Current Progress on the Semileptonic Form Factors for $B \to D^{(*)}\ell\nu$ Decay using the Oktay-Kronfeld Action

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Introduction

- To find semileptonic form factors (SFF) for $B \to D^* \ell \nu$ (e.g. $h_{A_1}(w)$, $h_{A_2}(w)$, $\cdots$), we need the data analysis on 2-point correlation functions.

- Using results for the 2pt function data analysis as input parameters, we do the 3pt function data analysis to obtain SFF.

- We report our recent progress in the complete analysis on 2pt functions.

- We report progress on preliminary data analysis on 3pt functions.

- In this talk, the key point is that we establish a reliable methodology of data analysis on 2pt and 3pt functions.

Measurement information

Content: we provide information on numerical study.

- MILC HISQ ensemble with $N_f = 2+1+1$ [PRD **87** 054505 (2013)]
  Ensemble ID = a12m220

a (fm)	$N_s^3 \times N_t$	M_π (MeV)	am_ℓ	am_s	am_c	N_{cfg}
0.1184(10)	$32^3 \times 64$	216.9(2)	0.00507	0.0507	0.628	1000

- Hopping parameters for the Oktay-Kronfeld action for valance heavy quarks
  $\kappa_{crit} = 0.051218$
  $\kappa_b = 0.04070$
  $\kappa_c = 0.048613$
- Valance strange quark mass for HISQ action
  $am_x = am_s = 0.0507$

Fitting function for 2pt correlator fit

Key point: we show fitting functional form for 2pt correlator fit.

- **Motivation:** Results of 2pt correlator fits are used as input parameters for 3pt correlator fits.

- Functional form of the $m+n$ fit for 2pt correlators

$$\begin{aligned} f(t) &= g(t) + g(T-t)\,, \\ g(t) &= A_0\, e^{-E_0\, t} \Big[1 + R_2\, e^{-\Delta E_2\, t} \Big(1 + R_4\, e^{-\Delta E_4\, t} \Big(\cdots \Big(1 + R_{2m-2}\, e^{-\Delta E_{2m-2}\, t} \Big) \cdots \Big) \Big) \\ &\qquad - (-1)^t\, R_1\, e^{-\Delta E_1\, t} \Big(1 + R_3\, e^{-\Delta E_3\, t} \Big(\cdots \Big(1 + R_{2n-1}\, e^{-\Delta E_{2n-1}\, t} \Big) \cdots \Big) \Big) \Big] \end{aligned}$$

- m (n): number of even (odd) time-parity states included in the fit
  - $E_{-1} \equiv E_0$: ground state energy ($A_{-1} \equiv A_0$)
  - $\Delta E_i \equiv E_i - E_{i-2}\,, \qquad R_i \equiv \dfrac{A_i}{A_{i-2}}$

Flow chart of sequential Bayesian method for 2pt fit

Key point: flow chart of our sequential Bayesian method

1 Do the 1st fitting.
ex) 1+0 fit with 2 parameters: $\{A_0, E_0\}$

2 Feed the previous fit results as prior information for the next fit.
ex) 1+1 fit with $\{A_0, E_0, R_1, \Delta E_1\}$, using prior info. on $\{A_0, E_0\}$

3 Do stability test and find optimal prior widths.
ex) stability test determines optimal prior widths on $\{A_0, E_0\}$.

4 Save the next fit results (*e.g.* 1+1 fit) into the previous fit.

5 Choose the 2+1 fit as the next fit.

6 Go back to Step **2**.
ex) 1+0 fit $\rightarrow$ 1+1 fit $\rightarrow$ 2+1 fit $\rightarrow$ 2+2 fit $\rightarrow \cdots$.

Initial guess for the χ^2 minimizer (χ^2-IG)

Key point: A good initial guess for the χ^2 minimizer

- We use the BFGS algorithm (one of the quasi-Newton methods) for the χ^2 minimizer. [BFGS = Broyden-Fletcher-Goldfarb-Shanno]
- Any Newton method needs an initial guess (χ^2-IG).
- A good initial guess is within the radius of convergence near the minimum.
- A bad initial guess is out of the radius of convergence.
- If we choose the bad initial guess by accident, then the number of iteration will increase dramatically.
- Example [PoS (LAT2021) 136]
    - Number of iteration of good initial guess = 327
    - Number of iteration of bad initial guess = 1627
- For a good initial guess, we use a multi-dimensional Newton method combined with the scanning method.

Initial guess for the Newton method (N-IG)

Key point: A good initial guess for the Newton method

- To obtain a good initial guess χ^2-IG with p time slices for p unknown parameters, we solve the following equations

$$\frac{f(t_i) - C(t_i)}{C(t_i)} = 0 \quad \text{for} \quad i = 1, \cdots, p$$

$f(t_i)$ = fit function, $C(t_i)$ = raw-data.

- The Newton method also need its own initial guess (N-IG).
  - To obtain a good initial guess (N-IG) within the radius of convergence, we use a scanning method.
  - Refer to [PoS (LAT2021) 136] on details.

Flow chart of $m+n$ fit I

Key point: We show flow chart of $m+n$ fit

1 We choose p time slices under the following constraints.

- t_{min} (lower bound of fit range) should be included.
- # of even time slices = # of odd time slices
- $p = 2(m+n)$

Ex) 2+2 fit with fit range: $3 \leq t \leq 29$

- $p = 8$: A_0^g, E_0^g, $\{R_j^g, \Delta E_j^g\}$ $(j = 1, 2, 3)$
- $t_{\text{min}} = 3$ must be included in odd time slice set.
- # of odd time slices $= 13 \Rightarrow {}_{13}\text{C}_3$
- # of even time slices $= 13 \Rightarrow {}_{13}\text{C}_4$
- Total # of the possible time slice combinations = 204,490

$$ {}_{13}\text{C}_3 \times {}_{13}\text{C}_4 = 204{,}490 $$

(continues in the next page)

Flow chart of $m+n$ fit II

2 Run the Newton method as follows,

① Take i-th time slice combination randomly. (initially, $i = 1$)
② Recycle fit results from the previous fit ($[m-1]+n$ or $m+[n-1]$) to set part of the initial guess for the Newton method.
③ Use the scanning method to set the remaining part of the initial guess for the Newton method.
④ Run the Newton method.
⑤ If the Newton method finds a root, save it. If it fails, abandon it.
⑥ Repeat the loop (① --- ⑥) with $(i+1)$-th time slice combination.

3 For each good initial guess (χ^2-IG) obtained by the Newton method, perform the least χ^2 fitting to produce a set of χ^2 distribution.

4 Sort values of χ^2/d.o.f. and find the global minimum and local minima.

Stability test for optimal prior width

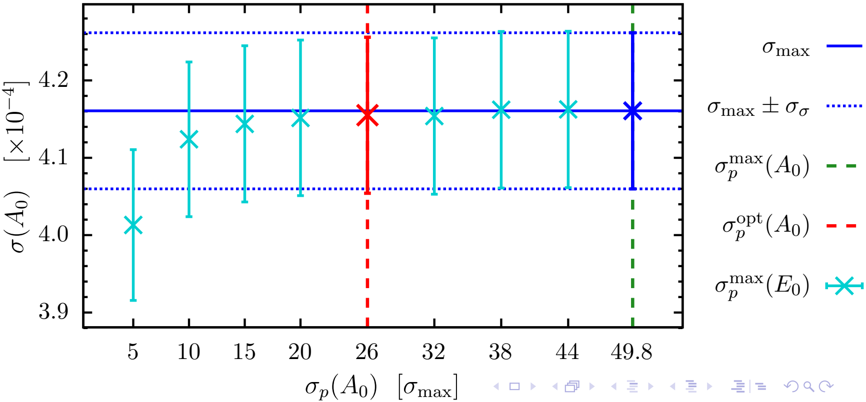
- We use sequential Bayesian method to the 2pt correlator fit.
- For example, in 2+1 fit, there are six fit parameters: A_0, E_0, $\{R_i, \Delta E_i\}$ $(i = 1, 2)$.
- By construction, $\{R_2, \Delta E_2\}$ includes any contamination from all the excited states with even time-parity, which protect the ground state signal $\{A_0, E_0\}$.
- Hence, 2+1 fit is a minimal set of the fits which allows the stability test for $\{A_0, E_0\}$, because $\{R_1, \Delta E_1\}$ and $\{R_2, \Delta E_2\}$ absorb all the unwanted contamination from the excited states.
- In stability test, we find the optimal prior widths (σ_p^{opt}) such that

$$\sigma_p^{\text{opt}} = \min(\sigma_p) \quad \text{for} \quad \forall \sigma_p \in \{\sigma | f_i(\sigma) = \lim_{\sigma_t \to \infty} f_i(\sigma_t)\},$$

$\{f_i(\sigma_p)\}$ = fit results with prior width σ_p.

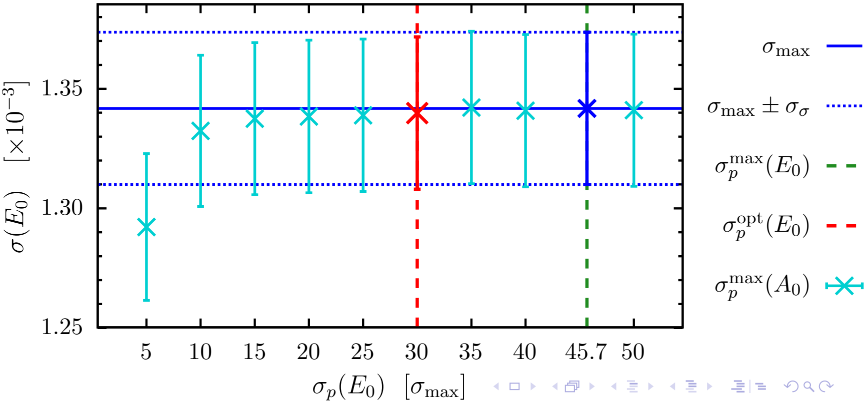
Stability test for optimal prior width (ex: $\sigma(A_0)$)

- $\sigma_p^{\text{opt}} = \min(\sigma_p)$ for $\forall \sigma_p \in \{\sigma | f_i(\sigma) = \lim_{\sigma_t \to \infty} f_i(\sigma_t)\}$, $\{f_i(\sigma_p)\}$ = fit results with prior width σ_p.
- $\sigma_{\text{max}} \equiv \sigma(A_0, \sigma_p^{\text{max}}(A_0), \sigma_p^{\text{max}}(E_0))$



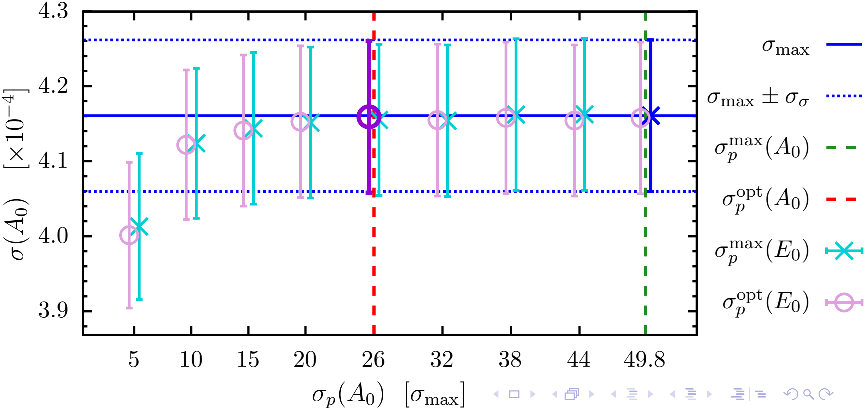
Stability test for optimal prior width (ex: $\sigma(E_0)$)

- $\sigma_p^{\text{opt}} = \min(\sigma_p) \quad \text{for} \quad \forall \sigma_p \in \{\sigma | f_i(\sigma) = \lim_{\sigma_t \to \infty} f_i(\sigma_t)\}$,
  $\{f_i(\sigma_p)\}$ = fit results with prior width σ_p.
- $\sigma_{\text{max}} \equiv \sigma(E_0, \sigma_p^{\text{max}}(A_0), \sigma_p^{\text{max}}(E_0))$



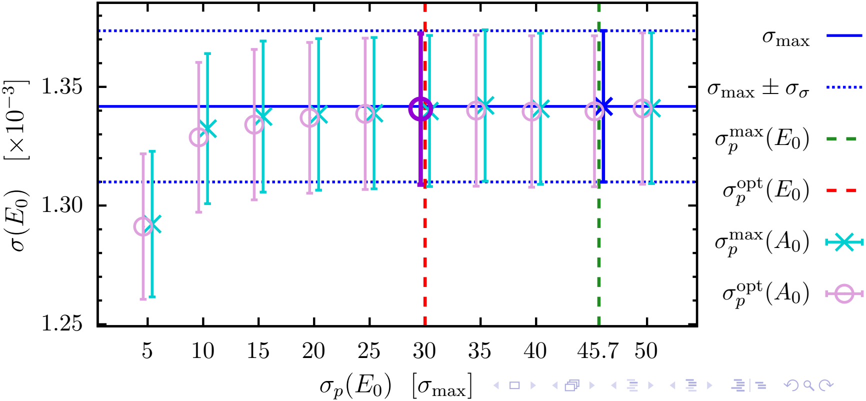
Stability test for optimal prior width (ex: $\sigma(A_0)$)

- $\sigma_p^{\text{opt}} = \min(\sigma_p)$ for $\forall \sigma_p \in \{\sigma | f_i(\sigma) = \lim_{\sigma_t \to \infty} f_i(\sigma_t)\}$, $\{f_i(\sigma_p)\}$ = fit results with prior width σ_p.
- $\sigma_{\max} \equiv \sigma(A_0, \sigma_p^{\max}(A_0), \sigma_p^{\max}(E_0))$



Stability test for optimal prior width (ex: $\sigma(E_0)$)

- $\sigma_p^{\text{opt}} = \min(\sigma_p)$ for $\forall \sigma_p \in \{\sigma | f_i(\sigma) = \lim_{\sigma_t \to \infty} f_i(\sigma_t)\}$,
  $\{f_i(\sigma_p)\}$ = fit results with prior width σ_p.
- $\sigma_{\text{max}} \equiv \sigma(E_0, \sigma_p^{\text{max}}(A_0), \sigma_p^{\text{max}}(E_0))$



Results of B meson 2pt correlator fit

Key point: we provide fit results for B meson 2pt correlator fit (2+2 fit)

- Fit range: $3 \leq t \leq 29$

B 2pt	prior info. on A_0, E_0		final step	
	fit result	prior info.	fit result	prior info.
A_0	0.01727(35)	0.1725(1043)	0.01727(35)	none
E_0	2.0450(18)	2.0449(450)	2.0450(18)	none
R_1	0.639(81)	0.646(418)	0.639(75)	0.646(418)
ΔE_1	0.241(14)	0.242(166)	0.241(13)	0.242(166)
R_2	1.888(76)	1.879(1879)	1.888(72)	1.879(1879)
ΔE_2	0.477(22)	0.475(475)	0.477(21)	0.475(475)
R_3	2.05(44)	none	2.05(40)	2.05(205)
ΔE_3	0.57(14)	none	0.57(13)	0.57(57)
χ^2/d.o.f.	0.4200(85)		0.4200(85)	

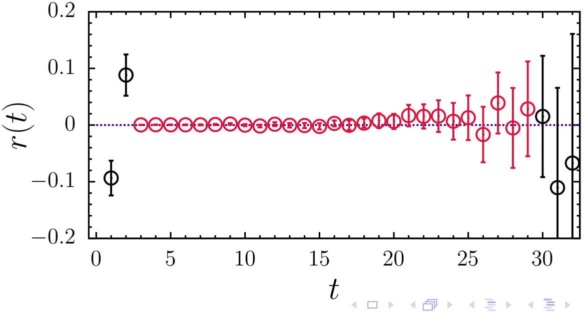
Residual plot of B meson 2pt correlator fit

Key point: we provide residual plot for B meson 2pt fit (2+2 fit)

- Here, the residual is defined as

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}$$

where $C(t)$ is the correlator data and $f(t)$ is the fit function.



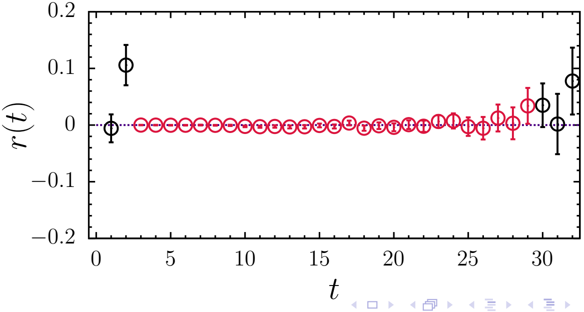
Residual plot of D^* meson 2pt correlator fit

Key point: we provide residual plot for D^* meson 2pt fit (2+2 fit)

- Here, the residual is defined as

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}$$

where $C(t)$ is the correlator data and $f(t)$ is the fit function.



Data analysis on 3pt correlation function

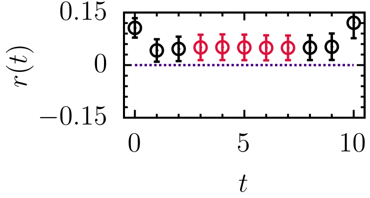
3pt correlator fit

Key point: summary of our method on 3pt correlator fit

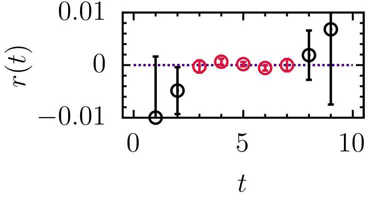
- For the B meson, we use results for 2+2 fit on the 2pt correlator as input parameters of 3pt correlator fit.
- Likewise, for the D^* meson, we use results for 2+2 fit on the 2pt correlator as input parameters of 3pt correlator fit.
- Hence, the remaining fitting is linear.
- We determine the remaining coefficients with a linear fit.
  $\Rightarrow$ simple linear algebra.

Diagonal approximation on the 3pt correlator fit

- Residual plots $r(t) = \frac{C(t) - f(t)}{|C(t)|}$



(a) full covariance fit

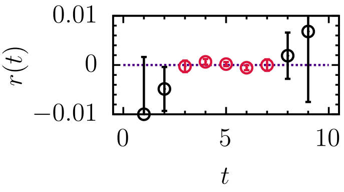


(b) diagonal approximation

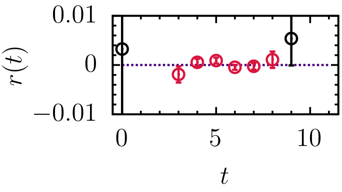
- Off-diagonal elements of $\rho(t_\alpha, t_\beta)$ (correlation matrix) are close to one.
- Therefore, we use diagonal approximation in this talk.
  - To resolve this problem, we should increase our statistics.
    $\Rightarrow$ full covariance fit

Residual plots for $B \to D^*$ at the $\mathcal{O}\left(\lambda_b^0 \lambda_c^0\right)$ level (diag)

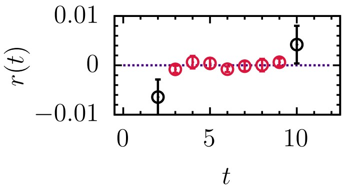
Key point: residual plots for $B \to D^*$ channel at the $\mathcal{O}\left(\lambda_b^0 \lambda_c^0\right)$ level



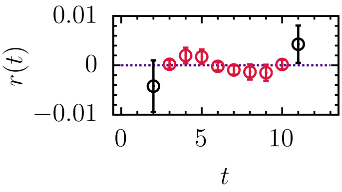
(c) $T_{\text{sep}} = 10$



(d) $T_{\text{sep}} = 11$



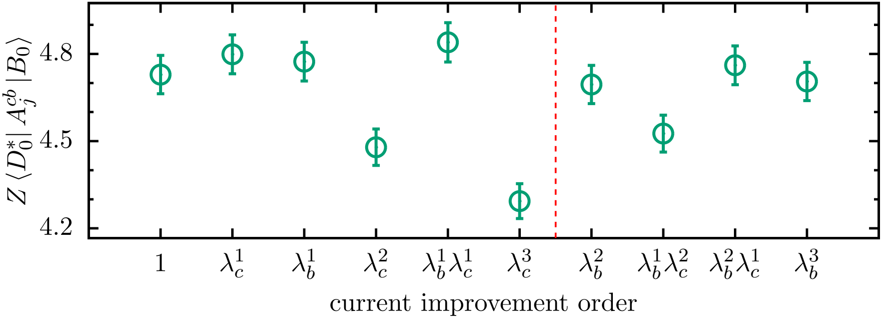
(e) $T_{\text{sep}} = 12$



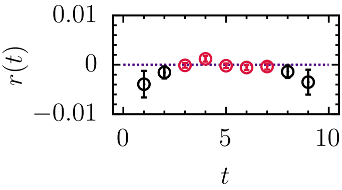
(f) $T_{\text{sep}} = 13$

$\langle D_0^* | A_j^{cb} | B_0 \rangle$ on current improvement order ($B \to D^*$)

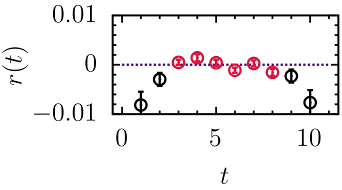
Key point: we show plot on $\langle D_0^* | A_j^{cb} | B_0 \rangle$ versus curr. imp. order



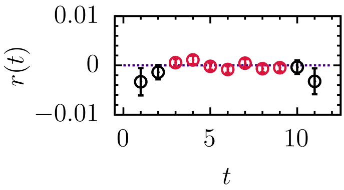
Key point: residual plots for $B \to B$ channel at the $\mathcal{O}\left(\lambda_b^0\right)$ level



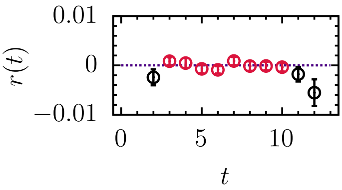
(g) $T_{\text{sep}} = 10$



(h) $T_{\text{sep}} = 11$



(i) $T_{\text{sep}} = 12$



(j) $T_{\text{sep}} = 13$

Summary and To-do list

Key point: we provide summary and to-do list.

- We have reported our final 2pt fit result on the B meson and D^* meson.
- We have reported our 3pt fit result on the $B \to D^*$ and $B \to B$ channel.
  - Results on $D^* \to B$ and $D^* \to D^*$ are similar with them.
  - The off-diagonal elements of correlation matrix $\rho(t_\alpha, t_\beta)$ is close to one.
  - Hence, we have reported our preliminary result with diagonal approximation.
  - To resolve this problem, we should increase our statistics for the full covariance fitting.
- For the semileptonic form factor, we need to calculate the matching factor in perturbative one-loop level.

Thank you for listening!

Backup slides

$B \to D^*$ 3pt correlator fit function (1)

Key point: fitting functional form for $B \to D^*$ channel

- For $B \to D^*$, our $[2{+}2]_B\,[2{+}2]_{D^*}$ fit function for given T_{sep} is

$$f^{B\to D^*}_{T_{\text{sep}}}\left(\langle D^*_a|\,A^{cb}_j\,|B_b\rangle\,;t\right)$$

$$= \begin{pmatrix} h^{D^*}_0(t) \\ h^{D^*}_1(t) \\ h^{D^*}_2(t) \\ h^{D^*}_3(t) \end{pmatrix}^T \begin{pmatrix} \langle D^*_0|\,A^{cb}_j\,|B_0\rangle & 0 & \langle D^*_0|\,A^{cb}_j\,|B_2\rangle & 0 \\ 0 & \langle D^*_1|\,A^{cb}_j\,|B_1\rangle & 0 & \langle D^*_1|\,A^{cb}_j\,|B_3\rangle \\ \langle D^*_2|\,A^{cb}_j\,|B_0\rangle & 0 & \langle D^*_2|\,A^{cb}_j\,|B_2\rangle & 0 \\ 0 & \langle D^*_3|\,A^{cb}_j\,|B_1\rangle & 0 & \langle D^*_3|\,A^{cb}_j\,|B_3\rangle \end{pmatrix} \begin{pmatrix} h^B_0(T_{\text{sep}}-t) \\ h^B_1(T_{\text{sep}}-t) \\ h^B_2(T_{\text{sep}}-t) \\ h^B_3(T_{\text{sep}}-t) \end{pmatrix}$$

$$\begin{aligned} &= \langle D^*_0|\,A^{cb}_j\,|B_0\rangle\; h^{D^*}_0(t)\,h^B_0(T_{\text{sep}}-t) + \langle D^*_0|\,A^{cb}_j\,|B_2\rangle\; h^{D^*}_0(t)\,h^B_2(T_{\text{sep}}-t) \\ &+ \langle D^*_1|\,A^{cb}_j\,|B_1\rangle\; h^{D^*}_1(t)\,h^B_1(T_{\text{sep}}-t) + \langle D^*_1|\,A^{cb}_j\,|B_3\rangle\; h^{D^*}_1(t)\,h^B_3(T_{\text{sep}}-t) \\ &+ \langle D^*_2|\,A^{cb}_j\,|B_0\rangle\; h^{D^*}_2(t)\,h^B_0(T_{\text{sep}}-t) + \langle D^*_2|\,A^{cb}_j\,|B_2\rangle\; h^{D^*}_2(t)\,h^B_2(T_{\text{sep}}-t) \\ &+ \langle D^*_3|\,A^{cb}_j\,|B_1\rangle\; h^{D^*}_3(t)\,h^B_1(T_{\text{sep}}-t) + \langle D^*_3|\,A^{cb}_j\,|B_3\rangle\; h^{D^*}_3(t)\,h^B_3(T_{\text{sep}}-t) \end{aligned}$$

- $h^{D^*}_a(t)$ and $h^B_b(T_{\text{sep}}-t)$ $(a, b = 0, 1, 2, 3)$ are given in the next page.

$B \to D^*$ 3pt correlator fit function (2)

Key point: fitting functional form for $B \to D^*$ channel

- For $B \to D^*$ 3pt $[2{+}2]_B\,[2{+}2]_{D^*}$ fit, $h_a^{D^*}$ and h_b^B $(a, b = 0, 1, 2, 3)$ are

$$h_0^B\,(T_{\rm sep} - t) = \sqrt{A_0^B}\; e^{-E_0^B\,(T_{\rm sep}-t)}$$

$$h_1^B\,(T_{\rm sep} - t) = \sqrt{A_0^B R_1^B}\; e^{-\left(E_0^B+\Delta E_1^B\right)(T_{\rm sep}-t)}\,(-1)^{T_{\rm sep}-t+1}$$

$$h_2^B\,(T_{\rm sep} - t) = \sqrt{A_0^B R_2^B}\; e^{-\left(E_0^B+\Delta E_2^B\right)(T_{\rm sep}-t)}$$

$$h_3^B\,(T_{\rm sep} - t) = \sqrt{A_0^B R_1^B R_3^B}\; e^{-\left(E_0^B+\Delta E_1^B+\Delta E_3^B\right)(T_{\rm sep}-t)}\,(-1)^{T_{\rm sep}-t+1}$$

$$h_0^{D^*}(t) = \sqrt{A_0^{D^*}}\; e^{-E_0^{D^*}\,t}$$

$$h_1^{D^*}(t) = \sqrt{A_0^{D^*} R_1^{D^*}}\; e^{-\left(E_0^{D^*}+\Delta E_1^{D^*}\right)t}\,(-1)^{t+1}$$

$$h_2^{D^*}(t) = \sqrt{A_0^{D^*} R_2^{D^*}}\; e^{-\left(E_0^{D^*}+\Delta E_2^{D^*}\right)t}$$

$$h_3^{D^*}(t) = \sqrt{A_0^{D^*} R_1^{D^*} R_3^{D^*}}\; e^{-\left(E_0^{D^*}+\Delta E_1^{D^*}+\Delta E_3^{D^*}\right)t}\,(-1)^{t+1}$$

where A_0^X, E_0^X, $\{R_j^X, \Delta E_j^X\}$ $(X = B, D^*)$ are input parameters found at 2pt correlator fit.

3pt correlator fit (linear fit) for $B \to D^*$

Key point: linear fit method for $B \to D^*$ channel

- For the least χ^2 fitting of 3pt correlator, we solve

$$\begin{pmatrix} g_1(t_\alpha) V^{-1}_{\alpha\beta} g_1(t_\beta) & g_1(t_\alpha) V^{-1}_{\alpha\beta} g_2(t_\beta) & g_1(t_\alpha) V^{-1}_{\alpha\beta} g_3(t_\beta) & \cdots \\ g_2(t_\alpha) V^{-1}_{\alpha\beta} g_1(t_\beta) & g_2(t_\alpha) V^{-1}_{\alpha\beta} g_2(t_\beta) & g_2(t_\alpha) V^{-1}_{\alpha\beta} g_3(t_\beta) & \cdots \\ g_3(t_\alpha) V^{-1}_{\alpha\beta} g_1(t_\beta) & g_3(t_\alpha) V^{-1}_{\alpha\beta} g_2(t_\beta) & g_3(t_\alpha) V^{-1}_{\alpha\beta} g_3(t_\beta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \langle D_0^* | A_j^{cb} | B_0 \rangle \\ \langle D_0^* | A_j^{cb} | B_2 \rangle \\ \langle D_1^* | A_j^{cb} | B_1 \rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} g_1(t_\alpha) V^{-1}_{\alpha\beta} C(t_\beta) \\ g_2(t_\alpha) V^{-1}_{\alpha\beta} C(t_\beta) \\ g_3(t_\alpha) V^{-1}_{\alpha\beta} C(t_\beta) \\ \vdots \end{pmatrix}$$

  - $g_1(t) = h_0^{D^*}(t)\, h_0^B(T_{\text{sep}} - t)$, $g_2(t) = h_0^{D^*}(t)\, h_2^B(T_{\text{sep}} - t)$, etc.
  - $V^{-1}_{\alpha\beta} = V^{-1}(t_\alpha, t_\beta)$: inversed covariance matrix
  - $C(t)$: $B \to D^*$ 3pt correlator data at time slice t

- This is came from

$$\frac{\partial \chi^2}{\partial \lambda_p} = \frac{\partial f(\lambda; t_\alpha)}{\partial \lambda_p} V^{-1}_{\alpha\beta} \{ f(\lambda; t_\beta) - C(t_\beta) \} = g_p(t_\alpha) V^{-1}_{\alpha\beta} \{ f(\lambda; t_\beta) - C(t_\beta) \} = 0 \,.$$

  - $f(\lambda; t_\alpha) = \sum_{p=1}^{8} \lambda_p g_p(t_\alpha)$: 3pt fit function
  - $\lambda_1 = \langle D_0^* | A_j^{cb} | B_0 \rangle$, $\lambda_2 = \langle D_0^* | A_j^{cb} | B_2 \rangle$, $\lambda_3 = \langle D_1^* | A_j^{cb} | B_1 \rangle$, etc.

Reason for the usage of diagonal approximation (1)

Key point: we report the reason for the usage of diagonal approximation

- The off-diagonal elements of the covariance matrix, $V(t_\alpha, t_\beta)$ where $\alpha \neq \beta$, are too large.

- To check it clearly, we introduce the correlation matrix $\rho(t_\alpha, t_\beta)$ which is defined as

$$\rho(t_\alpha, t_\beta) \equiv \frac{V(t_\alpha, t_\beta)}{\sigma(t_\alpha)\,\sigma(t_\beta)} \quad \text{where} \quad \sigma(t_\alpha) = \sqrt{V(t_\alpha, t_\alpha)}$$

where we note that $\rho(t_\alpha, t_\alpha)$, the diagonal elements of ρ, is unity by definition.

Reason for the usage of diagonal approximation (2)

Key point: we report the reason for the usage of diagonal approximation

- Note that we use simultaneous fit over $T_{\text{sep}} = 10, 11, 12, 13$ where T_{sep} is source-sink time separation.
- We call the block diagonal matrices of ρ for each T_{sep} as $\rho_{10,10}$, $\rho_{11,11}$, $\rho_{12,12}$ and $\rho_{13,13}$.
- Then we have

$$\rho_{10,10} = \begin{pmatrix} 1.0000 & 0.9866 & 0.9687 & 0.9519 & 0.9342 \\ 0.9866 & 1.0000 & 0.9890 & 0.9745 & 0.9578 \\ 0.9687 & 0.9890 & 1.0000 & 0.9906 & 0.9771 \\ 0.9519 & 0.9745 & 0.9906 & 1.0000 & 0.9918 \\ 0.9342 & 0.9578 & 0.9771 & 0.9918 & 1.0000 \end{pmatrix}$$

$$\rho_{11,11} = \begin{pmatrix} 1.0000 & 0.9860 & 0.9690 & 0.9514 & 0.9326 & 0.9184 \\ 0.9860 & 1.0000 & 0.9896 & 0.9743 & 0.9571 & 0.9431 \\ 0.9690 & 0.9896 & 1.0000 & 0.9900 & 0.9757 & 0.9635 \\ 0.9514 & 0.9743 & 0.9900 & 1.0000 & 0.9912 & 0.9808 \\ 0.9326 & 0.9571 & 0.9757 & 0.9912 & 1.0000 & 0.9936 \\ 0.9184 & 0.9431 & 0.9635 & 0.9808 & 0.9936 & 1.0000 \end{pmatrix}$$

Reason for the usage of diagonal approximation (3)

Key point: we report the reason for the usage of diagonal approximation

- and

$$\rho_{12,12} = \begin{pmatrix} 1.0000 & 0.9842 & 0.9619 & 0.9408 & 0.9204 & 0.8989 & 0.8821 \\ 0.9842 & 1.0000 & 0.9866 & 0.9684 & 0.9493 & 0.9294 & 0.9140 \\ 0.9619 & 0.9866 & 1.0000 & 0.9881 & 0.9724 & 0.9555 & 0.9403 \\ 0.9408 & 0.9684 & 0.9881 & 1.0000 & 0.9897 & 0.9754 & 0.9615 \\ 0.9204 & 0.9493 & 0.9724 & 0.9897 & 1.0000 & 0.9907 & 0.9790 \\ 0.8989 & 0.9294 & 0.9555 & 0.9754 & 0.9907 & 1.0000 & 0.9926 \\ 0.8821 & 0.9140 & 0.9403 & 0.9615 & 0.9790 & 0.9926 & 1.0000 \end{pmatrix}$$

$$\rho_{13,13} = \begin{pmatrix} 1.0000 & 0.9840 & 0.9628 & 0.9412 & 0.9186 & 0.8995 & 0.8836 & 0.8664 \\ 0.9840 & 1.0000 & 0.9872 & 0.9688 & 0.9479 & 0.9298 & 0.9146 & 0.8977 \\ 0.9628 & 0.9872 & 1.0000 & 0.9879 & 0.9706 & 0.9536 & 0.9378 & 0.9210 \\ 0.9412 & 0.9688 & 0.9879 & 1.0000 & 0.9892 & 0.9751 & 0.9599 & 0.9441 \\ 0.9186 & 0.9479 & 0.9706 & 0.9892 & 1.0000 & 0.9909 & 0.9784 & 0.9634 \\ 0.8995 & 0.9298 & 0.9536 & 0.9751 & 0.9909 & 1.0000 & 0.9920 & 0.9802 \\ 0.8836 & 0.9146 & 0.9378 & 0.9599 & 0.9784 & 0.9920 & 1.0000 & 0.9923 \\ 0.8664 & 0.8977 & 0.9210 & 0.9441 & 0.9634 & 0.9802 & 0.9923 & 1.0000 \end{pmatrix}$$

- As we see from $\rho_{10,10}$, $\rho_{11,11}$, $\rho_{12,12}$ and $\rho_{13,13}$, due to the strong correlation, we cannot use full covariance matrix in the least χ^2 fitting.
- For the full covariance fitting, we should increase statistics.
- Today we report our results on the diagonal approximation.

$B \to B$ 3pt correlator fit function (1)

Key point: fitting functional form for $B \to B$ channel

- For $B \to B$, our $[2{+}2]_B\,[2{+}2]_B$ fit function for given T_{sep} is

$$f^{B\to B}_{T_{\text{sep}}}\left(\langle B_a|\, V_4^{bb}\,|B_b\rangle\,;t\right)$$

$$= \begin{pmatrix} h_0^B(t) \\ h_1^B(t) \\ h_2^B(t) \\ h_3^B(t) \end{pmatrix}^T \begin{pmatrix} \langle B_0|\, V_4^{bb}\,|B_0\rangle & 0 & \langle B_0|\, V_4^{bb}\,|B_2\rangle & 0 \\ 0 & \langle B_1|\, V_4^{bb}\,|B_1\rangle & 0 & \langle B_1|\, V_4^{bb}\,|B_3\rangle \\ \langle B_2|\, V_4^{bb}\,|B_0\rangle & 0 & \langle B_2|\, V_4^{bb}\,|B_2\rangle & 0 \\ 0 & \langle B_3|\, V_4^{bb}\,|B_1\rangle & 0 & \langle B_3|\, V_4^{bb}\,|B_3\rangle \end{pmatrix} \begin{pmatrix} h_0^B(T_{\text{sep}}-t) \\ h_1^B(T_{\text{sep}}-t) \\ h_2^B(T_{\text{sep}}-t) \\ h_3^B(T_{\text{sep}}-t) \end{pmatrix}$$

$$= \langle B_0|\, V_4^{bb}\,|B_0\rangle\; h_0^B(t)\, h_0^B(T_{\text{sep}}-t) + \langle B_1|\, V_4^{bb}\,|B_1\rangle\; h_1^B(t)\, h_1^B(T_{\text{sep}}-t)$$
$$+ \langle B_2|\, V_4^{bb}\,|B_2\rangle\; h_2^B(t)\, h_2^B(T_{\text{sep}}-t) + \langle B_3|\, V_4^{bb}\,|B_3\rangle\; h_3^B(t)\, h_3^B(T_{\text{sep}}-t)$$
$$+ \langle B_0|\, V_4^{bb}\,|B_2\rangle\; \left[h_0^B(t)\, h_2^B(T_{\text{sep}}-t) + h_2^B(t)\, h_0^B(T_{\text{sep}}-t)\right]$$
$$+ \langle B_1|\, V_4^{bb}\,|B_3\rangle\; \left[h_1^B(t)\, h_3^B(T_{\text{sep}}-t) + h_3^B(t)\, h_1^B(T_{\text{sep}}-t)\right]$$

- We find 6 fit params, using $\langle B_i|\, V_4^{bb}\,|B_j\rangle^* = \langle B_j|\, V_4^{bb}\,|B_i\rangle$ $(i \neq j)$.
- $h_a^B(t)$ and $h_b^B(T_{\text{sep}}-t)$ $(a, b = 0, 1, 2, 3)$ are given in the next page.

$B \to B$ 3pt correlator fit function (2)

Key point: fitting functional form for $B \to B$ channel

- For $B \to B$ 3pt $[2{+}2]_B\,[2{+}2]_B$ fit, $h_a^B\,(T_{\rm sep} - t)$ and $h_b^B\,(t)$ are

$$h_0^B\,(T_{\rm sep} - t) = \sqrt{A_0^B}\; e^{-E_0^B\,(T_{\rm sep}-t)}$$

$$h_1^B\,(T_{\rm sep} - t) = \sqrt{A_0^B R_1^B}\; e^{-\left(E_0^B+\Delta E_1^B\right)(T_{\rm sep}-t)}\,(-1)^{T_{\rm sep}-t+1}$$

$$h_2^B\,(T_{\rm sep} - t) = \sqrt{A_0^B R_2^B}\; e^{-\left(E_0^B+\Delta E_2^B\right)(T_{\rm sep}-t)}$$

$$h_3^B\,(T_{\rm sep} - t) = \sqrt{A_0^B R_1^B R_3^B}\; e^{-\left(E_0^B+\Delta E_1^B+\Delta E_3^B\right)(T_{\rm sep}-t)}\,(-1)^{T_{\rm sep}-t+1}$$

$$h_0^B\,(t) = \sqrt{A_0^B}\; e^{-E_0^B\,t}$$

$$h_1^B\,(t) = \sqrt{A_0^B R_1^B}\; e^{-\left(E_0^B+\Delta E_1^B\right)t}\,(-1)^{t+1}$$

$$h_2^B\,(t) = \sqrt{A_0^B R_2^B}\; e^{-\left(E_0^B+\Delta E_2^B\right)t}$$

$$h_3^B\,(t) = \sqrt{A_0^B R_1^B R_3^B}\; e^{-\left(E_0^B+\Delta E_1^B+\Delta E_3^B\right)t}\,(-1)^{t+1}$$

where A_0^B, E_0^B, $\{R_j^B,\, \Delta E_j^B\}$ are input parameters found at 2pt correlator fit.

3pt correlator fit (linear fit) for $B \to B$ (1)

Key point: linear fit method for $B \to B$ channel

- For the least χ^2 fitting of 3pt correlator, we solve

$$\begin{pmatrix} g_1(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_1(t_\beta) & g_1(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_1(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_3(t_\beta) & g_1(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_1(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_5(t_\beta) & g_1(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_6(t_\beta) \\ g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_1(t_\beta) & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_3(t_\beta) & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_5(t_\beta) & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_6(t_\beta) \\ g_3(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_1(t_\beta) & g_3(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_3(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_3(t_\beta) & g_3(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_3(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_5(t_\beta) & g_3(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_6(t_\beta) \\ g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_1(t_\beta) & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_3(t_\beta) & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_5(t_\beta) & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_6(t_\beta) \\ g_5(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_1(t_\beta) & g_5(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_5(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_3(t_\beta) & g_5(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_5(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_5(t_\beta) & g_5(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_6(t_\beta) \\ g_6(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_1(t_\beta) & g_6(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_6(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_3(t_\beta) & g_6(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_6(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_5(t_\beta) & g_6(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_6(t_\beta) \end{pmatrix} \begin{pmatrix} \langle B_0 | V_4^{bb} | B_0 \rangle \\ \langle B_0 | V_4^{bb} | B_2 \rangle \\ \langle B_1 | V_4^{bb} | B_1 \rangle \\ \langle B_1 | V_4^{bb} | B_3 \rangle \\ \langle B_2 | V_4^{bb} | B_2 \rangle \\ \langle B_3 | V_4^{bb} | B_3 \rangle \end{pmatrix} = \begin{pmatrix} g_1(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_3(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_5(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_6(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \end{pmatrix}$$

where $\mathcal{V}^{-1}_{\alpha\beta} = \mathcal{V}^{-1}(t_\alpha, t_\beta)$ is inversed covariance matrix, $C(t)$ is correlator data at t and

$$\begin{aligned}
g_1(t) &= h_0^B(t)\, h_0^B(T_{\text{sep}} - t) && = g_{1;T_{\text{sep}}} = \text{constant in } t \\
g_2(t) &= h_0^B(t)\, h_2^B(T_{\text{sep}} - t) + h_2^B(t)\, h_0^B(T_{\text{sep}} - t) \\
g_3(t) &= h_1^B(t)\, h_1^B(T_{\text{sep}} - t) && = g_{3;T_{\text{sep}}} = \text{constant in } t \\
g_4(t) &= h_1^B(t)\, h_3^B(T_{\text{sep}} - t) + h_3^B(t)\, h_1^B(T_{\text{sep}} - t) \\
g_5(t) &= h_2^B(t)\, h_2^B(T_{\text{sep}} - t) && = g_{5;T_{\text{sep}}} = \text{constant in } t \\
g_6(t) &= h_3^B(t)\, h_3^B(T_{\text{sep}} - t) && = g_{6;T_{\text{sep}}} = \text{constant in } t
\end{aligned}$$

3pt correlator fit (linear fit) for $B \to B$ (2)

Key point: linear fit method for $B \to B$ channel

- Then the linear equation can be written as

$$\begin{pmatrix} g_{1;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{1;T_{sep}} & g_{1;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_{1;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{3;T_{sep}} & g_{1;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_{1;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{5;T_{sep}} & g_{1;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{6;T_{sep}} \\ g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_{1;T_{sep}} & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_{3;T_{sep}} & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_{5;T_{sep}} & g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_{6;T_{sep}} \\ g_{3;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{1;T_{sep}} & g_{3;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_{3;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{3;T_{sep}} & g_{3;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_{3;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{5;T_{sep}} & g_{3;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{6;T_{sep}} \\ g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_{1;T_{sep}} & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_{3;T_{sep}} & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_{5;T_{sep}} & g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}g_{6;T_{sep}} \\ g_{5;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{1;T_{sep}} & g_{5;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_{5;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{3;T_{sep}} & g_{5;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_{5;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{5;T_{sep}} & g_{5;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{6;T_{sep}} \\ g_{6;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{1;T_{sep}} & g_{6;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_2(t_\beta) & g_{6;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{3;T_{sep}} & g_{6;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_4(t_\beta) & g_{6;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{5;T_{sep}} & g_{6;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}g_{6;T_{sep}} \end{pmatrix} \begin{pmatrix} \langle B_0|V_4^{bb}|B_0\rangle \\ \langle B_0|V_4^{bb}|B_2\rangle \\ \langle B_1|V_4^{bb}|B_1\rangle \\ \langle B_1|V_4^{bb}|B_3\rangle \\ \langle B_2|V_4^{bb}|B_2\rangle \\ \langle B_3|V_4^{bb}|B_3\rangle \end{pmatrix} = \begin{pmatrix} g_{1;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_2(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_{3;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_4(t_\alpha)\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_{5;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \\ g_{6;T_{sep}}\mathcal{V}^{-1}_{\alpha\beta}C(t_\beta) \end{pmatrix}$$

- The 6×6 matrix in the L.H.S. is non-singular if we use four different $T_{\text{sep}} = 10, 11, 12, 13$ (simultaneous fit).

- Note that the t dependence remains in $g_2(t)$ and $g_4(t)$.

HQET orders on λ_b and λ_c

$$\lambda_b = \frac{\Lambda}{2m_b}, \qquad \lambda_c = \frac{\Lambda}{2m_c} \tag{1}$$

$$1 = 1.00000000000000\text{e+}00$$
$$\lambda_c^1 \approx 1.95618153364632\text{e-}01$$
$$\lambda_b^1 \approx 5.94813228646205\text{e-}02$$
$$\lambda_c^2 \approx 3.82664619257888\text{e-}02$$
$$\lambda_b^1\lambda_c^1 \approx 1.16356265384625\text{e-}02$$
$$\lambda_c^3 \approx 7.48561461772081\text{e-}03$$
$$\lambda_b^2 \approx 3.53802776972523\text{e-}03$$
$$\lambda_b^1\lambda_c^2 \approx 2.27613977669455\text{e-}03$$
$$\lambda_b^2\lambda_c^1 \approx 6.92102458866437\text{e-}04$$
$$\lambda_b^3 \approx 2.10446572075019\text{e-}04$$

$D^* \to B$ 3pt correlator fit function (1)

- For $D^* \to B$, our $[2{+}2]_{D^*}\,[2{+}2]_B$ fit function for given T_{sep} is

$$
\begin{aligned}
&f_{T_{\text{sep}}}^{D^*\to B}\left(\langle B_a|\, A_j^{bc}\,|D_b^*\rangle\,;t\right)\\
&= \begin{pmatrix} h_0^B(t)\\ h_1^B(t)\\ h_2^B(t)\\ h_3^B(t)\end{pmatrix}^T
\begin{pmatrix}
\langle B_0|\, A_j^{bc}\,|D_0^*\rangle & 0 & \langle B_0|\, A_j^{bc}\,|D_2^*\rangle & 0\\
0 & \langle B_1|\, A_j^{bc}\,|D_1^*\rangle & 0 & \langle B_1|\, A_j^{bc}\,|D_3^*\rangle\\
\langle B_2|\, A_j^{bc}\,|D_0^*\rangle & 0 & \langle B_2|\, A_j^{bc}\,|D_2^*\rangle & 0\\
0 & \langle B_3|\, A_j^{bc}\,|D_1^*\rangle & 0 & \langle B_3|\, A_j^{bc}\,|D_3^*\rangle
\end{pmatrix}
\begin{pmatrix} h_0^{D^*}(T_{\text{sep}}-t)\\ h_1^{D^*}(T_{\text{sep}}-t)\\ h_2^{D^*}(T_{\text{sep}}-t)\\ h_3^{D^*}(T_{\text{sep}}-t)\end{pmatrix}\\
&= \langle B_0|\, A_j^{bc}\,|D_0^*\rangle\; h_0^B(t)\, h_0^{D^*}(T_{\text{sep}}-t) + \langle B_0|\, A_j^{bc}\,|D_2^*\rangle\; h_0^B(t)\, h_2^{D^*}(T_{\text{sep}}-t)\\
&+ \langle B_1|\, A_j^{bc}\,|D_1^*\rangle\; h_1^B(t)\, h_1^{D^*}(T_{\text{sep}}-t) + \langle B_1|\, A_j^{bc}\,|D_3^*\rangle\; h_1^B(t)\, h_3^{D^*}(T_{\text{sep}}-t)\\
&+ \langle B_2|\, A_j^{bc}\,|D_0^*\rangle\; h_2^B(t)\, h_0^{D^*}(T_{\text{sep}}-t) + \langle B_2|\, A_j^{bc}\,|D_2^*\rangle\; h_2^B(t)\, h_2^{D^*}(T_{\text{sep}}-t)\\
&+ \langle B_3|\, A_j^{bc}\,|D_1^*\rangle\; h_3^B(t)\, h_1^{D^*}(T_{\text{sep}}-t) + \langle B_3|\, A_j^{bc}\,|D_3^*\rangle\; h_3^B(t)\, h_3^{D^*}(T_{\text{sep}}-t)
\end{aligned}
$$

- $h_a^{D^*}(t)$ and $h_b^B(T_{\text{sep}}-t)$ $(a, b = 0, 1, 2, 3)$ are given in the next page.

$D^* \to B$ 3pt correlator fit function (2)

For $D^* \to B$ 3pt $[2{+}2]_{D^*}$ $[2{+}2]_B$ fit, h_a^B and $h_b^{D^*}$ $(a, b = 0, 1, 2, 3)$ are

$$h_0^{D^*}(T_{\text{sep}} - t) = \sqrt{A_0^{D^*}}\, e^{-E_0^{D^*}(T_{\text{sep}}-t)}$$

$$h_1^{D^*}(T_{\text{sep}} - t) = \sqrt{A_0^{D^*} R_1^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_1^{D^*}\right)(T_{\text{sep}}-t)} (-1)^{T_{\text{sep}}-t+1}$$

$$h_2^{D^*}(T_{\text{sep}} - t) = \sqrt{A_0^{D^*} R_2^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_2^{D^*}\right)(T_{\text{sep}}-t)}$$

$$h_3^{D^*}(T_{\text{sep}} - t) = \sqrt{A_0^{D^*} R_1^{D^*} R_3^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_1^{D^*}+\Delta E_3^{D^*}\right)(T_{\text{sep}}-t)} (-1)^{T_{\text{sep}}-t+1}$$

$$h_0^B(t) = \sqrt{A_0^B}\, e^{-E_0^B t}$$

$$h_1^B(t) = \sqrt{A_0^B R_1^B}\, e^{-\left(E_0^B+\Delta E_1^B\right)t} (-1)^{t+1}$$

$$h_2^B(t) = \sqrt{A_0^B R_2^B}\, e^{-\left(E_0^B+\Delta E_2^B\right)t}$$

$$h_3^B(t) = \sqrt{A_0^B R_1^B R_3^B}\, e^{-\left(E_0^B+\Delta E_1^B+\Delta E_3^B\right)t} (-1)^{t+1}$$

where A_0^X, E_0^X, $\{R_j^X, \Delta E_j^X\}$ $(X = B, D^*)$ are input parameters determined from 2pt correlator fit.

3pt correlator fit (linear fit): $D^* \to B$

- For the least χ^2 fitting of 3pt correlator, we solve

$$\begin{pmatrix} g_1(t_\alpha) V^{-1}_{\alpha\beta} g_1(t_\beta) & g_1(t_\alpha) V^{-1}_{\alpha\beta} g_2(t_\beta) & g_1(t_\alpha) V^{-1}_{\alpha\beta} g_3(t_\beta) & \cdots \\ g_2(t_\alpha) V^{-1}_{\alpha\beta} g_1(t_\beta) & g_2(t_\alpha) V^{-1}_{\alpha\beta} g_2(t_\beta) & g_2(t_\alpha) V^{-1}_{\alpha\beta} g_3(t_\beta) & \cdots \\ g_3(t_\alpha) V^{-1}_{\alpha\beta} g_1(t_\beta) & g_3(t_\alpha) V^{-1}_{\alpha\beta} g_2(t_\beta) & g_3(t_\alpha) V^{-1}_{\alpha\beta} g_3(t_\beta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \langle B_0 | A_j^{bc} | D_0^* \rangle \\ \langle B_0 | A_j^{bc} | D_2^* \rangle \\ \langle B_1 | A_j^{bc} | D_1^* \rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} g_1(t_\alpha) V^{-1}_{\alpha\beta} C(t_\beta) \\ g_2(t_\alpha) V^{-1}_{\alpha\beta} C(t_\beta) \\ g_3(t_\alpha) V^{-1}_{\alpha\beta} C(t_\beta) \\ \vdots \end{pmatrix}$$

  where $g_1(t) = h_0^B(t)\, h_0^{D^*}(T_{\text{sep}} - t)$, $g_2(t) = h_0^B(t)\, h_2^{D^*}(T_{\text{sep}} - t)$, $g_3(t) = h_1^B(t)\, h_1^{D^*}(T_{\text{sep}} - t)$, ... , $V^{-1}_{\alpha\beta} = V^{-1}(t_\alpha, t_\beta)$ is inversed covariance matrix, $C(t)$ is correlator data at t.

- This is came from

$$\frac{\partial \chi^2}{\partial \lambda_p} = \frac{\partial f(\lambda; t_\alpha)}{\partial \lambda_p} V^{-1}_{\alpha\beta} \{ f(\lambda; t_\beta) - C(t_\beta) \} = g_p(t_\alpha) V^{-1}_{\alpha\beta} \{ f(\lambda; t_\beta) - C(t_\beta) \} = 0$$

  where $f(\lambda; t_\alpha) = \sum_{p=1}^{8} \lambda_p g_p(t_\alpha)$ and $\lambda_1 = \langle B_0 | A_j^{bc} | D_0^* \rangle$, $\lambda_2 = \langle B_0 | A_j^{bc} | D_2^* \rangle$, etc.

$D^* \to D^*$ 3pt correlator fit function (1)

- For $D^* \to D^*$, our $[2{+}2]_{D^*}\,[2{+}2]_{D^*}$ fit function for given T_{sep} is

$$
\begin{aligned}
&f_{T_{\text{sep}}}^{D^*\to D^*}\left(\langle D_a^*|\,V_4^{cc}\,|D_b^*\rangle\,;t\right)\\
&=\begin{pmatrix} h_0^{D^*}(t)\\ h_1^{D^*}(t)\\ h_2^{D^*}(t)\\ h_3^{D^*}(t)\end{pmatrix}^T
\begin{pmatrix}
\langle D_0^*|\,V_4^{cc}\,|D_0^*\rangle & 0 & \langle D_0^*|\,V_4^{cc}\,|D_2^*\rangle & 0\\
0 & \langle D_1^*|\,V_4^{cc}\,|D_1^*\rangle & 0 & \langle D_1^*|\,V_4^{cc}\,|D_3^*\rangle\\
\langle D_2^*|\,V_4^{cc}\,|D_0^*\rangle & 0 & \langle D_2^*|\,V_4^{cc}\,|D_2^*\rangle & 0\\
0 & \langle D_3^*|\,V_4^{cc}\,|D_1^*\rangle & 0 & \langle D_3^*|\,V_4^{cc}\,|D_3^*\rangle
\end{pmatrix}
\begin{pmatrix} h_0^{D^*}(T_{\text{sep}}-t)\\ h_1^{D^*}(T_{\text{sep}}-t)\\ h_2^{D^*}(T_{\text{sep}}-t)\\ h_3^{D^*}(T_{\text{sep}}-t)\end{pmatrix}\\
&=\langle D_0^*|\,V_4^{cc}\,|D_0^*\rangle\; h_0^{D^*}(t)\,h_0^{D^*}(T_{\text{sep}}-t)+\langle D_1^*|\,V_4^{cc}\,|D_1^*\rangle\; h_1^{D^*}(t)\,h_1^{D^*}(T_{\text{sep}}-t)\\
&+\langle D_2^*|\,V_4^{cc}\,|D_2^*\rangle\; h_2^{D^*}(t)\,h_2^{D^*}(T_{\text{sep}}-t)+\langle D_3^*|\,V_4^{cc}\,|D_3^*\rangle\; h_3^{D^*}(t)\,h_3^{D^*}(T_{\text{sep}}-t)\\
&+\langle D_0^*|\,V_4^{cc}\,|D_2^*\rangle\left[h_0^{D^*}(t)\,h_2^{D^*}(T_{\text{sep}}-t)+h_2^{D^*}(t)\,h_0^{D^*}(T_{\text{sep}}-t)\right]\\
&+\langle D_1^*|\,V_4^{cc}\,|D_3^*\rangle\left[h_1^{D^*}(t)\,h_3^{D^*}(T_{\text{sep}}-t)+h_3^{D^*}(t)\,h_1^{D^*}(T_{\text{sep}}-t)\right]
\end{aligned}
$$

- We find 6 fit params, using $\langle D_i^*|\,V_4^{cc}\left|D_j^*\right\rangle^* = \left\langle D_j^*\right|V_4^{cc}\,|D_i^*\rangle\ (i\neq j)$.
- $h_a^{D^*}(t)$ and $h_b^{D^*}(T_{\text{sep}}-t)$ $(a,b=0,1,2,3)$ are given in the next page.

$D^* \to D^*$ 3pt correlator fit function (2)

For $D^* \to D^*$ 3pt $[2{+}2]_{D^*}$ $[2{+}2]_{D^*}$ fit, $h_a^{D^*}(T_{\text{sep}} - t)$ and $h_b^{D^*}(t)$ are

$$h_0^{D^*}(T_{\text{sep}} - t) = \sqrt{A_0^{D^*}}\, e^{-E_0^{D^*}(T_{\text{sep}}-t)}$$

$$h_1^{D^*}(T_{\text{sep}} - t) = \sqrt{A_0^{D^*} R_1^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_1^{D^*}\right)(T_{\text{sep}}-t)} (-1)^{T_{\text{sep}}-t+1}$$

$$h_2^{D^*}(T_{\text{sep}} - t) = \sqrt{A_0^{D^*} R_2^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_2^{D^*}\right)(T_{\text{sep}}-t)}$$

$$h_3^{D^*}(T_{\text{sep}} - t) = \sqrt{A_0^{D^*} R_1^{D^*} R_3^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_1^{D^*}+\Delta E_3^{D^*}\right)(T_{\text{sep}}-t)} (-1)^{T_{\text{sep}}-t+1}$$

$$h_0^{D^*}(t) = \sqrt{A_0^{D^*}}\, e^{-E_0^{D^*} t}$$

$$h_1^{D^*}(t) = \sqrt{A_0^{D^*} R_1^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_1^{D^*}\right) t} (-1)^{t+1}$$

$$h_2^{D^*}(t) = \sqrt{A_0^{D^*} R_2^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_2^{D^*}\right) t}$$

$$h_3^{D^*}(t) = \sqrt{A_0^{D^*} R_1^{D^*} R_3^{D^*}}\, e^{-\left(E_0^{D^*}+\Delta E_1^{D^*}+\Delta E_3^{D^*}\right) t} (-1)^{t+1}$$

where A_0^B, E_0^B, $\{R_j^B, \Delta E_j^B\}$ are input parameters determined from 2pt correlator fit.

3pt correlator fit (linear fit): $D^* \to D^*$

- For the least χ^2 fitting of 3pt correlator, we solve

$$\begin{pmatrix} g_1(t_\alpha) V_{\alpha\beta}^{-1} g_1(t_\beta) & g_1(t_\alpha) V_{\alpha\beta}^{-1} g_2(t_\beta) & g_1(t_\alpha) V_{\alpha\beta}^{-1} g_3(t_\beta) & \cdots \\ g_2(t_\alpha) V_{\alpha\beta}^{-1} g_1(t_\beta) & g_2(t_\alpha) V_{\alpha\beta}^{-1} g_2(t_\beta) & g_2(t_\alpha) V_{\alpha\beta}^{-1} g_3(t_\beta) & \cdots \\ g_3(t_\alpha) V_{\alpha\beta}^{-1} g_1(t_\beta) & g_3(t_\alpha) V_{\alpha\beta}^{-1} g_2(t_\beta) & g_3(t_\alpha) V_{\alpha\beta}^{-1} g_3(t_\beta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \langle D_0^*| V_4^{cc} |D_0^*\rangle \\ \langle D_0^*| V_4^{cc} |D_2^*\rangle \\ \langle D_1^*| V_4^{cc} |D_1^*\rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} g_1(t_\alpha) V_{\alpha\beta}^{-1} C(t_\beta) \\ g_2(t_\alpha) V_{\alpha\beta}^{-1} C(t_\beta) \\ g_3(t_\alpha) V_{\alpha\beta}^{-1} C(t_\beta) \\ \vdots \end{pmatrix}$$

  where $g_1(t) = h_0^{D^*}(t)\, h_0^{D^*}(T_{\text{sep}} - t)$, $g_2(t) = h_0^{D^*}(t)\, h_2^{D^*}(T_{\text{sep}} - t)$, $g_3(t) = h_1^{D^*}(t)\, h_1^{D^*}(T_{\text{sep}} - t)$, ... , $V_{\alpha\beta}^{-1} = V^{-1}(t_\alpha, t_\beta)$ is inversed covariance matrix, $C(t)$ is correlator data at t.
- This is came from

$$\frac{\partial \chi^2}{\partial \lambda_p} = \frac{\partial f(\lambda; t_\alpha)}{\partial \lambda_p} V_{\alpha\beta}^{-1} \{f(\lambda; t_\beta) - C(t_\beta)\} = g_p(t_\alpha) V_{\alpha\beta}^{-1} \{f(\lambda; t_\beta) - C(t_\beta)\} = 0$$

  where $f(\lambda; t_\alpha) = \sum_{p=1}^{6} \lambda_p g_p(t_\alpha)$ and $\lambda_1 = \langle D_0^*| V_4^{cc} |D_0^*\rangle$, $\lambda_2 = \langle D_0^*| V_4^{cc} |D_2^*\rangle$, etc.