The International Symposium on Lattice Field Theory 2023

# Advancing real-time Yang-Mills: towards real-time observables from first principles

*Based on JHEP 06 (2023) 011, [2212.08602],*
+ preliminary results

Paul Hotzy
in collaboration with Kirill Boguslavski and David I. Müller

Institute for Theoretical Physics, TU Wien

August 03, 2023

$\int dk\, \Pi$
Doktoratskolleg
Particles and Interactions

TU WIEN
TECHNISCHE UNIVERSITÄT WIEN
Vienna | Austria

FWF

# Motivation of this endeavor

- There is **no first principle description** of the quark-gluon plasma (QGP).
- Stages of the QGP evolution are described by different models with **limited applicability** (classical statistical approximation, kinetic theory, holography, . . . ).
- Matching of these models requires **QCD transport coefficients** as input from theory (viscosity coefficients for hydrodynamics, $\hat{q}$, $\kappa$ for jets and heavy quarks, . . . ).
- Direct computations of such QCD real-time observables are difficult due to the **complex action problem** (next slide).
- Potential impact on other fields: QCD at finite chemical potential, cold quantum gases, compact stars, many-body physics, . . . .

# Observables in the real-time formalism

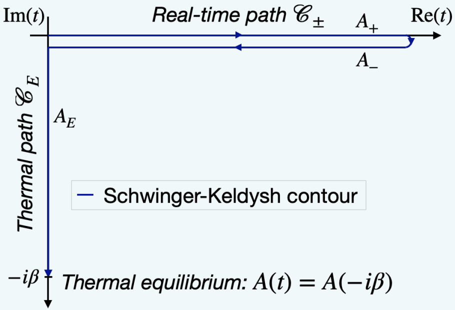
▷ **Path integral expression** for expectation values:

$$\langle \mathscr{O}[A] \rangle = \frac{1}{Z} \int \mathcal{D}A_E \, e^{-S_E[A_E]} \int \mathcal{D}A_+ \mathcal{D}A_- \, \mathbf{e^{iS[A_+, A_-]}} \, \mathcal{O}[A_+, A_-, A_E]$$

▷ **Yang-Mills action:**

$S_{\mathrm{YM}} = -\frac{1}{4} \int_{\mathscr{C}_\pm, \mathscr{C}_E} d^4x F_a^{\mu\nu} F_{\mu\nu}^a$

▷ **Correlation functions of $T^{\mu\nu}$**:
- speed of sound
- bulk- and shear-viscosity
- spectral functions



# Curing the sign problem

**Real-time path introduces a complex weight $\mathbf{e^{iS[A_+,A_-]}}$**

**$\rightarrow$ results in a hard sign problem!**

- *Various methods tackle the sign problem:*
  Reweighting, contour deformation, analytic continuation, Taylor expansion, Lefschetz thimbles, spectral reconstruction, **complex Langevin (CL) method**, ...

- **Ideas behind the CL method:**
  - adding an additional d.o.f., the Langevin time $\theta$: $A^a_\mu(x) \rightsquigarrow A^a_\mu(\theta, x)$
  - complexification of the gauge fields: $\mathfrak{su}(N_c) \rightsquigarrow \mathfrak{sl}(N_c, \mathbb{C})$ (non-compact!)
  - Gauge fields are treated as a stochastic process w.r.t $\theta$

# Complex Langevin in a nutshell

- CL equation describes the **stochastic process in $\theta$**

$$\partial_\theta A^a_\mu(\theta, x) = i\frac{\delta S_{\mathrm{YM}}}{\delta A^a_\mu(t,x)} + \eta^a_\mu(\theta, x)$$

- Gaussian distributed noise term

$$\langle \eta^a_\mu(\theta, t, \mathbf{x})\rangle = 0,$$
$$\langle \eta^a_\mu(\theta, t, \mathbf{x})\eta^b_\nu(\theta', t', \mathbf{x}')\rangle = 2\delta(\theta-\theta')\delta(t-t')\delta^{(d-1)}(\mathbf{x}-\mathbf{x}')\delta^{ab}\delta_{\mu\nu}$$

- **Correspondence with Fokker-Planck equation** shows that $A^a_\mu(\theta\to\infty, x)$ is described by $\rho[A] = \mathcal{N}\exp\left[-S_{\mathrm{YM}}[A]\right]$.

**CL bypasses the sign problem by sampling at late $\theta$**

$$\langle \mathcal{O}[A]\rangle = \int \mathcal{D}A\,\rho[A]\mathcal{O}[A] \approx \lim_{\theta_0\to\infty}\frac{1}{T}\int_{\theta_0}^{\theta_0+T} d\theta\,\mathcal{O}[A(\theta)]$$

# Complex Langevin on the lattice

- Link variables and plaquette variables

$$U_{x,\mu} \simeq \exp\left[iga_\mu A_\mu\big(x+\hat{\mu}/2\big)\right] \in \mathrm{SU}(N_c) \rightsquigarrow \mathrm{SL}(N_c,\mathbb{C}),$$
$$U_{x,\mu\nu} = U_{x,\mu}U_{x+\hat{\mu},\nu}U^{-1}_{x+\hat{\nu},\mu}U^{-1}_{x,\nu}$$

- Wilson plaquette action: $S_{\mathrm{W}}[U] = \frac{1}{2N_c}\sum_{x,\mu\neq\nu}\beta_{\mu\nu}\mathrm{Tr}\left[U_{x,\mu\nu}-1\right]$
- Coupling constants: $\beta_{0i} = -\frac{2N_c}{g^2}\frac{a_s}{a_{t,k}}, \quad \beta_{ij} = +\frac{2N_c}{g^2}\frac{\bar{a}_{t,k}}{a_s}$
- Averaged lattice spacing (time reversability): $\bar{a}_{t,k} = (a_{t,k}+a_{t,k+1})/2$

## Update step (Euler-Maruyama) for the link variables

$$U_{x,\mu}(\theta+\epsilon) = \exp\left[it^a\left(i\Gamma_\mu\epsilon\, D^a_{x,\mu}S_{\mathrm{W}} + \sqrt{\Gamma_\mu\epsilon}\,\eta^a_{x,\mu}(\theta)\right)\right]U_{x,\mu}(\theta)$$

→ Field independent kernel function $\Gamma_\mu$ leaves stationary solution intact but can be utilised to improve convergence and stability!

# The fall and rise of complex Langevin

- **CL suffers from two types of instabilities [1]:**

  - *Runaways*: Straying too far into the "bulk" of complex manifold leads to a numerical blowup of the solution.

  - *Wrong convergence*: Breaking of assumption of the method during evolution can lead to distorted results.

- Several methods have been introduced to mitigate these issues: adaptive step size [2], gauge cooling [3], dynamical stabilization [4], *kernels* [5], ...

## Anisotropic kernel

- We introduce an anisotropic kernel to alleviate instabilities

$$\Gamma_t = |a_t|^2/a_s^2,\ \Gamma_s = 1$$

- *Motivation*: Based on a careful rederivation of the CL equation on complex time contours [6].

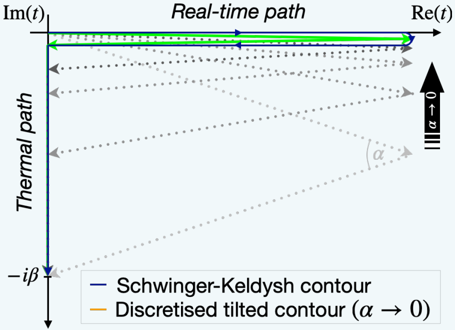
# Our simulation approach

- ▷ Discrete path integral needs to be regularized

  → we tilt the SK-contour

  → instabilities more sever for smaller tilt angles

- ▷ Our kernel alleviates instabilities for increasing anisotropies!

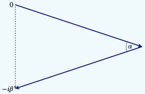


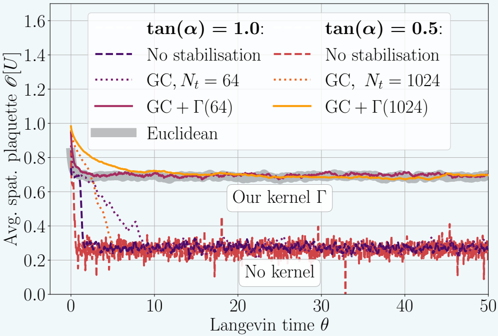

**Instabilities for smaller tilt angles can be counteracted by larger lattice anisotropies $a_s/|a_t|$!**

# Average spatial plaquette

Lattice parameters:
$L = N_t \times 4^3$
$g = 1, \beta = 1/T = 4$

- Spatial average plaquette:
  $\mathcal{O}[U] = \frac{1}{6N_c N_x} \sum_{x,i\neq j} \mathrm{Tr}\left[U_{x,ij}\right]$
- $t$-indep. in thermal equilibrium $\rightarrow$ **comparison to Euclidean results**
- Reproduction of studies on isosceles contours [7]




☒ Not-stabilised simulation:
Wrong convergence result after short time.

☑ Stabilised simulation:
GC and the anisotropic kernel reproduces Euclidean results.

$\Longrightarrow$ *Existing methods are not enough to stabilise the simulations, however the introduction of our kernel did!*

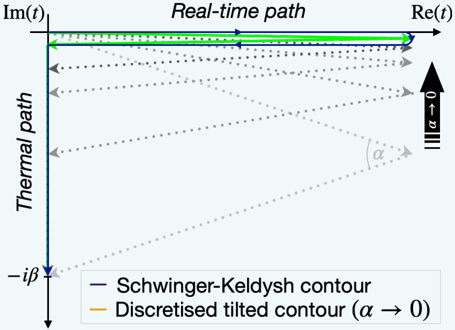
# Correlations of the magnetic energy density

Lattice parameters:
$L = 64 \times 16^3$
$g = 0.5, \beta = 1/T = 1$

- Calc. of corr. fcts. on the real-time path requires extrapolation $\alpha \to 0$
- We are interested in the magnetic energy density:

$$B^2(t,x) = \frac{1}{4}\sum_{i<j} F_{ij}^2(t,x)$$

- Clover leafs for the calc. of $F_{\mu\nu}$




As a first study, we calculate the unequal time correlation function of $B^2(t,x)$ (summed over all $x$):

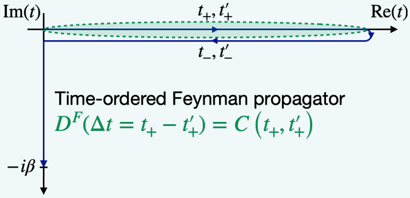
$$C(t,t') = \sum_x C(t,t',x) = \sum_x \langle B^2(t,x) B^2(t',x) \rangle \propto \sum_k \tilde{C}(t,t',k).$$

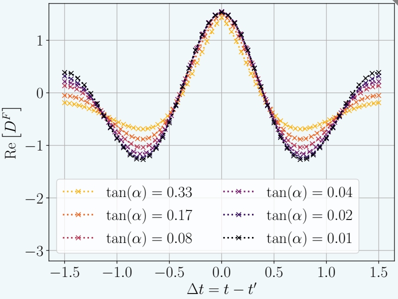
$\rightarrow$ improves statistics, shorter runtimes

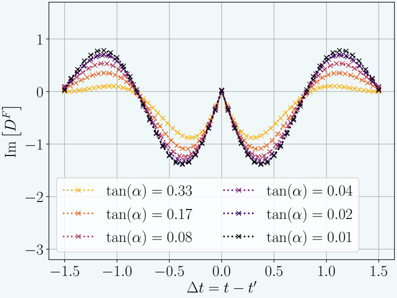
# Limit towards the SK-contour $\alpha \to 0$

Preliminary

- Decreasing tilt angle leads to "converging" time-ordered Feynman propagator $D^F$.
- Justification for extrapolation $\alpha \to 0$.



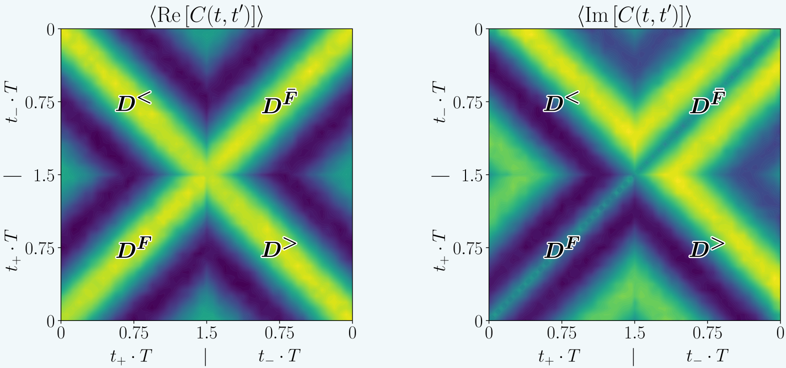



# Extrapolation to the SK contour



- Real-time correlation functions for 3+1D Yang-Mills theory
- Different quadrants are analytically related

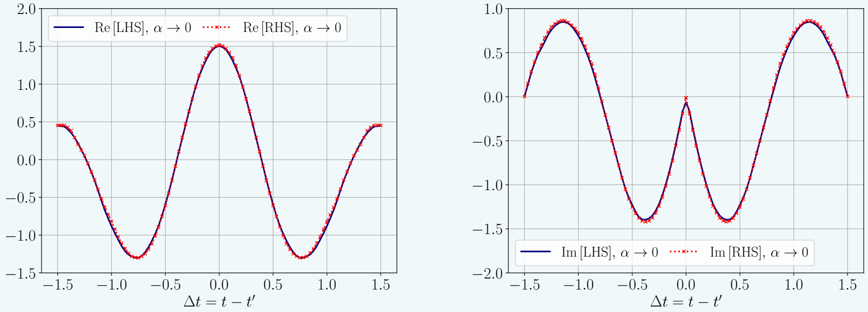
$$\underbrace{D^F(t,t')}_{\text{LHS}} = \underbrace{\theta(t-t')D^>(t,t') + \theta(t'-t)D^<(t,t')}_{\text{RHS}}$$



# Feynman propagators and Wightman functions

Preliminary

- Relation for Feynman propagator and Wightman function *satisfied* in the limit $\alpha \to 0$

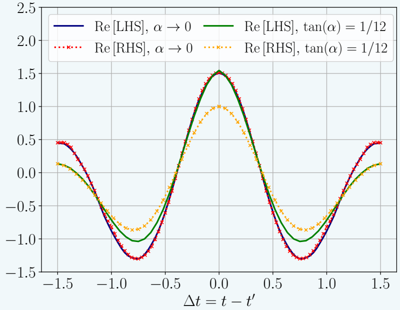


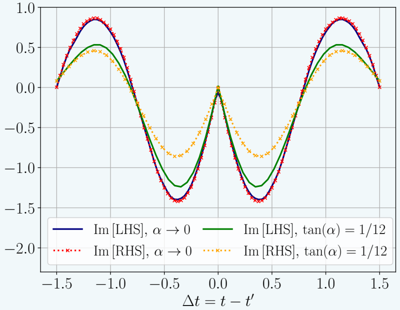
# Feynman propagators and Wightman functions

Preliminary

- Relation for Feynman propagator and Wightman function *satisfied* in the limit $\alpha \to 0$
- *Not satisfied* for non-vanishing tilt angles

  $\rightarrow$ **non-trivial property fulfilled; confirms our simulation approach!**



# Conclusion and outlook

## Conclusion

- **First direct calculation of unequal time correlation functions** for 3+1D lattice Yang-Mills theory
  - Correlations of magnetic energy density
  - Calculations done in the weak coupling regime (so far)
- New anisotropic kernel enables complex Langevin simulations for **real times larger than the inverse temperature**
- Extrapolation to Schwinger-Keldysh contour successful
  - Propagators are consistent $\rightsquigarrow$ *confirms our simulation approach*

## Outlook

- Repeat calculations at moderate coupling
  - higher anisotropies needed
  - development of further stabilization techniques
- Renormalization of lattice spacings for real-time contours (see our poster)
- Computation of shear and bulk viscosities, spectral functions

# Thank you for your attention!

# References


G. Aarts, F. A. James, E. Seiler, and I.-O. Stamatescu, *Complex Langevin: Etiology and Diagnostics of its Main Problem*, *Eur. Phys. J. C* **71** (2011) 1756, [arXiv:1101.3270].

G. Aarts, F. A. James, E. Seiler, and I.-O. Stamatescu, *Adaptive stepsize and instabilities in complex Langevin dynamics*, *Phys. Lett. B* **687** (2010) 154–159, [arXiv:0912.0617].

E. Seiler, D. Sexty, and I.-O. Stamatescu, *Gauge cooling in complex Langevin for QCD with heavy quarks*, *Phys. Lett. B* **723** (2013) 213–216, [arXiv:1211.3709].

F. Attanasio and B. Jäger, *Dynamical stabilisation of complex Langevin simulations of QCD*, *Eur. Phys. J. C* **79** (2019), no. 1 16, [arXiv:1808.04400].

K. Okano, L. Schulke, and B. Zheng, *Kernel controlled complex Langevin simulation: Field dependent kernel*, *Phys. Lett. B* **258** (1991) 421–426.

K. Boguslavski, P. Hotzy, and D. I. Müller, *Stabilizing complex Langevin for real-time gauge theories with an anisotropic kernel*, *JHEP* **06** (2023) 011, [arXiv:2212.08602].

J. Berges, S. Borsanyi, D. Sexty, and I. O. Stamatescu, *Lattice simulations of real-time quantum fields*, *Phys. Rev. D* **75** (2007) 045007, [hep-lat/0609058].

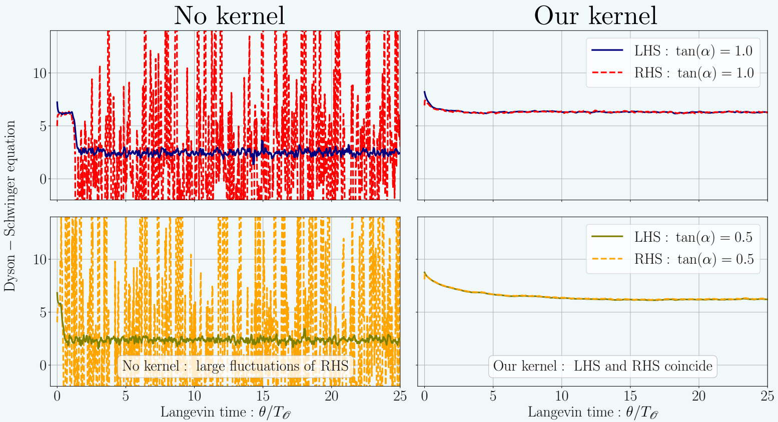
# Backup slides

# Dyson-Schwinger equations as probe for stability

- Self-consistency check of link configuration

$$\frac{2(N_c^2-1)}{N_c}\langle \mathrm{ReTr}(U_{x,ij})\rangle = \frac{i}{2N_c}\sum_{|\rho|\neq i}\beta_{i\rho}\left\langle \mathrm{ReTr}\left[(U_{x,i\rho}-U^{-1}_{x,i\rho})U_{x,ij}\right]\right\rangle$$

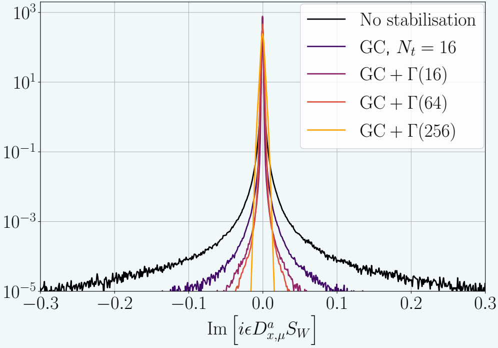
- RHS is very sensitive → probe for stability



# Localized histogram of the drift terms

- ▷ Localised histograms of $\mathrm{Im}\left[iDS_W\right]$ indicate correct convergence
- ▷ No skirts of histograms using our kernel with sufficiently large $N_t$

- ▷ Gauge cooling helps, but skirts are still present
- ▷ Increasing $N_t$ without our kernel does *not* improve the stability



# Euclidean correlation function

Preliminary

- Euclidean correlation function $C(\tau, \tau')$ along thermal path
- Coinciding with correlation function from Euclidean simulation
- Independence of tilt angle $\alpha$