Chiral condensate from the spectrum of the staggered Dirac operator

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Based on the work in collaboration with:
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Introduction

Two LECs for LO of ChPT with two light flavours

\[ \Sigma \equiv - \lim_{m_u, m_d \to 0} \langle \bar{u}u \rangle ; \quad F_\pi \equiv \lim_{m_u, m_d \to 0} \frac{1}{M_\pi} \langle \Omega | \bar{u}\gamma_4\gamma_5 d | \pi (\vec{p} = \vec{0}) \rangle \]

Content

- Determination of \( \Sigma \) in \( N_f = 2 + 1 \) QCD with staggered fermions, by using the mode number approach based on the Banks–Casher relation [L. Giusti, M. Lüscher (2009)]

- Check of our computation by means of methods based on the pion mass \( M_\pi \) and the topological susceptibility \( \chi = \langle Q^2 \rangle / V \)

Plot taken from [FLAG Review, 2021]
Determination of LCPs

Lattice setup: $N_f = 2 + 1$ QCD on different LCPs, tree-level Symanzik-improved action (gauge), rooted stout staggered Dirac operator

Simulation parameters: $\beta$, $m_u = m_d \equiv m_l$, $m_s$

- Starting point on physical LCP [Y. Aoki et al. (2009); S. Borsányi et al. (2010, 2014)]

\[
(\beta^{(\text{phys})}, m_l^{(\text{phys})}, m_s^{(\text{phys})}) \rightarrow M_{\pi}^{(\text{phys})} \simeq 135 \text{ MeV}, \quad R \equiv \frac{m_l}{m_s} = R^{(\text{phys})} \simeq \frac{1}{28.15}
\]

- Other LCPs used

\[
(\beta^{(\text{phys})}, m_l^{(\text{phys})}, m_s^{(\text{phys})}) \rightarrow (\beta^{(\text{phys})}, m_l, m_s^{(\text{phys})}) \quad m_l = 4, 6, 9 \quad m_l^{(\text{phys})}
\]

- $m_s$ kept at the physical point for all LCPs

- Scale setting with $w_0$ parameter based on Gradient Flow [S. Borsányi et al. (2012)], appropriate staggered correlator for $M_\pi$
Banks-Casher relation

Low-lying part of the spectrum of the Dirac operator \( \leftrightarrow \) chiral condensate \( \Sigma \)

\[
\lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda, m) = \frac{\Sigma}{\pi}
\]

\( \rho(\lambda, m) \to \) spectral density

- The **mode number** is a more convenient quantity to work with

\[
\langle \nu(M) \rangle = V \int_{-M}^{M} \rho(\lambda, m) d\lambda = \frac{2}{\pi} V \Sigma M
\]

- Staggered mode number \( \langle \nu_{\text{stag}} \rangle \) has to be divided by the number of tastes: \( \langle \nu(M) \rangle = \langle \nu_{\text{stag}}(M) \rangle / 4 \)

- Since \( \langle \nu(M) \rangle \), \( M/m_s \) and \( \Sigma m_s \) are RG-invariant quantities, we can rewrite the previous Eq. in the following manifestly-RG-invariant way:

\[
\langle \nu(M/m_s) \rangle = \frac{\langle \nu_{\text{stag}}(M/m_s) \rangle}{4} = \frac{2}{\pi} V [\Sigma m_s] \left( \frac{M}{m_s} \right)
\]
Σ from the mode number - 2

- **Idea:** look at the spectral density for a common plateau → linearity in $\langle \nu(M) \rangle$
- Same fit range for all LCPs to properly perform **continuum + chiral limit**
- **Fit range:** $M/m_s \in [0.075, 0.150]$
- $\Sigma_R$ [MeV] from $\Sigma m_s$ using $m_s = 92.4(1.5)$ MeV ($N_f = 2 + 1$ QCD, stag. fermions, $\overline{\text{MS}}$ ren. scheme, $\mu = 2$ GeV) [C.T.H. Davies *et al.* (2010)]

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![Graph](image_url)
Continuum limit at fixed $R \rightarrow \Sigma^{1/3}_R(a, R) = \Sigma^{1/3}_R(R) + c_1(R) a^2 + o(a^2)$

Chiral limit $\rightarrow \Sigma^{1/3}_R(R) = \Sigma^{1/3}_R + c_2 R + o(R)$

Final result from $\langle \nu \rangle$: $\Sigma^{1/3}_R = 276(4) \text{ MeV}$
Gell-Mann–Oakes–Renner relation \((m_u = m_d \equiv m_l)\)

\[
M_\pi^2 = 2 \frac{\Sigma}{F_\pi^2} m_l = 2 \left( \frac{\Sigma m_s}{F_\pi^2} \right) R
\]

\(F_\pi = 87.6(6.5)\) MeV (computed in this work)

Final result from \(M_\pi\): \(\Sigma_R^{1/3} = 267(15)\) MeV
- \( \chi \) computed with spectral projectors (SP) definitions [L. Giusti, M. Lüscher (2009); C. Alexandrou et al. (2018); C. Bonanno et al. (2019); A. Athenodorou et al. (2022)]

- SP definition → more control on systematics
LO ChPT prediction for $\chi$ ($m_u = m_l, m_l \equiv m_l$) → $\chi = \frac{1}{2} \Sigma m_l = \frac{1}{2} \left[ \Sigma m_s \right] R$

- SP point at $R^{(\text{phys})}$ from [A.Athenodorou et al. (2022)]
- results well described by a linear function in $R$
- vanishing chiral limit for $\chi$ within errors
- $m_s = 92.4(1.5) \text{ MeV}$ [C.T.H. Davies et al. (2010)]

Final result from $\chi$: $\Sigma_R^{1/3} = 297(20) \text{ MeV}$
Conclusions

- We have addressed the computation of the SU(2) chiral condensate from a staggered discretization of $N_f = 2 + 1$ QCD
- We apply the Giusti-Lüscher method based on the Banks–Casher relation to the staggered case
- We obtain agreeing results with the ones based on the pion mass and the topological susceptibility

<table>
<thead>
<tr>
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<th>$\Sigma_{R}^{1/3}$ [MeV]</th>
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<tbody>
<tr>
<td>$\langle \nu \rangle$</td>
<td>276(4)</td>
</tr>
<tr>
<td>$M_\pi$</td>
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</tr>
<tr>
<td>$\chi$</td>
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Global fit of $\Sigma_{R}^{1/3}(R)$, $M_\pi^2(R)$, $\chi(R)$, imposing a single fit parameter for $\Sigma_{R}^{1/3}$, provides

$$\Sigma_{R}^{1/3} = 270(3) \text{ MeV}$$

Agreement with world-average ($N_f = 2 + 1$ QCD) of latest FLAG review:

$$\Sigma_{\text{FLAG}}^{1/3} = 272(5) \text{ MeV}$$
Backup slides
Simulations at $m_l = 4 \, m_l^{(\text{phys})}$, $\beta = 3.868$, $a = 0.0964$ fm, $N_s = 24, 32, 40$ ($M_0 \ell = 3, 4, 5$) where $M_0 \simeq 2M_\pi^{(\text{phys})}$, $\ell = N_s a$

- $\Sigma^{1/3}_R = 285(2), 283(2), 283(2)$ MeV, for, respectively, $M_0 \ell = 3, 4, 5$ in the fit range $M/m_s \in [0.075, 0.150]$
Let \( P(t) = \sum_{\vec{x}} P(t, \vec{x}) \) be the staggered interpolating operator of the physical pion, we expect that

\[
C_\pi(t) = \langle P(t)P(0) \rangle_{t \to \infty} \sim A_\pi \left[ e^{-M_\pi t} + e^{-M_\pi(\ell-t)} \right]
\]

where

\[
A_\pi = \left| \langle \Omega | \bar{u} \gamma_5 d | \pi (\vec{p} = \vec{0}) \rangle \right|^2 / M_\pi.
\]

\( F_\pi \) definition + PCAC relation [S. Borsányi et al. (2013)]

\[
F_\pi = m_l \sqrt{\frac{A_\pi}{M_\pi^3}} = m_s R \sqrt{\frac{A_\pi}{M_\pi^3}}
\]
Computation of $F_\pi - 2$

- Continuum limit at fixed $R \to F_\pi(a, R) = F_\pi(R) + b_1(R) a^2 + o(a^2)$
- Chiral limit $\to F_\pi(R) = F_\pi + b_2 R + o(R)$

**Final result:** $F_\pi = 87.6(6.5) \text{ MeV}$
Lattice version of index theorem for staggered fermions ($\mathbb{P}_M$ is the projector on the space spanned by the eigenstates of $\mathcal{D}_{stag}$ with eigenvalues $|\lambda| \leq M$):

$$Q_{SP, \text{bare}}^{stag} = \frac{1}{2^{d/2}} \text{Tr}\{\Gamma_5 \mathbb{P}_M\} = \frac{1}{2^{d/2}} \sum_{|\lambda_k| \leq M} u_k^\dagger \Gamma_5 u_k; \quad \mathcal{D}_{stag} u_k = i\lambda_k u_k$$

Multiplicative renormalization of $Q_{SP, \text{bare}}^{stag}$ [C. Bonanno et al. (2019)]:

$$Q_{SP}^{stag} = Z_{SP} Q_{SP, \text{bare}}^{stag} = \sqrt{\frac{\langle \text{Tr}\{\mathbb{P}_M\} \rangle}{\langle \text{Tr}\{\Gamma_5 \mathbb{P}_M \Gamma_5 \mathbb{P}_M\} \rangle}} Q_{SP, \text{bare}}^{stag}$$

Topological susceptibility: $\chi_{SP} = \frac{1}{V} \left\langle \left( Q_{SP}^{stag} \right)^2 \right\rangle$