Nucleon Electromagnetic Polarizabilities from Four-point Correlation Functions

Speaker: Xuan-He Wang

In collaboration with: Xu Feng, Luchang Jin

Peking University

August 4th, 2023
Motivations

• Polarizabilities are **fundamental** parameters of hadron structure

• Characterize the second-order response of a proton to an EM field,

\[ H^{(2)}_{\text{eff}} = -\frac{4\pi}{2} \alpha_E E^2 - \frac{4\pi}{2} \beta_M B^2 \]

• Our Method: calculating a proton **4-point** correlation function on lattice, can be used in multiple applications\(^1\), \(^2\).

---

\(^1\) Y Fu et al. Lattice QCD Calculation of the Two-Photon Exchange Contribution to the Muonic-Hydrogen Lamb Shift[J]. 2022  
\(^2\) P X Ma et al. Lattice Calculation of Electroweak Radiative Corrections of Neutron Beta Decays, arXiv: 2308.xxxx
Recent Researches on Polarizabilities

- Summary for electric polarizability $\alpha_E$ of the proton

Only one former Lattice result for proton $\alpha_E^p$

---

1 F Hagelstein, Nucleon Polarizabilities and Compton Scattering as Playground for Chiral Perturbation Theory[J]. 2020
Recent Researches on Polarizabilities

- Summary for electric polarizability $\alpha_E$ of the neutron\(^1\)

![Graph showing comparison between Lattice and $\chi$PT results for electric polarizability $\alpha_{E1n}$](image)

Lattice results seem lower than others?

- Calculate on physical pion mass
- Calculate with new method (4-point function)

---

\(^1\)F Hagelstein, Nucleon Polarizabilities and Compton Scattering as Playground for Chiral Perturbation Theory[J]. 2020
Doubly Virtual Compton Scattering

• Unpolarized doubly virtual Compton scattering

\[ T^{\mu\nu}(P, q) = T_{\text{Born}}^{\mu\nu} + \frac{8\pi M}{e^2} \left[ -\beta M K_1^{\mu\nu} + (\alpha_E + \beta M) K_2^{\mu\nu} \right] \]

\[ K_1^{\mu\nu} = q^\mu q^\nu - g^{\mu\nu} q^2 \]

\[ K_2^{\mu\nu} = \frac{1}{M^2} \left[ (P^\mu q^\nu + P^\nu q^\mu) P \cdot q - g^{\mu\nu} (P \cdot q)^2 - P^\mu P^\nu q^2 \right] \]

➢ Born term: Thomson scattering

➢ Polarizability terms: Rayleigh scattering

Intermediate states: \( N \)

Intermediate states: \( N^*, N\pi, \Delta, \cdots \)

• Polarizabilities: pure excited states contribution.
Extraction from 4-point Function

- Compton tensor
  \[ T^\mu\nu = \int d^4x \ e^{iqx} \langle N | J^\mu(x, t) J^\nu(0) | N \rangle = T^{\mu\nu}_{\text{Born}} + \frac{8\pi M}{e^2} \left[ -\beta M K_1^{\mu\nu} + (\alpha_E + \beta M) K_2^{\mu\nu} \right] \]

- With nucleon momentum \( P = (M, 0) \) and photon momentum \( q = (q_0, 0) \):
  \[
  \alpha_E = \frac{1}{3} \left( \frac{\partial T_{\mu\nu}}{\partial q_0^2} - \frac{\partial T_{\text{Born}}^{\mu\nu}}{\partial q_0^2} \right) \bigg|_{q_0 \to 0}
  \]
  \[
  = \frac{e^2}{4\pi} \left( 1 + \frac{\kappa^2}{4M^3} + \frac{r_E^2}{3M} \right) + \frac{e^2}{4\pi} \int_{|t|<t_s} d^4x \left( -\frac{t^2}{12M} \right) H_{ii}(x, t)
  \]

  \[ H_{ii}(x, t) = \langle N | J^i(x, t) J^i(0) | N \rangle \]

  ➢ For neutron: \( \alpha_E^{(n)} = \frac{e^2 \kappa_{(n)}^2}{4\pi 4M^3} + \frac{e^2}{4\pi} \int_{|t|<t_s} d^4x \left( -\frac{t^2}{12M} \right) H_{ii}^{(n)}(x, t) \)
Ensemble for Calculation

- Domain Wall Fermion ensemble generated by RBC/UKQCD\(^1\).
- Random field selection method\(^2,3\) is used.

<table>
<thead>
<tr>
<th>Ensembles</th>
<th>(m_\pi[\text{MeV}])</th>
<th>(m_\rho[\text{MeV}])</th>
<th>L/a</th>
<th>T/a</th>
<th>a[fm]</th>
<th>(N_{\text{conf}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>24D</td>
<td>141.7(2)</td>
<td>935(5)</td>
<td>24</td>
<td>64</td>
<td>0.1944</td>
<td>207</td>
</tr>
</tbody>
</table>

\(^2\) Y Li et al. Field sparsening for the construction of correlation functions in lattice QCD[J]. 2021.
4-point Function Calculated on Lattice

- Compton tensor extracted from nucleon 4-point function

\[ \langle p | J^\mu(x) J^\nu(0) | p \rangle \Rightarrow \langle N(t + \Delta t_1) J^\mu(t, x) J^\nu(0) \bar{N}(-\Delta t_2) \rangle \]

- Two nucleon operators and two vector current operators placed on different time slices, separated by \( \Delta t \) and \( t \).
Feynman Diagrams in 4-point Function
Examination of 4-point Function: Charge Radius

$$\langle N|J^0(x, t)J^0(0)|N\rangle \xrightarrow{\text{long distance}} \int \frac{d^3 Q}{(2\pi)^3} \frac{M(E + M)}{E} G_E^2(Q^2)e^{i px} e^{-(E-M)t}$$

$$\langle N|J^0(x, t)J^0(0)|N\rangle$$ as a function of $|x|$

Charge radius fitted at long distance $r > 1 fm$, with dipole model:

$$G_E(Q^2) = \frac{1}{1 + Q^2 \langle r_E^2 \rangle / 12}$$
Signal of 4-point Function i-i Component: \( H_{ii}(t, x) \)

\[
\alpha_E = \frac{e^2}{4\pi} \left( \frac{1 + \kappa^2}{4M^3} + \frac{\langle r_E^2 \rangle}{3M} \right) + \frac{e^2}{4\pi} \int_{|t|<t_s} d^4x \left( -\frac{t^2}{12M} \right) H_{ii}(x, t)
\]

\[\sum_x H(x, t) \] as a function of time separation \( t \)

Hadronic function mainly contribute in the region of \( t < 0.8 \) fm
Polarizability $\alpha_E$ from $H_{ii}(t, x)$

$$\alpha_E = \frac{e^2}{4\pi} \left( \frac{1 + \kappa^2}{4M^3} + \frac{\langle r_E^2 \rangle}{3M} \right) + \frac{e^2}{4\pi} \int_{|t|<t_s} d^4x \left( -\frac{t^2}{12M} \right) H_{ii}(x, t)$$

- $\sum_x H(x, t)$ as a function of time separation $t$

However, lattice predictions are significantly below the PDG value. Why?
Polarizability $\alpha_E$ from

$$\alpha_E = \frac{e^2}{4\pi} \left( \frac{1 + \kappa^2}{4M^3} + \left\langle \frac{r_E^2}{3M} \right\rangle \right) + \frac{e^2}{4\pi} \int_{|t|<} \sigma(x, t)$$

$\Sigma_x H(x, t)$ as a function of time sep:

However, lattice predictions are significantly below the PDG value. Why?
Nucleon polarizabilities and $N\pi$ scattering

Structure of hadronic function

$$
\int d^4x \left(-\frac{t^2}{6}\right) H_{ii}(x, t) = \int dt \left(-\frac{t^2}{6}\right) \sum_k \langle p|\psi_i(0)|k\rangle e^{-(E_k-M)t} \langle k|\psi_i(0)|p\rangle \\
= \frac{2}{3} \sum_k \frac{\langle p|\psi_i(0)|k\rangle\langle k|\psi_i(0)|p\rangle}{(E_k-M)^3}
$$

The dominant contribution is given by $|k\rangle = |N\pi\rangle$ ground intermediate states

$N\pi$ contribution seems negligible and is completely hidden by noise at long distance
Nucleon polarizabilities and \(N\pi\) scattering

Structure of hadronic function

\[
\int d^4x \left( -\frac{t^2}{6} \right) H_{ii}(x, t) = \int dt \left( -\frac{t^2}{6} \right) \sum_k \langle p | J_i(0) | k \rangle e^{-(E_k - M)t} \langle k | J_i(0) | p \rangle
\]

\[
= -\frac{2}{3} \sum_k \langle p | J_i(0) | k \rangle \langle k | J_i(0) | p \rangle \frac{1}{(E_k - M)^3}
\]

The dominant contribution is given by \(|k\rangle = |N\pi\rangle\) ground intermediate states

\[\text{Short distance:} \quad \text{High excited states} \]

\[\text{Long distance:} \quad |N\pi\rangle \text{ state} \]

\(N\pi\) contribution significantly enhanced by factor of \(t^2\)  

Need to calculate \(N\pi\) rescattering on lattice
Feynman Diagrams of $N\pi$ Rescattering

$I = 1/2$: $O_{N\pi}^{l_3=+\frac{1}{2}} = 0_p O_{\pi^0} - \sqrt{2} 0_n O_{\pi^+}$,  $O_{N\pi}^{l_3=-\frac{1}{2}} = \sqrt{2} 0_n O_{\pi^0} - 0_p O_{\pi^-}$

$I = 3/2$: $O_{N\pi}^{l_3=+\frac{1}{2}} = \sqrt{2} 0_p O_{\pi^0} + 0_n O_{\pi^+}$,  $O_{N\pi}^{l_3=-\frac{1}{2}} = 0_n O_{\pi^0} + \sqrt{2} 0_p O_{\pi^-}$
Results of $N\pi$ Scattering

- $N\pi$ scattering for $I_3 = +1/2$
  
  (similar for $I_3 = -1/2$ case)

  $$R = \frac{C_2^{N\pi}(t)}{C_2^N(t)C_2^\pi(t)} = \frac{A_{N\pi}}{A_N A_{\pi}} e^{-E_{N\pi}t} e^{-(M_N + M_{\pi})t} \approx R_0 (1 - \Delta E t)$$

  with $\Delta E = E_{N\pi} - M_N - M_{\pi}$

- Scattering for different isospin channel
  
  - $I = 1/2, \Delta E < 0$, attractive interaction
  - $I = 3/2, \Delta E > 0$, repulsive interaction

Preliminary

This work:

$\frac{1}{2} m_\pi = 0.101(45), a_0^{3/2} m_\pi = -0.131(26)$

Phenomenology$^1$:

$\frac{1}{2} m_\pi = 0.1699(20), a_0^{3/2} m_\pi = -0.0865(18)$

$< 2\sigma$ consistent

Summary

• All excited states calculated with 4-point correlation function, but the lowest $N\pi$ state contribution need to be calculated separately.

• Lattice QCD calculation of $N\gamma \rightarrow N\pi$ is necessary.

• Various other projects contains $N\pi$ scattering, like pion photoproduction…….