Chiral condensate from the spectrum of the staggered Dirac operator

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Based on the work in collaboration with:
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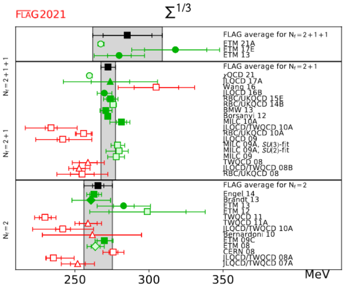
Two LECs for LO of ChPT with two light flavours

$$\Sigma \equiv - \lim_{m_u, m_d \to 0} \langle \overline{u} u \rangle \, ; \qquad F_\pi \equiv \lim_{m_u, m_d \to 0} \frac{1}{M_\pi} \langle \Omega | \, \overline{u} \gamma_4 \gamma_5 d \, | \pi(\vec{p} = \vec{0}) \rangle$$

Content

- Determination of Σ in $N_f = 2+1$ QCD with staggered fermions, by using the mode number approach based on the Banks–Casher relation [L. Giusti, M. Lüscher (2009)]
- Check of our computation by means of methods based on the pion mass M_π and the topological susceptibility $\chi = \langle Q^2 \rangle / V$



Plot taken from [FLAG Review, 2021]

Lattice setup: $N_f = 2+1$ QCD on different LCPs, tree-level Symanzik-improved action (gauge), rooted stout staggered Dirac operator

Simulation parameters: β, $m_u = m_d \equiv m_l$, m_s

- Starting point on physical LCP [Y. Aoki *et al.* (2009); S. Borsányi *et al.* (2010, 2014)]

$$(\beta^{(\mathrm{phys})}, m_l^{(\mathrm{phys})}, m_s^{(\mathrm{phys})}) \to M_\pi^{(\mathrm{phys})} \simeq 135 \text{ MeV}, \; R \equiv \frac{m_l}{m_s} = R^{(\mathrm{phys})} \simeq \frac{1}{28.15}$$

- Other LCPs used

$$(\beta^{(\mathrm{phys})}, m_l^{(\mathrm{phys})}, m_s^{(\mathrm{phys})}) \longrightarrow (\beta^{(\mathrm{phys})}, m_l, m_s^{(\mathrm{phys})}) \qquad m_l = 4, 6, 9 \; m_l^{(\mathrm{phys})}$$

- m_s kept at the physical point for all LCPs
- Scale setting with w_0 parameter based on Gradient Flow [S. Borsányi *et al.* (2012)], appropriate staggered correlator for M_π

Σ from the mode number - 1

Banks-Casher relation

Low-lying part of the spectrum of the Dirac operator $\rightleftarrows$ chiral condensate Σ

$$\lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda, m) = \frac{\Sigma}{\pi}$$

$\rho(\lambda, m) \to$ spectral density

- The mode number is a more convenient quantity to work with

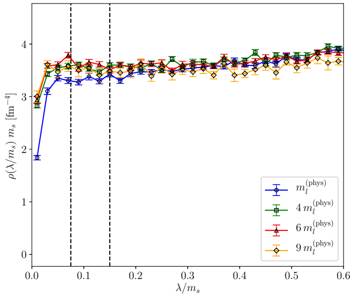
$$\langle\nu(M)\rangle = V\int_{-M}^{M}\rho(\lambda, m)d\lambda = \frac{2}{\pi}V\Sigma M$$

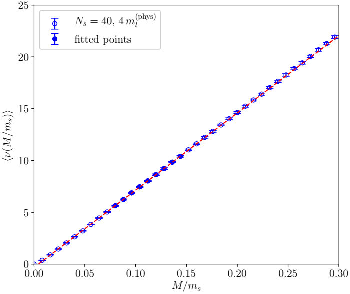
- Staggered mode number $\langle\nu_{\text{stag}}\rangle$ has to be divided by the number of tastes: $\langle\nu(M)\rangle = \langle\nu_{\text{stag}}(M)\rangle/4$
- Since $\langle\nu(M)\rangle$, M/m_s and Σm_s are RG-invariant quantities, we can rewrite the previous Eq. in the following manifestly-RG-invariant way:

$$\langle\nu(M/m_s)\rangle = \frac{\langle\nu_{\text{stag}}(M/m_s)\rangle}{4} = \frac{2}{\pi}V[\Sigma m_s]\left(\frac{M}{m_s}\right)$$

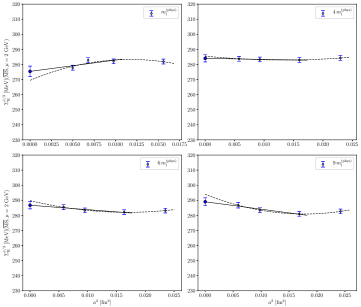
Σ from the mode number - 2

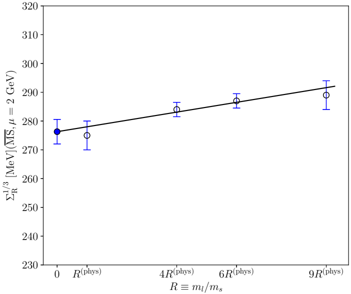
- Idea: look at the spectral density for a common plateau → linearity in $\langle \nu(M) \rangle$
- Same fit range for all LCPs to properly perform continuum + chiral limit
- Fit range: $M/m_s \in [0.075, 0.150]$
- Σ_R [MeV] from Σm_s using $m_s = 92.4(1.5)$ MeV ($N_f = 2+1$ QCD, stag. fermions, $\overline{\text{MS}}$ ren. scheme, $\mu = 2$ GeV) [C.T.H. Davies *et al.* (2010)]





- Continuum limit at fixed $R \to \Sigma_{\mathrm{R}}^{1/3}(a,R) = \Sigma_{\mathrm{R}}^{1/3}(R) + c_1(R)\,a^2 + o(a^2)$
- Chiral limit $\to \Sigma_{\mathrm{R}}^{1/3}(R) = \Sigma_{\mathrm{R}}^{1/3} + c_2\,R + o(R)$





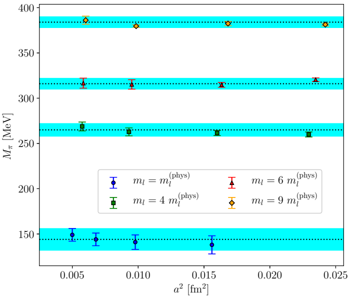
Final result from $\langle\nu\rangle$: $\Sigma_{\mathrm{R}}^{1/3} = 276(4)$ MeV

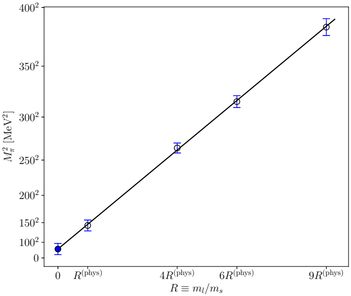
Σ from M_π

- Gell-Mann–Oakes–Renner relation $(m_u = m_d \equiv m_l)$

$$M_\pi^2 = 2\frac{\Sigma}{F_\pi^2} m_l = 2\left(\frac{\Sigma m_s}{F_\pi^2}\right) R$$

- $F_\pi = 87.6(6.5)$ MeV (computed in this work)

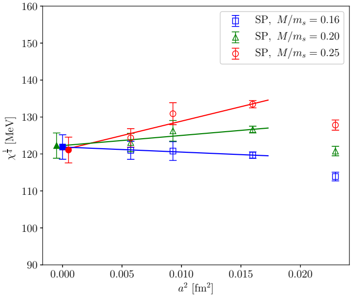


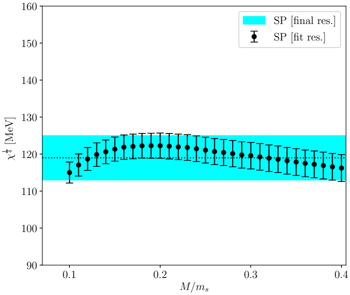


Final result from M_π: $\Sigma_{\mathrm{R}}^{1/3} = 267(15)$ MeV

Σ from the topological susceptibility - 1

- χ computed with spectral projectors (SP) definitions [L. Giusti, M. Lüscher (2009); C. Alexandrou *et al.* (2018); C. Bonanno *et al.* (2019); A. Athenodorou *et al.* (2022)]
- SP definition → more control on systematics

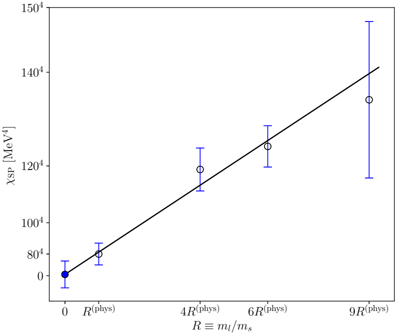




Σ from the topological susceptibility - 2

LO ChPT prediction for χ $(m_u = m_l \equiv m_l)$ $\rightarrow \chi = \frac{1}{2}\Sigma m_l = \frac{1}{2}[\Sigma m_s]R$

- SP point at $R^{(\text{phys})}$ from [A.Athenodorou *et al.* (2022)]
- results well described by a linear function in R
- vanishing chiral limit for χ within errors
- $m_s = 92.4(1.5)$ MeV [C.T.H. Davies *et al.* (2010)]



Final result from χ: $\Sigma_{\text{R}}^{1/3} = 297(20)$ MeV

Conclusions

- We have addressed the computation of the SU(2) chiral condensate from a staggered discretization of $N_f = 2+1$ QCD
- We apply the Giusti-Lüscher method based on the Banks–Casher relation to the staggered case
- We obtain agreeing results with the ones based on the pion mass and the topological susceptibility

	$\Sigma_R^{1/3}$ [MeV]
$\langle\nu\rangle$	276(4)
M_π	267(15)
χ	297(20)

Global fit of $\Sigma_R^{1/3}(R)$, $M_\pi^2(R)$, $\chi(R)$, imposing a single fit parameter for $\Sigma_R^{1/3}$, provides

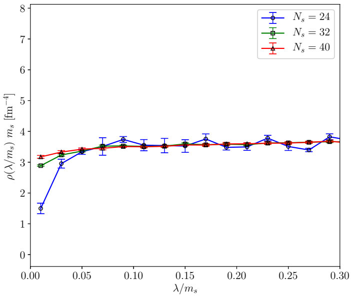
$$\boxed{\Sigma_R^{1/3} = 270(3)\ \text{MeV}}$$

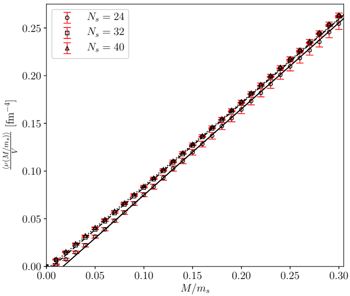
Agreement with world-average ($N_f = 2+1$ QCD) of latest FLAG review: $\Sigma_{\text{FLAG}}^{1/3} = 272(5)$ MeV

Backup slides

Σ from the mode number - Finite Size Effects

- Simulations at $m_l = 4\ m_l^{(\text{phys})}$, $\beta = 3.868$, $a = 0.0964$ fm, $N_s = 24,\ 32,\ 40$ ($M_0\ell = 3,\ 4,\ 5$) where $M_0 \simeq 2M_\pi^{(\text{phys})}$, $\ell = N_s a$
- $\Sigma_{\text{R}}^{1/3} = 285(2), 283(2), 283(2)$ MeV, for, respectively, $M_0\ell = 3, 4, 5$ in the fit range $M/m_s \in [0.075, 0.150]$





Computation of F_π - 1

Let $P(t) = \sum_{\vec{x}} P(t, \vec{x})$ be the staggered interpolating operator of the physical pion, we expect that

$$\boxed{C_\pi(t) = \langle P(t) P(0) \rangle \underset{t\to\infty}{\sim} A_\pi \left[e^{-M_\pi t} + e^{-M_\pi(\ell - t)} \right]}$$

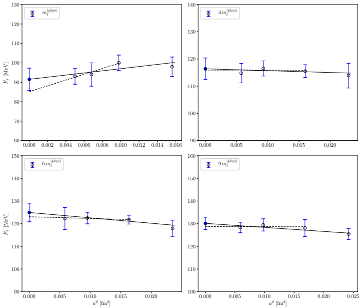
where

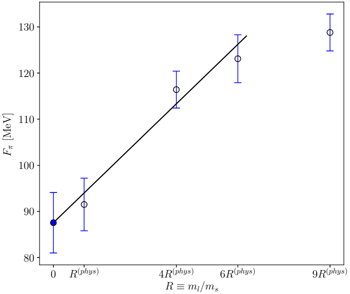
$$\boxed{A_\pi = |\, \langle \Omega |\, \overline{u} \gamma_5 d \,| \pi(\vec{p} = \vec{0}) \rangle \,|^2 / M_\pi .}$$

F_π definition + PCAC relation [S. Borsányi *et al.* (2013)]

$$\boxed{F_\pi = m_l \sqrt{\frac{A_\pi}{M_\pi^3}} = m_s R \sqrt{\frac{A_\pi}{M_\pi^3}}}$$

- Continuum limit at fixed $R \to F_\pi(a, R) = F_\pi(R) + b_1(R)\, a^2 + o(a^2)$
- Chiral limit $\to F_\pi(R) = F_\pi + b_2\, R + o(R)$





Final result: $F_\pi = 87.6(6.5)$ MeV

Spectral Projectors

Lattice version of index theorem for staggered fermions ($\mathbb{P}_M$ is the projector on the space spanned by the eigenstates of $\not{D}_{stag}$ with eigenvalues $|\lambda| \leq M$):

$$Q^{(\text{stag})}_{\text{SP, bare}} = \frac{1}{2^{d/2}}\,\text{Tr}\{\Gamma_5 \mathbb{P}_M\} = \frac{1}{2^{d/2}} \sum_{|\lambda_k| \leq M} u_k^\dagger \Gamma_5 u_k; \qquad \not{D}_{stag} u_k = i\lambda_k u_k$$

Multiplicative renormalization of $Q^{(\text{stag})}_{\text{SP, bare}}$ [C. Bonanno *et al.* (2019)]:

$$Q^{(\text{stag})}_{\text{SP}} = Z_{\text{SP}}\, Q^{(\text{stag})}_{\text{SP, bare}} = \sqrt{\frac{\langle \text{Tr}\{\mathbb{P}_M\}\rangle}{\langle \text{Tr}\{\Gamma_5 \mathbb{P}_M \Gamma_5 \mathbb{P}_M\}\rangle}}\, Q^{(\text{stag})}_{\text{SP, bare}}$$

Topological susceptibility: $\chi_{\text{SP}} = \frac{1}{V}\left\langle \left(Q^{(\text{stag})}_{\text{SP}}\right)^2 \right\rangle$