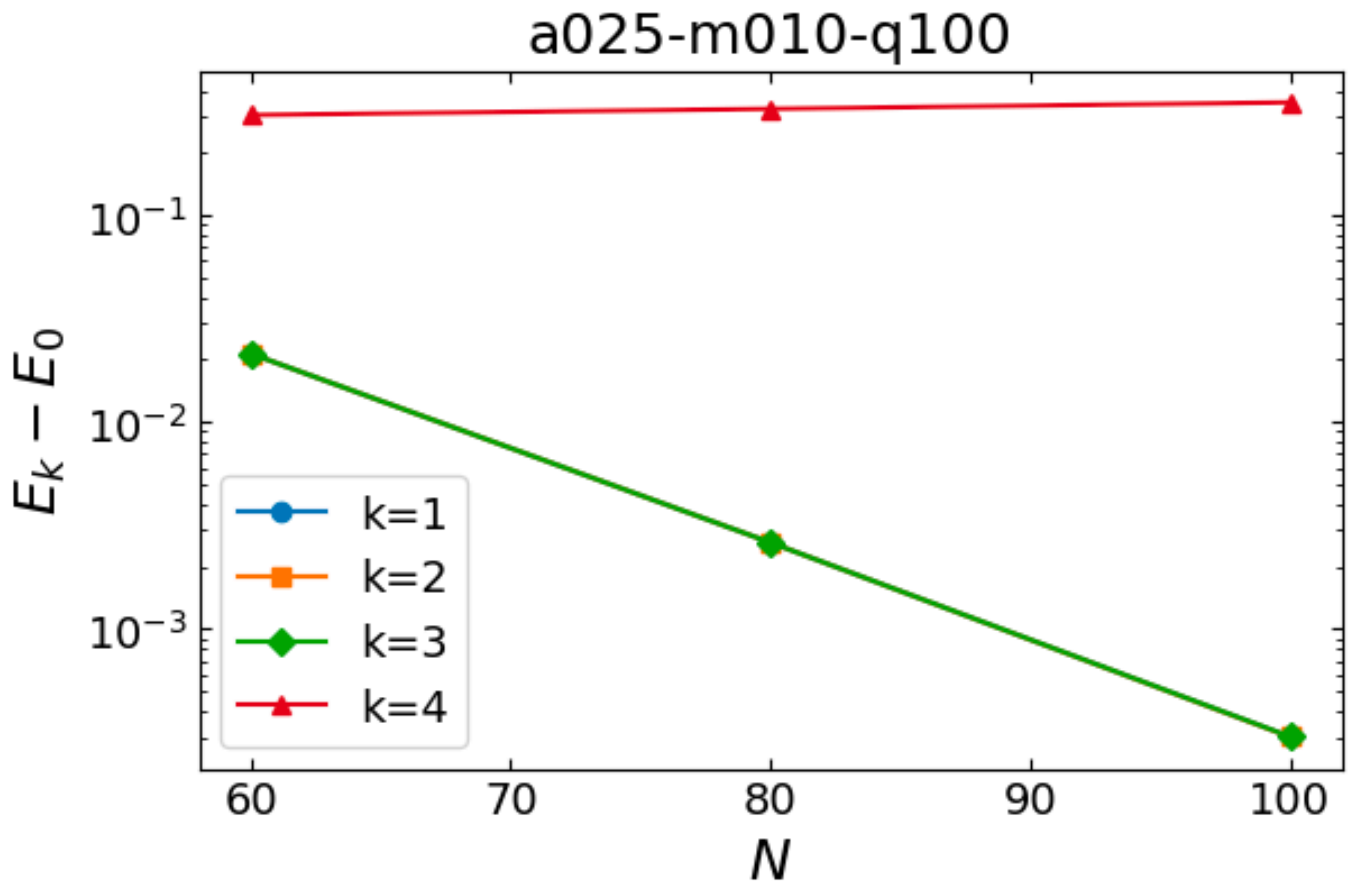
- $$P = \prod_{f=1}^{N_f} \left(\prod_{j=0}^{N/2-1} \sigma^z_{f,2j+1} \right) \left(\prod_{n=0}^{N-2} (\mathrm{SWAP})_{f;N-2-n,N-1-n} \right) \left(\prod_{n=0}^{N/2-1} (\mathrm{SWAP})_{f;n,N-1-n} \right)$$

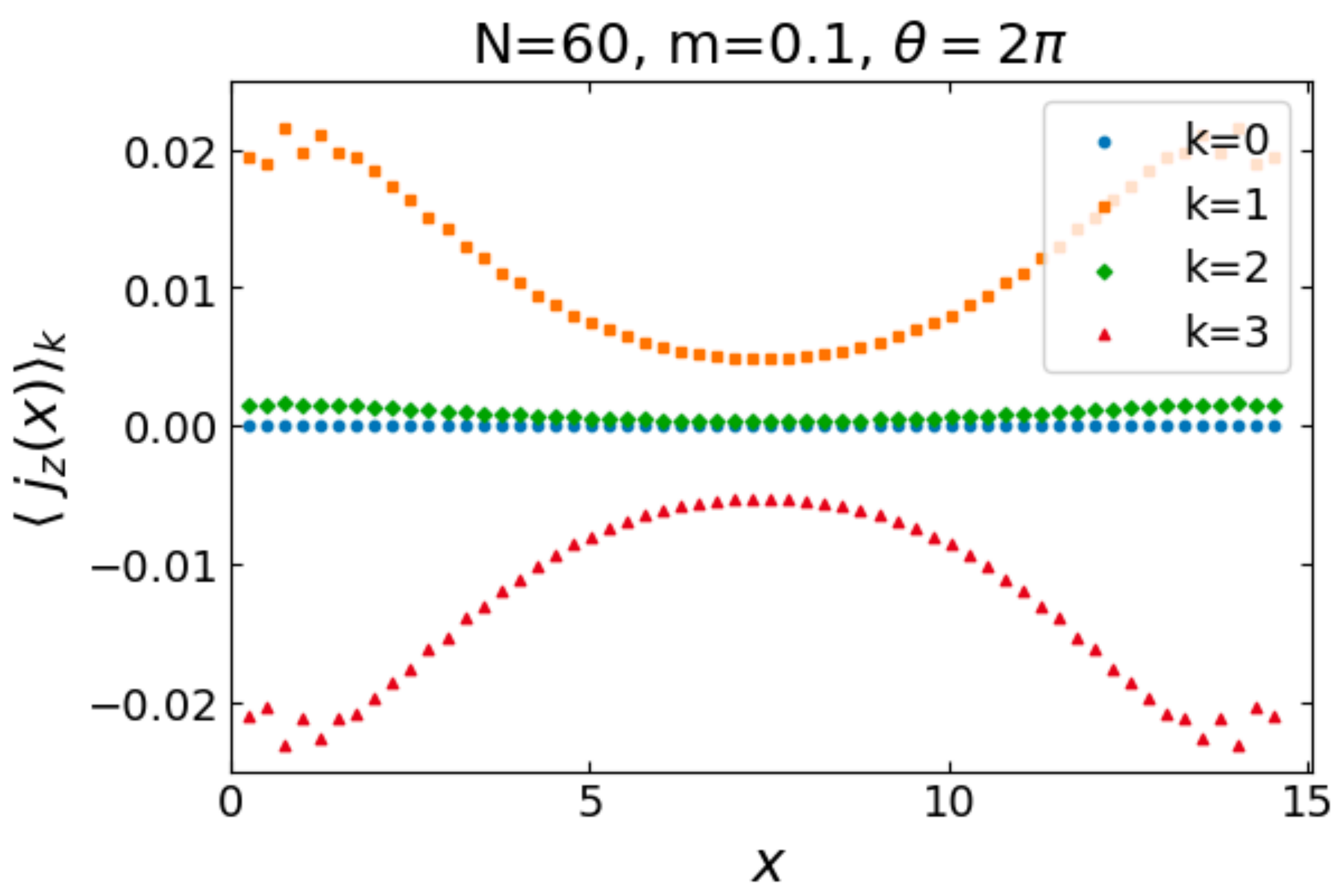
k	$\mathbf{J}^2$	J_z	G	P
0	0.00000003	-0.00000000	0.27984227	3.896×10^{-7}
13	0.00000003	0.00000000	0.27865844	1.273×10^{-7}
14	0.00000003	0.00000000	0.27508176	-2.765×10^{-8}
18	0.00000028	0.00000006	-0.27390909	-6.372×10^{-7}
22	0.00001537	0.00000115	0.26678987	7.990×10^{-8}
23	0.00003607	-0.00000482	-0.27664779	5.715×10^{-7}

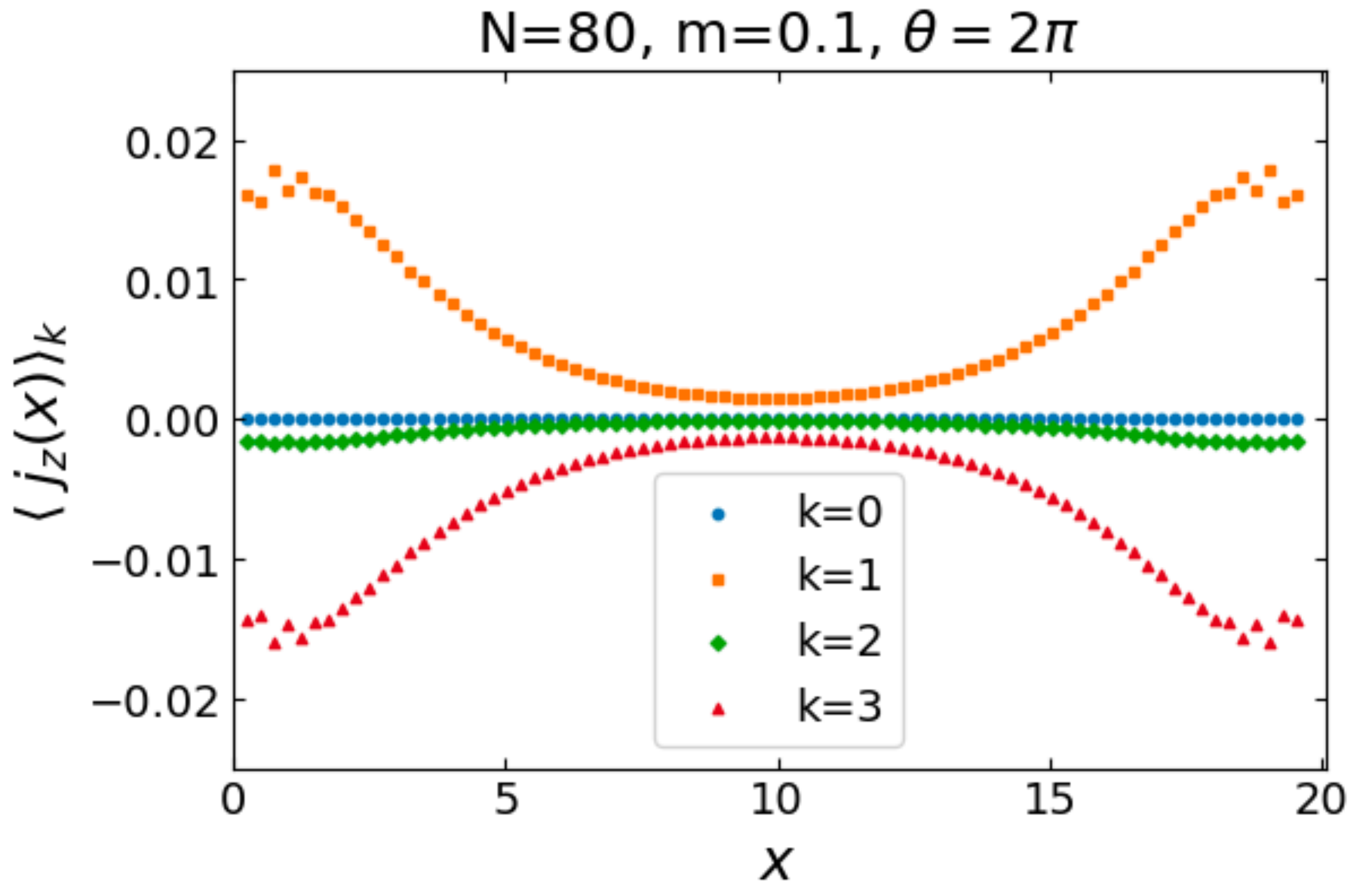
k	$\mathbf{J}^2$	J_z	G	P
1	2.00000004	0.99999997	0.27872443	-6.819×10^{-8}
2	2.00000012	-0.00000000	0.27872416	-6.819×10^{-8}
3	2.00000004	-0.99999996	0.27872443	-6.819×10^{-8}
4	2.00000007	0.99999999	0.27736066	7.850×10^{-8}
5	2.00000006	0.00000000	0.27736104	7.850×10^{-8}
6	2.00000009	-0.99999998	0.27736066	7.850×10^{-8}
7	2.00000010	1.00000000	0.27536687	-8.838×10^{-8}
8	2.00000002	0.00000000	0.27536702	-8.837×10^{-8}
9	2.00000007	-0.99999998	0.27536687	-8.838×10^{-8}
10	2.00000007	0.99999998	0.27356274	9.856×10^{-8}
11	2.00000005	0.00000001	0.27356277	9.856×10^{-8}
12	2.00000007	-0.99999999	0.27356274	9.856×10^{-8}
15	1.99999942	0.99999966	0.27173470	-1.077×10^{-7}
16	2.00000052	0.00000000	0.27173482	-1.077×10^{-7}
17	2.00000015	-1.00000003	0.27173470	-1.077×10^{-7}
19	2.00009067	1.00004377	0.27717104	-3.022×10^{-8}
20	2.00002578	-0.00000004	0.27717020	-3.023×10^{-8}
21	2.00003465	-1.00001622	0.27717104	-3.023×10^{-8}

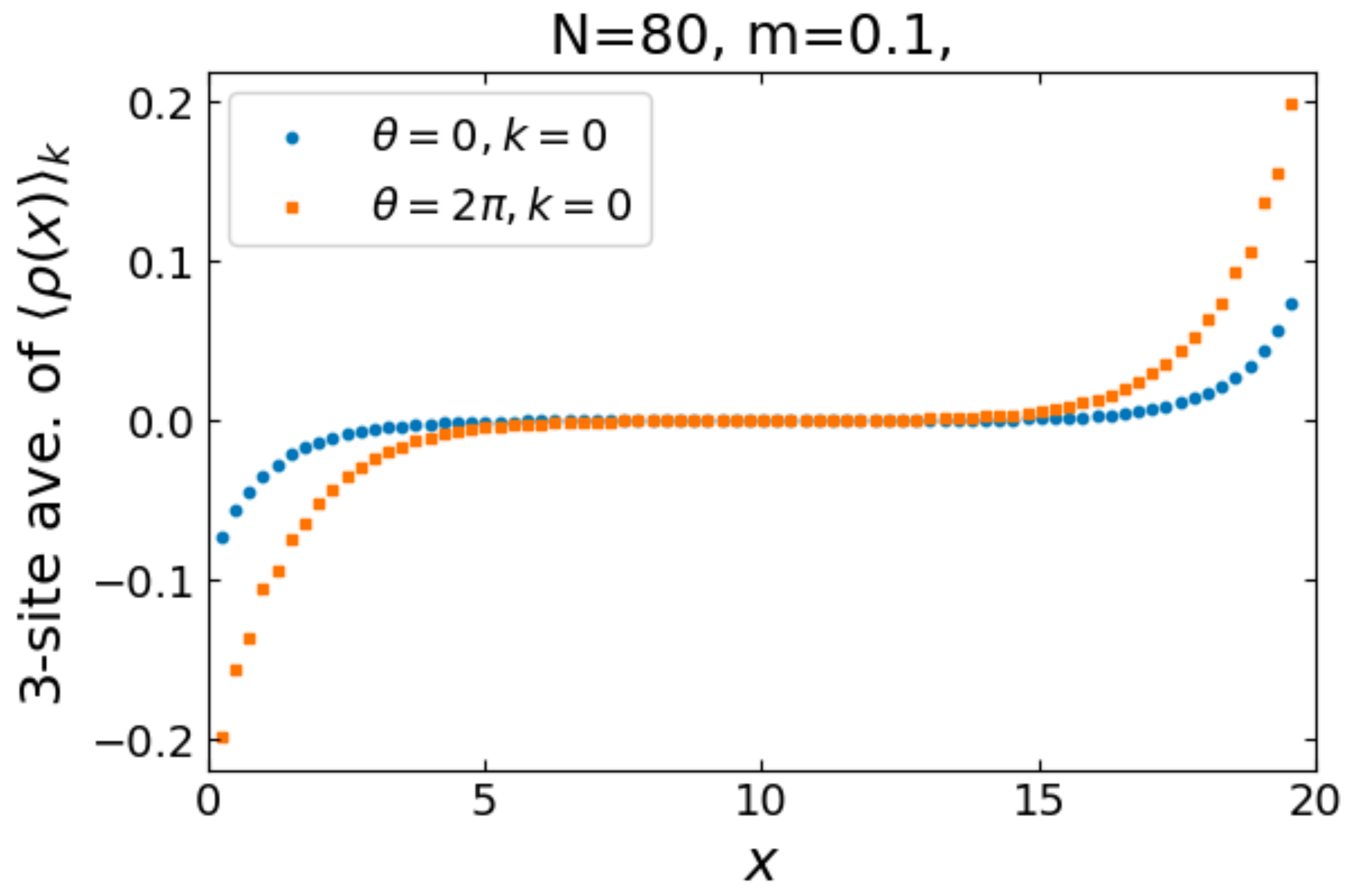
Degeneracy at $\theta = 2\pi$

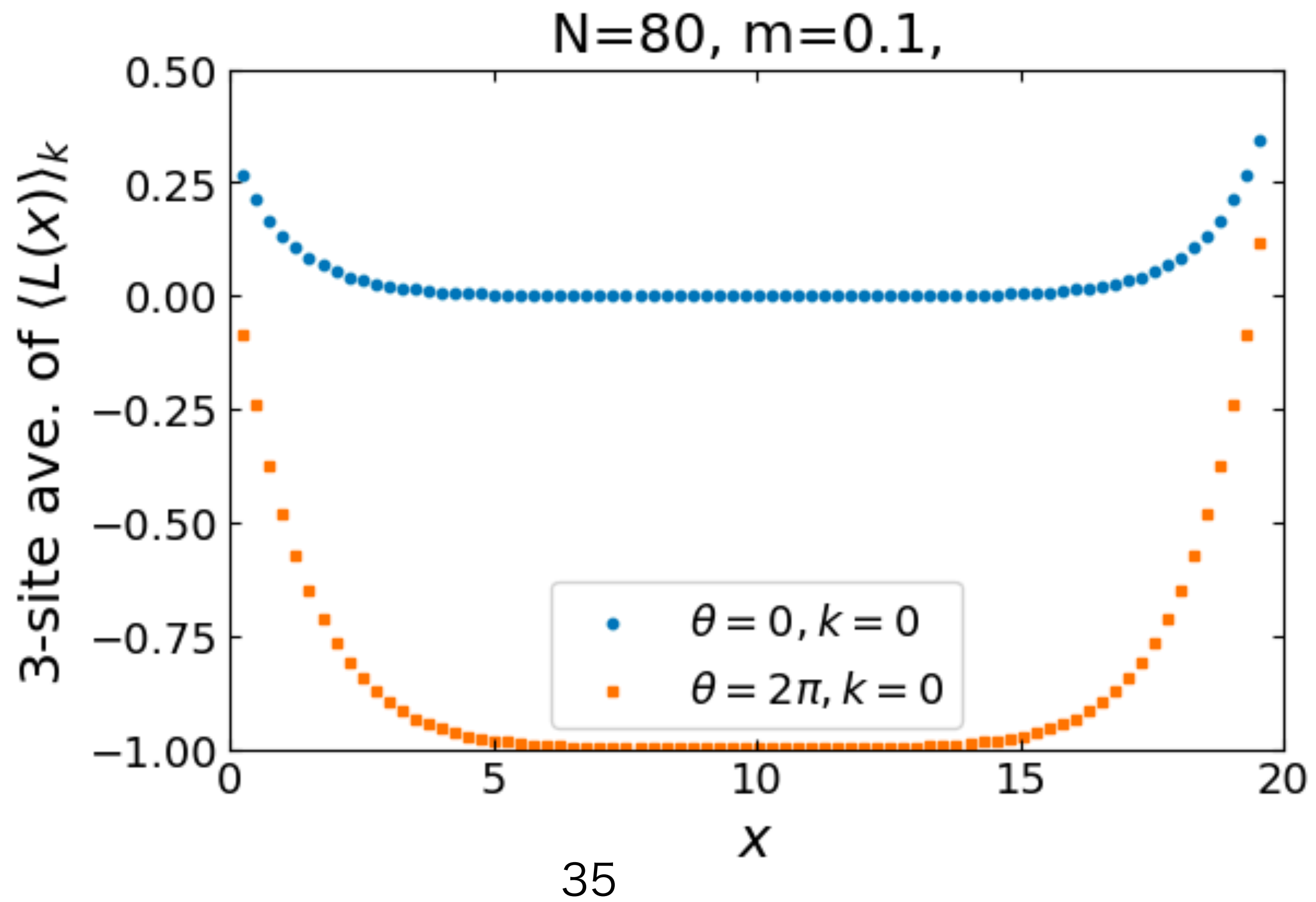
- energy gap $\sim \exp(-AL)$

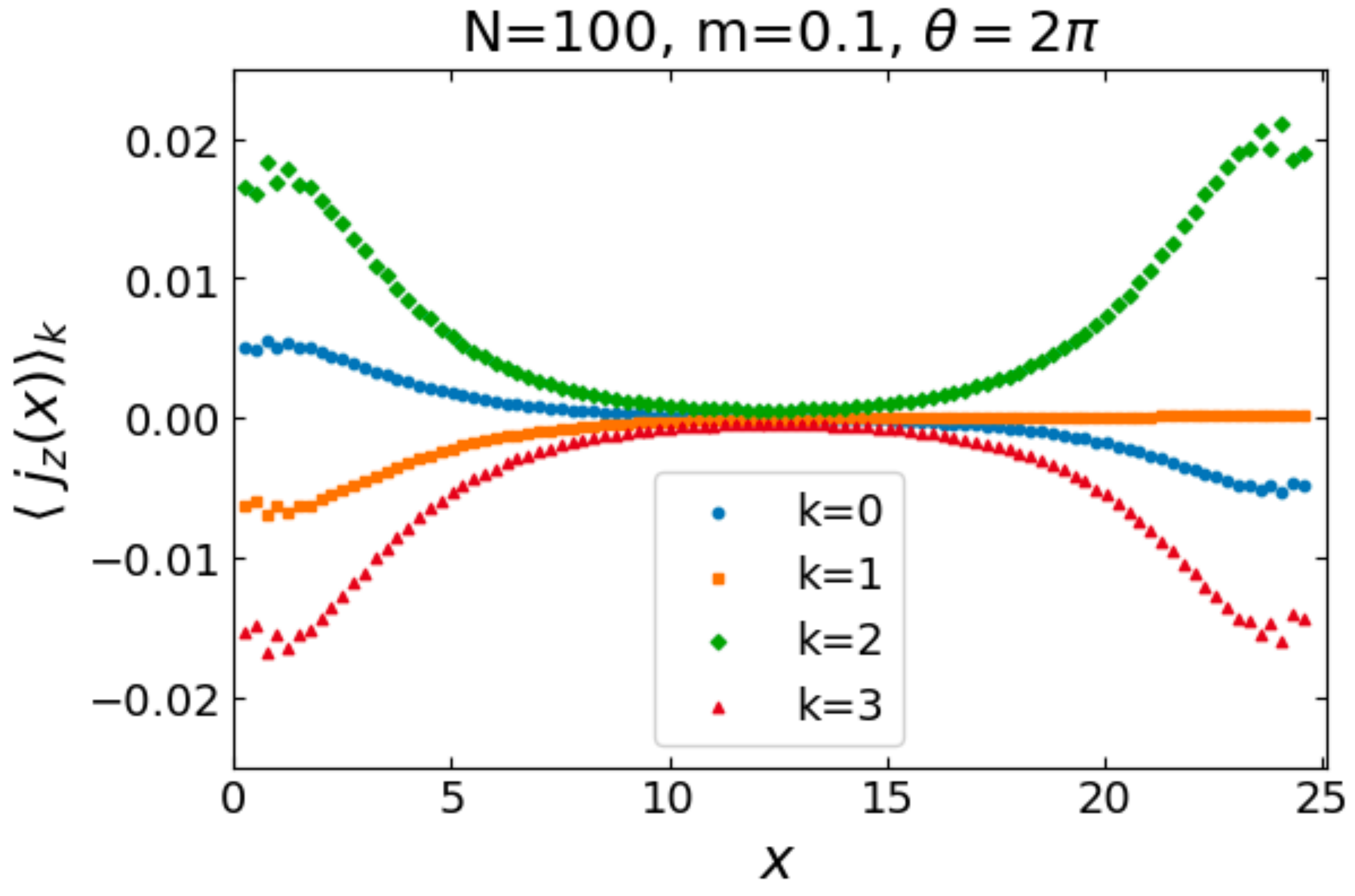






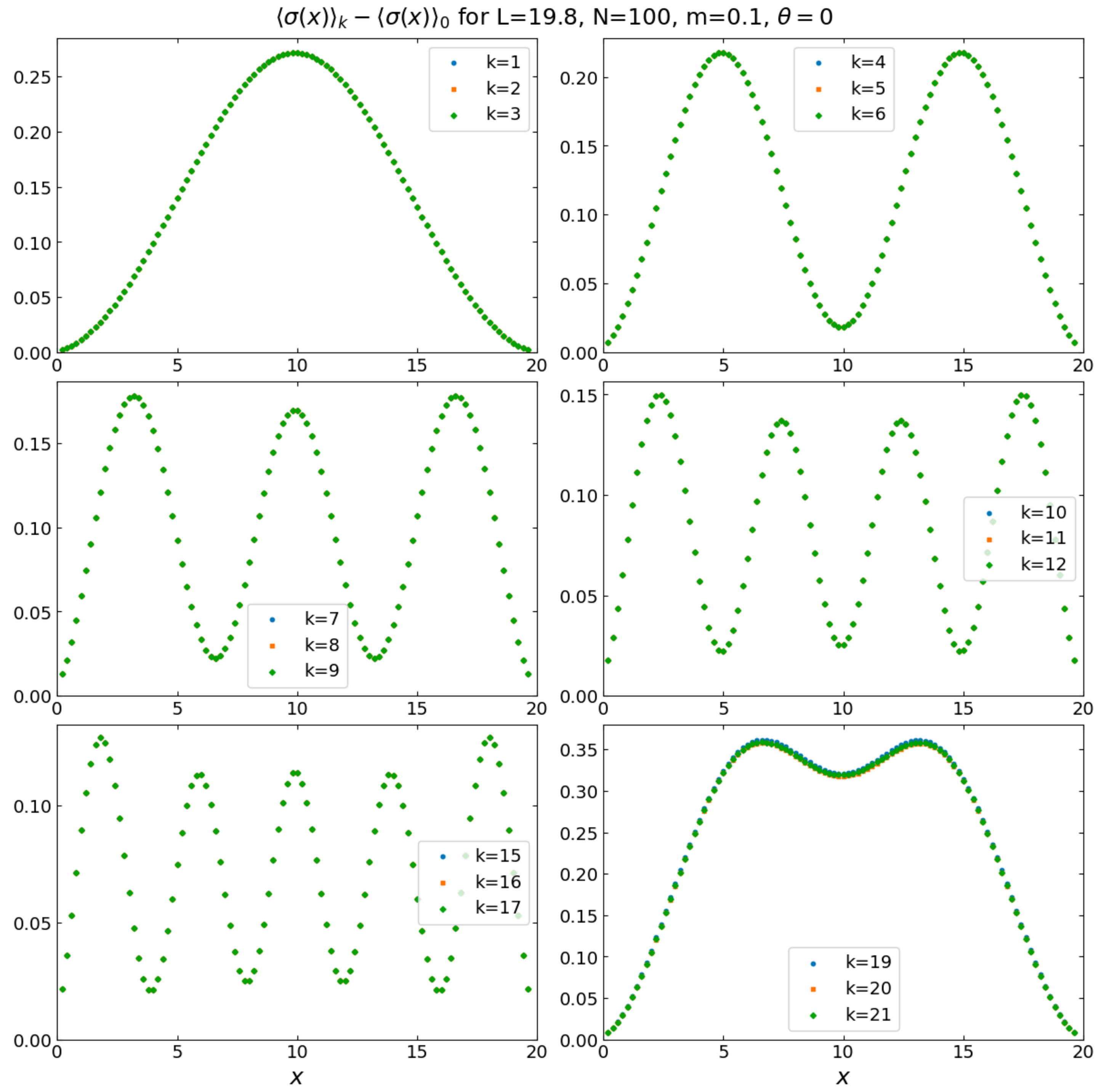


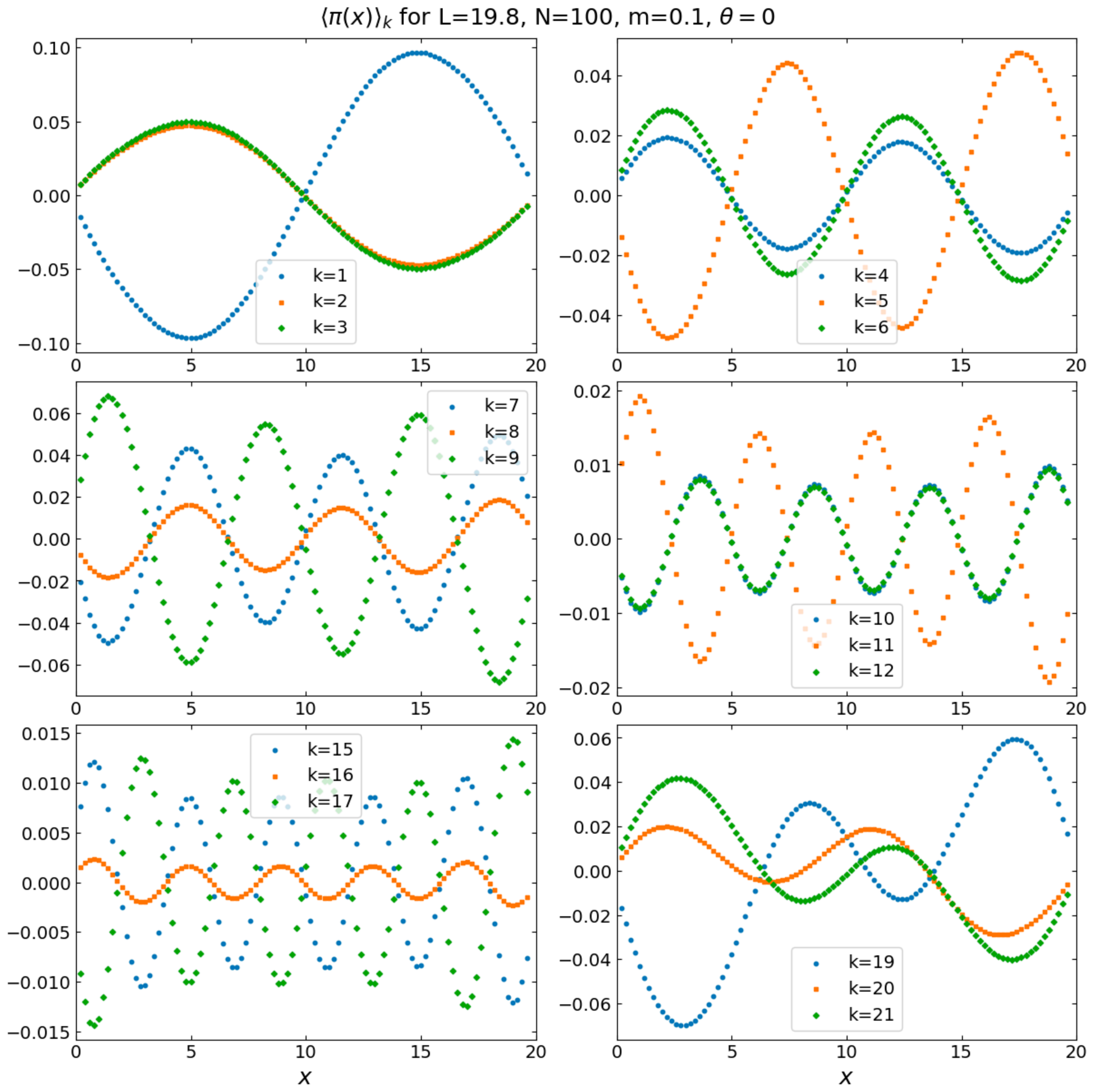


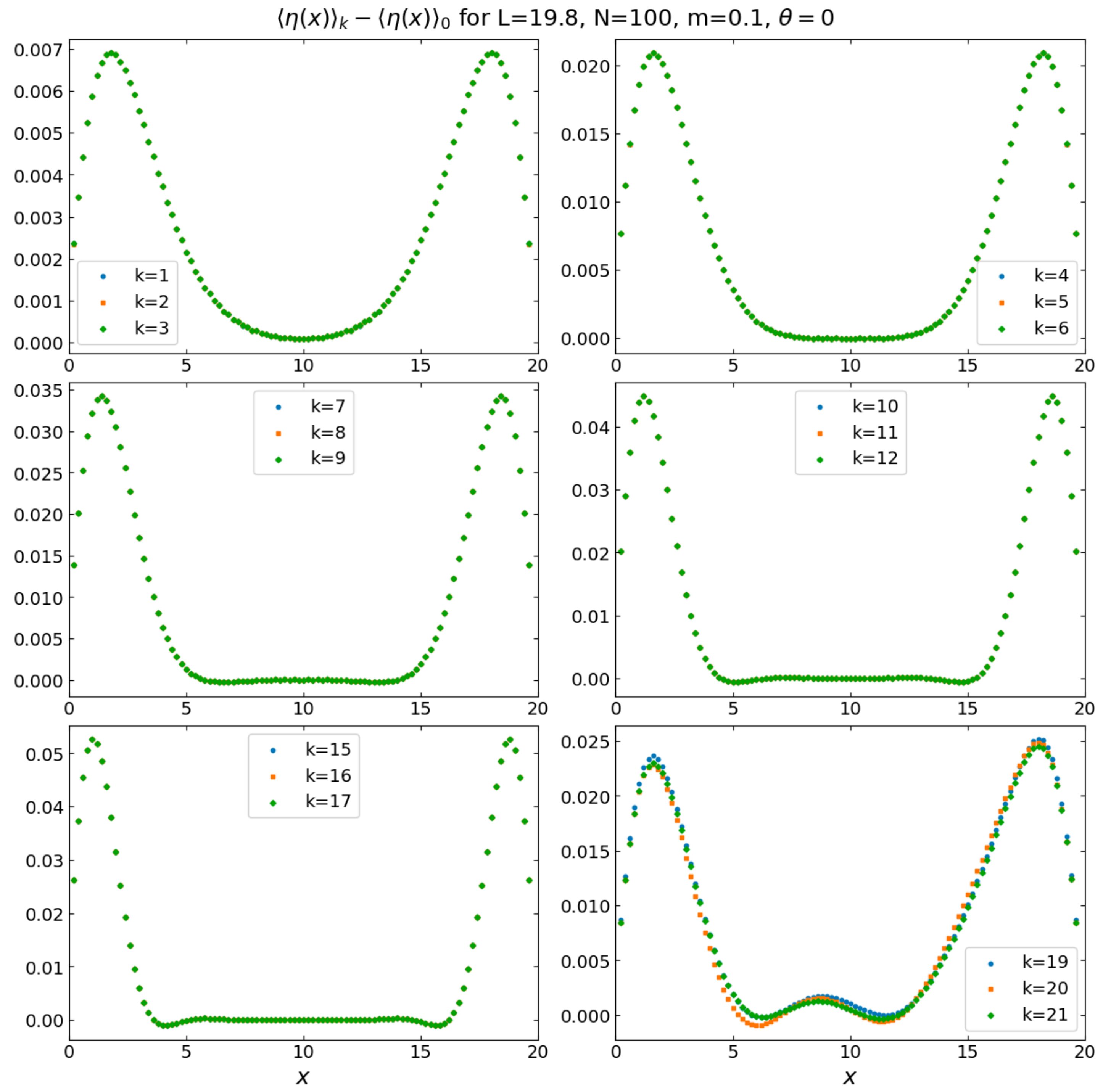


local observable

- local observables for the triplets
- scalar: $\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 = \sigma$ pseudo-scalar: $\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2 = \pi$, $\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2 = \eta$





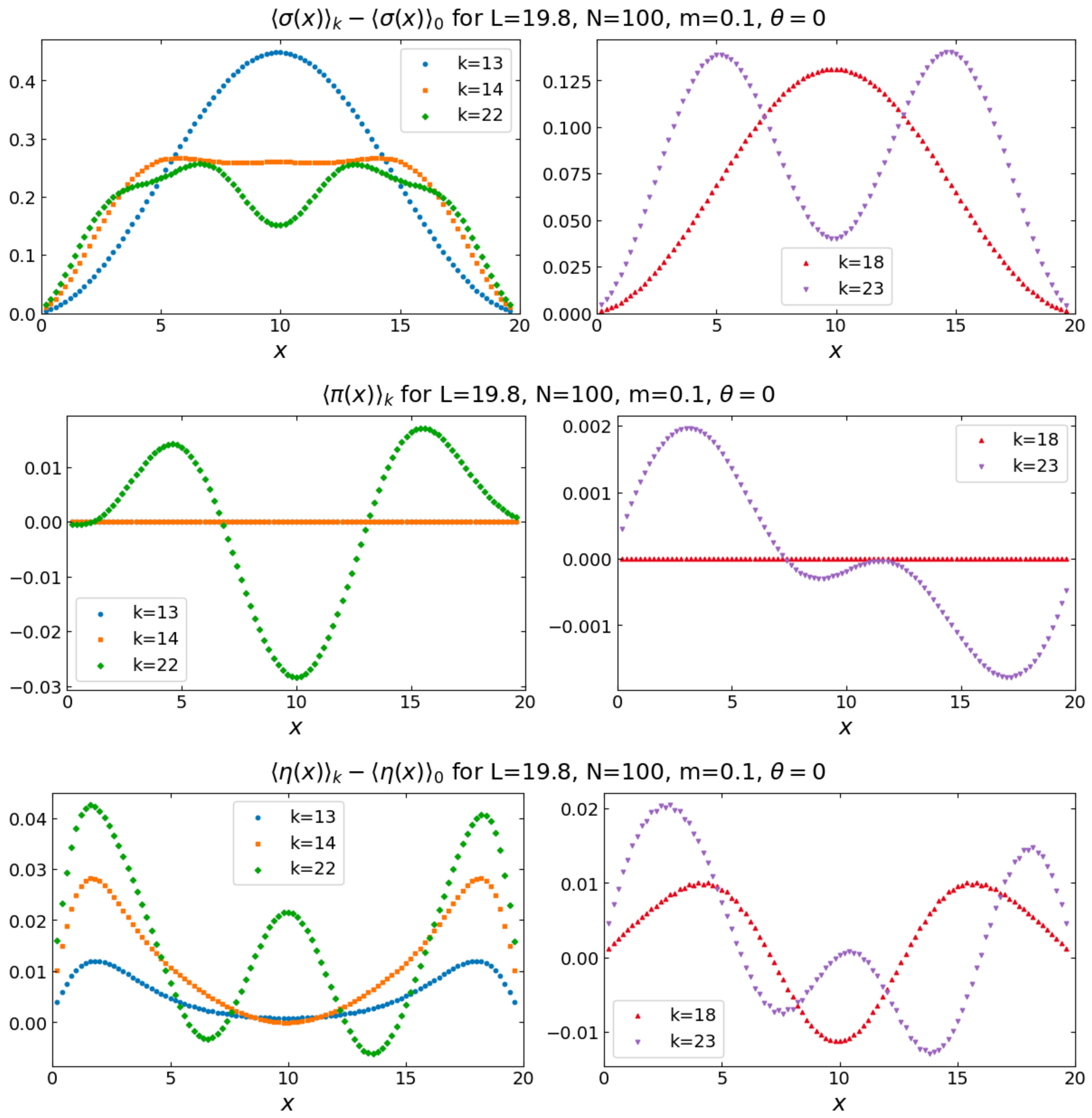


local observable

- local observables for singlets
- scalar: $\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 = \sigma$
- pseudo-scalar:

  $\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2 = \pi,$

  $\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2 = \eta$



local observable

- momentum ~ $K \in \pi\mathbb{Z}/L$
- $\kappa := \frac{L}{\pi}\sqrt{\langle K^2 \rangle_k - \langle K^2 \rangle_0}$

Jz basis

k	J^2	Jz	C	kappa
singlet				
0	0.00000003	-0.00000000	0.27984227	0.00000000
13	0.00000003	0.00000000	0.27865844	1.37084134
14	0.00000003	0.00000000	0.27508176	2.50790468
18	0.00000028	0.00000006	-0.27390909	1.00661137
22	0.00001537	0.00000115	0.26678988	3.48919571
23	0.00003607	-0.00000482	-0.27664779	2.06865557

Jz basis

k	J^2	Jz	C	kappa
triplet				
1	2.00000004	0.99999997	-0.00000003	1.04166142
2	2.00000012	-0.00000000	-0.27872414	1.04165391
3	2.00000004	-0.99999996	0.00000054	1.04166235
4	2.00000007	0.99999999	-0.00000040	2.03154499
5	2.00000006	0.00000000	-0.27736122	2.03154338
6	2.00000009	-0.99999998	-0.00000020	2.03154461
7	2.00000010	1.00000000	-0.00000017	2.98260402
8	2.00000002	0.00000000	-0.27536703	2.98260011
9	2.00000007	-0.99999998	-0.00000012	2.98259227
10	2.00000007	0.99999998	-0.00000019	3.91389249
11	2.00000005	0.00000001	-0.27356279	3.91388812
12	2.00000007	-0.99999999	0.00000015	3.91388523
15	1.99999942	0.99999966	-0.00000043	4.83458157
16	2.00000052	0.00000000	-0.27173463	4.83458062
17	2.00000015	-1.00000003	0.00000000	4.83457817
19	2.00009067	1.00004377	-0.00000116	1.82040459
20	2.00002578	-0.00000004	-0.27716678	1.82004384
21	2.00003465	-1.00001622	-0.00000059	1.82006827