Advancing real-time Yang-Mills: towards real-time observables from first principles

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+ preliminary results

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There is no first principle description of the quark-gluon plasma (QGP).

Stages of the QGP evolution are described by different models with limited applicability (classical statistical approximation, kinetic theory, holography, ...).

Matching of these models requires QCD transport coefficients as input from theory (viscosity coefficients for hydrodynamics, $\hat{q}$, $\kappa$ for jets and heavy quarks, ...).

Direct computations of such QCD real-time observables are difficult due to the complex action problem (next slide).

Potential impact on other fields: QCD at finite chemical potential, cold quantum gases, compact stars, many-body physics, ... .
Observables in the real-time formalism

- **Path integral expression** for expectation values:

\[
\langle \mathcal{O}[A] \rangle = \frac{1}{Z} \int \mathcal{D}A_E e^{-S_E[A_E]} \int \mathcal{D}A_+ \mathcal{D}A_- e^{iS[A_+, A_-]} \mathcal{O}[A_+, A_-, A_E]
\]

- **Yang-Mills action:**

\[
S_{YM} = -\frac{1}{4} \int_{\mathcal{C}_+, \mathcal{C}_E} d^4 x F_a^{\mu\nu} F_a^{\mu\nu}
\]

- **Correlation functions of** \(T^{\mu\nu}:

  - speed of sound
  - bulk- and shear-viscosity
  - spectral functions

- **Real-time path** \(\mathcal{C}_\pm\):

- **Thermal path** \(\mathcal{C}_E\):

  - **Schwinger-Keldysh contour**

- **Thermal equilibrium:** \(A(t) = A(-i\beta)\)
Real-time path introduces a complex weight \( e^{iS[A_+, A_-]} \)
\[ \rightarrow \text{results in a hard sign problem!} \]

- Various methods tackle the sign problem:
  Reweighting, contour deformation, analytic continuation, Taylor expansion, Lefschetz thimbles, spectral reconstruction, complex Langevin (CL) method, ...

- Ideas behind the CL method:
  - adding an additional d.o.f., the Langevin time \( \theta \): \( A^a_\mu(x) \sim A^a_\mu(\theta, x) \)
  - complexification of the gauge fields: \( \mathfrak{su}(N_c) \sim \mathfrak{sl}(N_c, \mathbb{C}) \) (non-compact!)
  - Gauge fields are treated as a stochastic process w.r.t \( \theta \)
Complex Langevin in a nutshell

- CL equation describes the **stochastic process in** $\theta$

\[
\partial_\theta A_\mu^a(\theta, x) = i \frac{\delta S_{YM}}{\delta A_\mu^a(t, x)} + \eta_\mu^a(\theta, x)
\]

- Gaussian distributed noise term

\[
\langle \eta_\mu^a(\theta, t, x) \rangle = 0,
\langle \eta_\mu^a(\theta, t, x) \eta_\nu^b(\theta', t', x') \rangle = 2\delta(\theta - \theta')\delta(t - t')\delta^{(d-1)}(x - x')\delta^{ab}\delta_{\mu\nu}
\]

- **Correspondence with Fokker-Planck equation** shows that $A_\mu^a(\theta \to \infty, x)$ is described by $\rho[A] = \mathcal{N} \exp[-S_{YM}[A]]$.

**CL bypasses the sign problem by sampling at late $\theta$**

\[
\langle \mathcal{O}[A] \rangle = \int \mathcal{D}A \rho[A] \mathcal{O}[A] \approx \lim_{\theta_0 \to \infty} \frac{1}{T} \int_{\theta_0}^{\theta_0+T} d\theta \mathcal{O}[A(\theta)]
\]
Complex Langevin on the lattice

- Link variables and plaquette variables

\[ U_{x,\mu} \simeq \exp \left[ i g a_\mu A_\mu (x + \hat{\mu}/2) \right] \in \text{SU}(N_c) \rightarrow \text{SL}(N_c, \mathbb{C}), \]

\[ U_{x,\mu\nu} = U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu} U_{x,\nu}^{-1} \]

- Wilson plaquette action: \( S_W[U] = \frac{1}{2N_c} \sum_{x,\mu \neq \nu} \beta_{\mu\nu} \text{Tr} [U_{x,\mu\nu} - 1] \)

- Coupling constants:

\[ \beta_0 = -\frac{2N_c}{g^2} \frac{a_s}{a_t,k}, \quad \beta_{ij} = +\frac{2N_c}{g^2} \frac{\bar{a}_{t,k}}{a_s} \]

- Averaged lattice spacing (time reversability):

\[ \bar{a}_{t,k} = \frac{a_{t,k} + a_{t,k+1}}{2} \]

**Update step (Euler-Maruyama) for the link variables**

\[ U_{x,\mu}(\theta + \epsilon) = \exp \left[ it^a \left( i \Gamma_\mu \epsilon D^a_{x,\mu} S_W + \sqrt{\Gamma_\mu \epsilon} \eta^a_{x,\mu}(\theta) \right) \right] U_{x,\mu}(\theta) \]

→ Field independent kernel function \( \Gamma_\mu \) leaves stationary solution intact but can be utilised to improve convergence and stability!
The fall and rise of complex Langevin

▷ CL suffers from two types of instabilities [1]:
  - Runaways: Straying too far into the “bulk” of complex manifold leads to a numerical blowup of the solution.
  - Wrong convergence: Breaking of assumption of the method during evolution can lead to distorted results.

▷ Several methods have been introduced to mitigate these issues:
  adaptive step size [2], gauge cooling [3], dynamical stabilization [4], kernels [5], ...

Anisotropic kernel

▷ We introduce an anisotropic kernel to alleviate instabilities

\[ \Gamma_t = |a_t|^2/a_s^2, \Gamma_s = 1 \]

▷ Motivation: Based on a careful rederivation of the CL equation on complex time contours [6].
Our simulation approach

- Discrete path integral needs to be regularized
  - we tilt the SK-contour
  - instabilities more severe for smaller tilt angles

- Our kernel alleviates instabilities for increasing anisotropies!

Instabilities for smaller tilt angles can be counteracted by larger lattice anisotropies $a_s/|a_t|$!
Average spatial plaquette

- Spatial average plaquette:
  \[ \mathcal{O}[U] = \frac{1}{6N_c N_x} \sum_{x, i \neq j} \text{Tr}[U_{x,ij}] \]

- \( t \)-indep. in thermal equilibrium \( \rightarrow \) comparison to Euclidean results

- Reproduction of studies on isosceles contours \([7]\)

\[ \tan(\alpha) = 1.0: \quad \tan(\alpha) = 0.5: \]

- Not-stabilised simulation:
  Wrong convergence result after short time.

- Stabilised simulation:
  GC and the anisotropic kernel reproduces Euclidean results.

\( \implies \) Existing methods are not enough to stabilise the simulations, however the introduction of our kernel did!
Correlations of the magnetic energy density

- Calc. of corr. fcts. on the real-time path requires extrapolation $\alpha \to 0$
- We are interested in the magnetic energy density:

$$B^2(t, x) = \frac{1}{4} \sum_{i<j} F_{ij}^2(t, x)$$

- Clover leafs for the calc. of $F_{\mu\nu}$

As a first study, we calculate the unequal time correlation function of $B^2(t, x)$ (summed over all $x$):

$$C(t, t') = \sum_x C(t, t', x) = \sum_x \langle B^2(t, x) B^2(t', x) \rangle \propto \sum_k \tilde{C}(t, t', k).$$

→ improves statistics, shorter runtimes
Limit towards the SK-contour $\alpha \to 0$

- Decreasing tilt angle leads to “converging” time-ordered Feynman propagator $D^F$.
- Justification for extrapolation $\alpha \to 0$.

\[ \Delta t = t - t' \]

\[ \text{Time-ordered Feynman propagator} \]
\[ D^F(\Delta t = t_+ - t'_+) = C(t_+, t'_+) \]

\[ \text{Re} [D^F] \quad \text{tan}(\alpha) = 0.33, 0.17, 0.08, 0.04, 0.02, 0.01 \]

\[ \text{Im} [D^F] \quad \text{tan}(\alpha) = 0.33, 0.17, 0.08, 0.04, 0.02, 0.01 \]
Extrapolation to the SK contour

- Real-time correlation functions for 3+1D Yang-Mills theory
- Different quadrants are analytically related

\[ D^F(t, t') = \theta(t - t') D^>(t, t') + \theta(t' - t) D^<(t, t') \]

\[ \text{LHS} = \theta(t - t') D^>(t, t') + \theta(t' - t) D^<(t, t') \]

\[ \text{RHS} \]
Feynman propagators and Wightman functions

Relation for Feynman propagator and Wightman function satisfied in the limit $\alpha \to 0$

\[ \Delta t = t - t' \]

- Real part of LHS, $\alpha \to 0$
- Real part of RHS, $\alpha \to 0$

- Imaginary part of LHS, $\alpha \to 0$
- Imaginary part of RHS, $\alpha \to 0$
- Relation for Feynman propagator and Wightman function *satisfied* in the limit $\alpha \rightarrow 0$

- *Not satisfied* for non-vanishing tilt angles

  $\rightarrow$ non-trivial property fulfilled; confirms our simulation approach!
Conclusion and outlook

**Conclusion**

▷ First direct calculation of unequal time correlation functions for 3+1D lattice Yang-Mills theory
  – Correlations of magnetic energy density
  – Calculations done in the weak coupling regime (so far)

▷ New anisotropic kernel enables complex Langevin simulations for real times larger than the inverse temperature

▷ Extrapolation to Schwinger-Keldysh contour successful
  – Propagators are consistent \(\Rightarrow\) confirms our simulation approach

**Outlook**

▷ Repeat calculations at moderate coupling
  – higher anisotropies needed
  – development of further stabilization techniques

▷ Renormalization of lattice spacings for real-time contours (see our poster)

▷ Computation of shear and bulk viscosities, spectral functions
Thank you for your attention!


Backup slides
Dyson-Schwinger equations as probe for stability

- Self-consistency check of link configuration

\[
\frac{2(N_c^2-1)}{N_c} \langle \text{ReTr}(U_{x,ij}) \rangle = \frac{i}{2N_c} \sum_{|\rho| \neq i} \beta_{i\rho} \langle \text{ReTr} [(U_{x,i\rho} - U_{x,i\rho}^{-1})U_{x,ij}] \rangle
\]

- RHS is very sensitive $\rightarrow$ probe for stability
Localized histogram of the drift terms

- Localised histograms of $\text{Im} [iD_{SW}]$ indicate correct convergence
- No skirts of histograms using our kernel with sufficiently large $N_t$

- Gauge cooling helps, but skirts are still present
- Increasing $N_t$ without our kernel does not improve the stability
Euclidean correlation function

- Euclidean correlation function $C(\tau, \tau')$ along thermal path
- Coinciding with correlation function from Euclidean simulation
- Independence of tilt angle $\alpha$

\[ \tan(\alpha) = \{0.01, 0.02, 0.08, 0.17, 0.33\} \]

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