The pion scalar form factor with $N_f = 2+1$ Wilson fermions

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Introduction

The pion scalar form factor and scalar radius

$$F_S^{\pi,l}(Q^2) = \langle \pi(p_f)| m_d \bar{d}d + m_u \bar{u}u |\pi(p_i)\rangle\rangle \,, \qquad \langle r_S^2\rangle_\pi^l = \frac{-6}{F_S^{\pi,l}(0)} \left. \frac{dF_S^{\pi,l}(Q^2)}{dQ^2} \right|_{Q^2=0} ,$$

are not directly accessible to experiment. However, the radius is relevant to the low-energy regime of QCD, i.e.

$$\frac{F_\pi}{\mathring{F}_\pi} = 1 + \frac{1}{6} M_\pi^2 \langle r_S^2\rangle_\pi^l + \frac{13 M_\pi^2}{192\pi^2 F_\pi^2} + \mathcal{O}(M_\pi^4) \,,$$

and

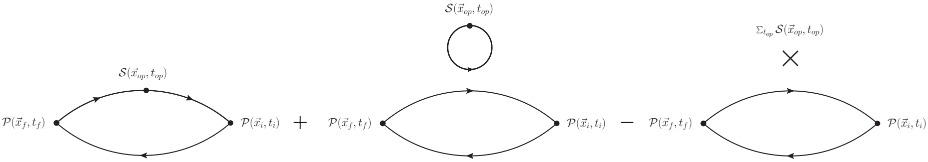
$$\langle r_S^2\rangle_\pi^l = \frac{1}{(4\pi f)^2} \left[-\frac{13}{6} + \left(6\bar{l}_4 + \log\left(\frac{M^2_{\pi,\mathrm{phys}}}{M_\pi^2} \right) \right) \right] .$$

Annals Phys. 158, 142 (1984)

- Only few lattice calculations exist; arguably none with fully controlled systematics.
  PRD 80 (2009) 034508 PRD 89 (2014) 9, 094503 PRD 93 (2016) 5, 054503 PRD 105, 054502 (2022)
- Even less is known for SU(3) χPT from the lattice.
- Scalar form factor itself typically only computed for very few momenta.
- Extraction of radius requires high momentum resolution.
- Calculation **computationally very demanding** due to significant quark-disconnected contribution.

Pion scalar form factor in lattice QCD

Study of the pion scalar form factor requires quark-connected and disconnected three-point functions:



where $\mathcal{P}(x) = u(x)\gamma_5 \bar{d}(x)$ and we make three choices for the scalar insertion:

$$\begin{aligned}
\mathcal{S}_l &= \bar{u}u + \bar{d}d && \rightarrow && F_S^{\pi,l}(Q^2) && \text{(light only)} \\
\mathcal{S}_8 &= \bar{u}u + \bar{d}d - 2\bar{s}s && \rightarrow && F_S^{\pi,8}(Q^2) && \text{(octet)} \\
\mathcal{S}_0 &= \bar{u}u + \bar{d}d + \bar{s}s && \rightarrow && F_S^{\pi,0}(Q^2) && \text{(singlet)}
\end{aligned}$$

- Basic building blocks to compute are **1pt**, **2pt** and **3pt** functions.
- Contributions from quark-disconnected diagrams can be large at small Q^2 (up to $\gtrsim$ 100% of conn piece).
- Last diagram (*vacuum expectation value*) only contributes at $Q^2 = 0$.
- Evaluation of 2+1 quark-disconnected diagram is very expensive (loops + large two-point statistics)

Quark-connected two- and three-point functions

Setup for the computation of quark-connected two-point and three-point function

$$C^{\rm 2pt}_{\mathcal{O}_1\mathcal{O}_2}(\vec{p},\vec{x}_i,t_f-t_i)=\sum_{\vec{x}_f} e^{i\vec{p}\cdot(\vec{x}_f-\vec{x}_i)}\langle \mathcal{O}_1(\vec{x}_f-\vec{x}_i,t_f-t_i)\mathcal{O}_2^\dagger(\vec{0},0)\rangle\,,$$

$$C^{\rm 3pt}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}(\vec{p}_f,\vec{q},\vec{x}_i,t_{\rm sep},t_{\rm ins})=\sum_{\vec{x}_f,\vec{x}_{op}} e^{i\vec{p}_f\cdot(\vec{x}_f-\vec{x}_i)}e^{i\vec{q}\cdot(\vec{x}_{op}-\vec{x}_i)}\langle \mathcal{O}_1(\vec{x}_f-\vec{x}_i,t_{\rm sep})\mathcal{O}_2(\vec{x}_{op}-\vec{x}_i,t_{\rm ins})\mathcal{O}_3^\dagger(\vec{0},0)\rangle\,.$$

where $t_{\rm sep}=t_f-t_i$ and $t_{\rm ins}=t_{op}-t_i$:

- Re-use point-to-all forward propagators in two- and three-point functions.
- Sequential sink method for three-point functions with $1\,{\rm fm}\lesssim t_{\rm sep}\lesssim 3.5\,{\rm fm}$ and $\vec{p}_f\in\{(0,0,0),\ (1,0,0)\}$.
- Truncated solver method gives speedup of 2-5. *Phys.Rev. D91 (2015) 11, 114511*
- For 2+1 disconnected diagrams compute additional two-point functions (currently 128-140 per config).
- **On periodic BC boxes:**
  - Sources randomy distributed; 8 sources for three-point functions per config.
- **On open BC boxes:**
  - Source setup symmetric around $T/2$ $\Rightarrow$ 4+4 sources for three-point functions at each $t_{\rm sep}$.
  - Additional two-point functions on the same timeslices.

Quark-disconnected contribution

Evaluation of quark-disconnected loops for some operator $\mathcal{O}_f(\vec{x}, t)$ (required here: $\mathcal{S}_{l,s}, \vec{x}, t)$)

$$L_{\mathcal{O}_f}(\vec{p}, t) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \langle \mathcal{O}_f(\vec{x}, t) \rangle_F \,,$$

carried out using **stochastic all-to-all propagators** and the following techniques:

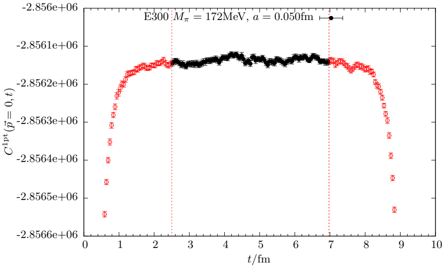
- One-end trick / frequency splitting to compute $L_1 - L_2, \ldots, L_{n-1} - L_n$, for $m_1 < m_2 < \ldots < m_n$
  EPJ C58, 261 (2008) *EPJ C79, 586 (2019)*
- Hopping parameter expansion + hierarchical probing for the heaviest quark, i.e. L_n
  PRD 89, 094503 (2014) *arXiv:1302.4018 [hep-lat]*

→ **Reach gauge noise level for all (ultra-)local and one-link displaced operators.**

- Statistical precision for $2+1$ diagrams limited only by two-point function measurements.
- For periodic BC: foward + backward averaging for all sources used for two-point functions.
- For open BC: Must stay away from boundary!

  Exclude timeslices on both sides: $t_{ex} \lesssim 2.5\,\mathrm{fm}$

Very challenging as statistics can be (severely) reduced depending on T, t_{sep} and t_{ex}.



Ensembles

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- $N_f = 2+1$ flavors of non-perturbatively improved Wilson clover fermions. *JHEP 1502 (2015) 043*
- Lüscher-Weisz gauge action *Commun.Math.Phys. 97 (1985)*
- Twisted mass regulator to suppress exceptional configurations. *PoS LATTICE2008 (2008) 049*
- Production is still ongoing: Additional ensembles + higher statistics planned.
- In particular, statistics for two-point functions for 2+1 diagrams may be increase by another factor 2–4.

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- **Ensembles cover four values of the lattice spacing a**
- Many different physical volumes with $L \approx 2.7 \ldots 6.2\,\mathrm{fm}$, $M_\pi L \gtrsim 4$.
- Pion masses range from $\sim 130\,\mathrm{MeV}$ to $\sim 350\,\mathrm{MeV}$
- Two very large and fine boxes at (near) physical quark mass.
  $\rightarrow$ High momentum resolution at small Q^2

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- **Measurements on $N_\mathrm{conf} \geq 500$ configurations on each ensemble.**

  However: Available configurations not yet exhausted on all ensembles.

  $\rightarrow$ Plan to double gauge statistics on **E250**, **E300** and **D200** in the near future.

- $N_\mathrm{meas}^\mathrm{3pt}$ always correponds to eight measurements per configuration.

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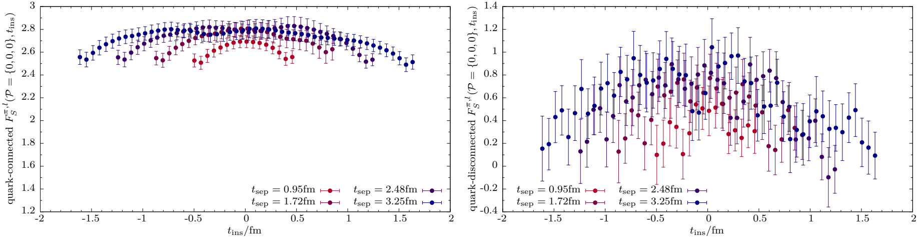
Effective form factor at $Q^2 = 0$



Light quark-connected and quark-disconnected effective form factors on E250 ($M_\pi = 132\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$); momentum labels: $\mathcal{P} \equiv (p_f^2, q^2, p_i^2)$

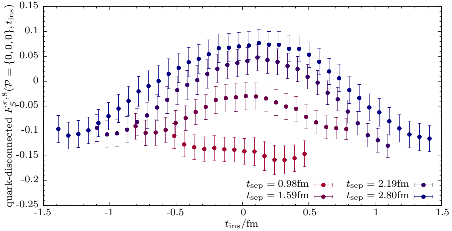
In a first step we use the **ratio method** to compute effective form factor

$$R(p_f^2, q^2, p_i^2, t_f - t_i, t_{op} - t_i) = \frac{C_3(p_f^2, q^2, p_i^2, t_f - t_i, t_{op} - t_i)}{C_2(p_f^2, t_f - t_i)} \sqrt{\frac{C_2(p_i^2, t_f - t_{op}) C_2(p_f^2, t_{op} - t_i) C_2(p_f^2, t_f - t_i)}{C_2(p_f^2, t_f - t_{op}) C_2(p_i^2, t_{op} - t_i) C_2(p_i^2, t_f - t_i)}}.$$

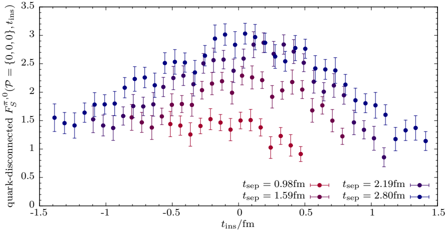
$\Rightarrow$ ground state matrix elements $\langle \pi(p_f^2) | S_f(q^2) | \pi(p_i^2) \rangle \sim F_S^{\pi,f}(Q^2)$ for $t_{op} - t_i \to \infty$ and $t_f - t_{op} \to \infty$.

- Quark-disconnected contribution very large at small Q^2. (Up to size of connected contribution!).
- Error entirely dominated by disconnected piece.

Octet vs. singlet contribution



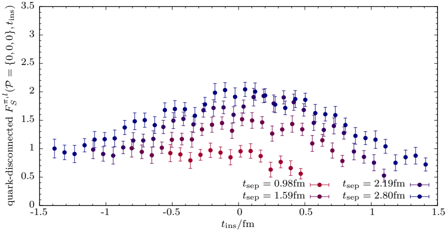
Octet quark-disconnected contribution on N451 ($M_\pi = 287\,\mathrm{MeV}$, $a = 0.076\,\mathrm{fm}$, $Q^2 = 0$)



Singlet quark-disconnected contribution on N451 ($M_\pi = 287\,\mathrm{MeV}$, $a = 0.076\,\mathrm{fm}$)

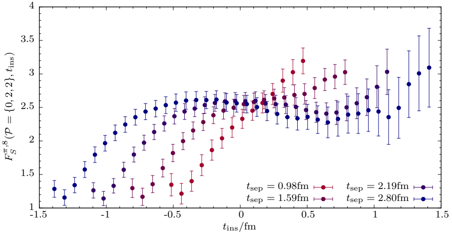
- Octet ($l - s$) combination is statistical very precise .
- Much cleaner signal than singlet contribution.
- Order of magnitude difference in size.
- Expected “hierarchy” for scalar radii:

$$\langle r_S^2 \rangle_\pi^8 < \langle r_S^2 \rangle_\pi^l < \langle r_S^2 \rangle_\pi^0$$

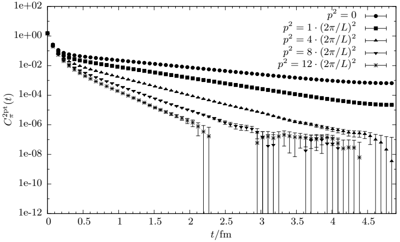


Light quark-disconnected contribution on N451 ($M_\pi = 287\,\mathrm{MeV}$, $a = 0.076\,\mathrm{fm}$)

Effective form factor at $Q^2 \neq 0$



Unrenormalized eff. form factor on D450 ($M_\pi = 218\,\mathrm{MeV}$, $a = 0.076\,\mathrm{fm}$)
momentum labels: $\mathcal{P} \equiv (p_f^2, q^2, p_i^2)$

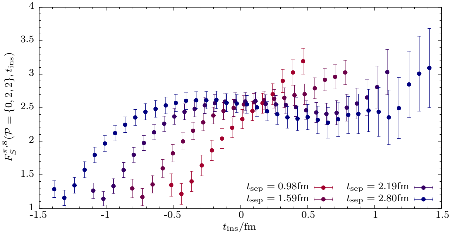


Two-point function at several values of p^2 (same ensemble)

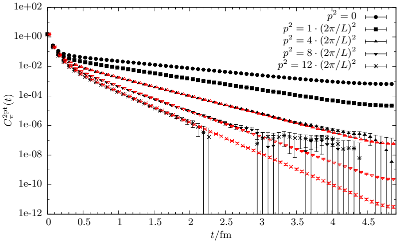
At non-zero momentum transfer $Q^2 > 0$:

- Two-point and three-point functions develop signal-to-noise problem as p_i^2 (p_f^2) increases.
- Problem more severe for two-point functions as they enter square-root in the ratio.
- For $t_{\rm sep} \gtrsim 2\,\mathrm{fm}$ the signal would be lost at fairly small p_i^2...

Effective form factor at $Q^2 \neq 0$



Unrenormalized eff. form factor on D450 ($M_\pi = 218\,\mathrm{MeV}$, $a = 0.076\,\mathrm{fm}$)
momentum labels: $\mathcal{P} \equiv (p_f^2, q^2, p_i^2)$



Two-point function at several values of p^2 (same ensemble)

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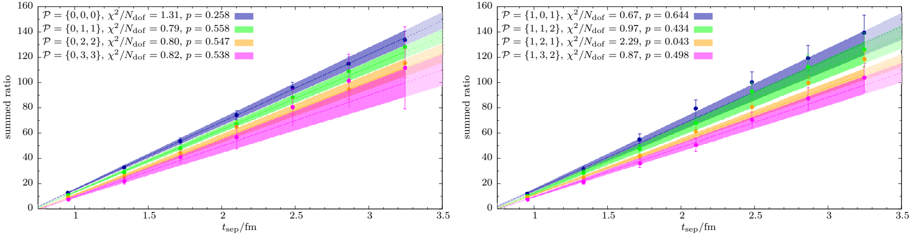
- For $t_{\rm sep} \gtrsim 2\,\mathrm{fm}$ the signal would be lost at fairly small p_i^2...

Solution: Replace two-point functions in ratio by fitted data for $p^2 \geq 2 \cdot (2\pi/L)^2$

Further suppression of excited states



Summation method fits to extract (unrenormalized) $F_S^{\pi,l}(Q^2)$ on E250 ($M_\pi = 132\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$); momentum labels: $\mathcal{P} \equiv (p_f^2, q^2, p_i^2)$

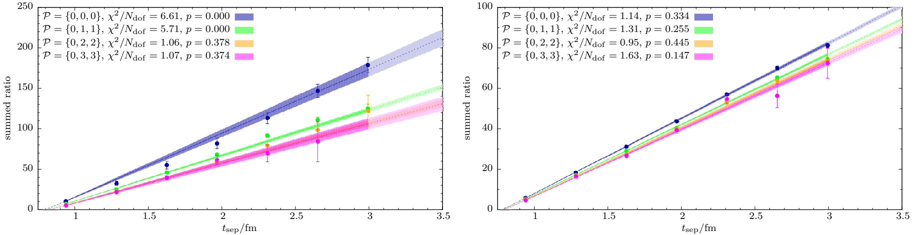
Summation method is used to further suppress excited states

$$S(p_f^2, q^2, p_i^2, t_{\rm sep}) = \sum_{t_{\rm ins}=t_0}^{t_{\rm sep}-t_0} R(p_f^2, q^2, p_i^2, t_{\rm sep}, t_{\rm ins}) = \mathrm{const} + \langle \pi(p_f^2)|S(q^2)|\pi(p_i^2)\rangle (t_{\rm sep} - t_0) + \mathcal{O}(e^{-\Delta t_{\rm sep}})$$

where Δ is the mass gap between ground and first excited state and we choose $t_0 = 0.4\,\mathrm{fm}$.

- Can vary $t_{\rm sep}^{\rm min}$, for now we (mostly) choose $t_{\rm sep}^{\rm min} \approx 1\,\mathrm{fm}$.
- We find that excluding timeslice by increasing t_0 improves signal at $Q^2 > 0$.

Further suppression of excited states cont'd



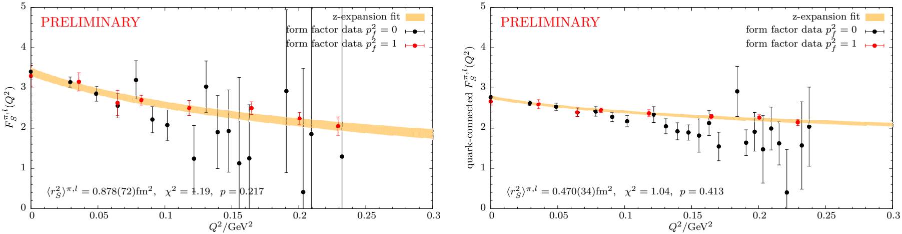
Left: Summation method fits to extract (unrenormalized) $F_S^{\pi,l}(Q^2)$ on C101 ($M_\pi = 222\,\mathrm{MeV}$, $a = 0.086\,\mathrm{fm}$); Right: quark-connected contribution; momentum labels: $\mathcal{P} \equiv (p_f^2, q^2, p_i^2)$

Application of summation method can be challenging on some of the boxes with open BC:

- Statistics available for 2+1 diagrams can be drastically reduced at $t_{\rm sep} \gtrsim 2\,\mathrm{fm}$ depending on $t_{\rm ex}$ and T.
- Systematic devation at small Q^2 for $t_{\rm sep} \gtrsim 2fm$ towards larger slopes ($\rightarrow$ overestimation of $F_S^{\pi}(Q^2)$).
- Mimics an excited state effect, but not seen on any of the periodic boxes even at physical M_π.
- Quark-connected contribution is unaffected as well.

Alternative approach: Simulatenous fit of three- and two-point functions (work in progress...)

Form factor parametrization: z-expansion



z-expansion fits to (unrenormalized) $F_S^{\pi,l}(Q^2)$ on E250 ($M_\pi = 132\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$). Left: full $F_S^{\pi,l}(Q^2)$; right: quark-connected only

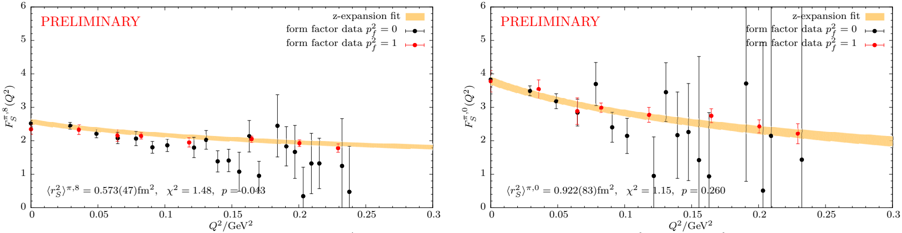
Use z-expansion for parametrization of the **unrenormalized** form factor and extraction of radii:

$$F_S^{\pi,f}(Q^2) = \sum_{n=0}^{N_z} a_n z^n\,, \qquad z = \frac{\sqrt{t_{\mathrm{cut}} + Q^2} - \sqrt{t_{\mathrm{cut}} - t_0}}{\sqrt{t_{\mathrm{cut}} + Q^2} + \sqrt{t_{\mathrm{cut}} - t_0}}\,, \qquad a_1 \sim \langle r_S^2\rangle_\pi^f = -\frac{6}{F_S^{\pi,f}(0)} \cdot \left.\frac{dF_S^{\pi,f}(Q^2)}{dQ^2}\right|_{Q^2=0}$$

Here $n = N_z = 1$, $t_{\mathrm{cut}} = 4M_\pi^2$ and "optimal" choice for $t_0 = t_{\mathrm{cut}}(1 - \sqrt{1 + Q^2_{\mathrm{max}}/t_{\mathrm{cut}}})$ *PRD 92, 013013 (2015)*

- Radius does not required renormalization.
- $\vec{p}_f = (1,0,0)$ gives much cleaner signal than $\vec{p}_f = (0,0,0)$ due to smaller possible p_i^2 at similar Q^2.
- Many more points $\vec{p}_f = (1,0,0)$ with large errors not included in fits and plots.
- **Very high momentum resolution at physical quark mass.**

Comparison of different flavor structures



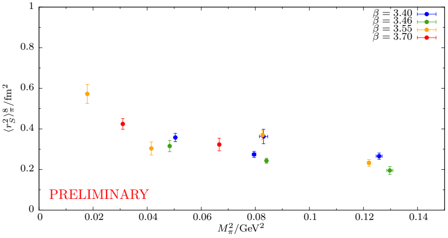
z-expansion fits to (unrenormalized) $F_S^{\pi,l}(Q^2)$ on E250 ($M_\pi = 132\,\mathrm{MeV}$, $a = 0.064\,\mathrm{fm}$). Left: $F_S^{\pi,8}(Q^2)$; right: $F_S^{\pi,0}(Q^2)$

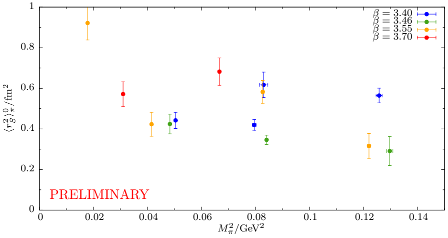
- Difference in individual octet and singlet contributions to $F_S^\pi(Q^2)$ clearly resolved.
- Results on two most chiral ensembles:

ID	$M_\pi\,/\,\mathrm{MeV}$	$a\,/\,\mathrm{fm}$	$\langle r_S^2\rangle_\pi^l\,/\,\mathrm{fm}^2$	$\langle r_S^2\rangle_\pi^8\,/\,\mathrm{fm}^2$	$\langle r_S^2\rangle_\pi^0\,/\,\mathrm{fm2}$
E250	132	0.064	0.878(72)	0.573(47)	0.922(83)
E300	174	0.050	0.522(49)	0.425(26)	0.572(60)

- Rather strong light quark mass dependence at small M_π as expected from χPT.
- Statistical errors for $r_S^{\pi,l}$ and $r_S^{\pi,0}$ typically $\lesssim 5\%$; for $r_S^{\pi,8} \lesssim 3\%$.
- Still need to study systematic effects due to choice of $Q^2_{\max}$, N_z, residual excited states (?) ...

Chiral behavior

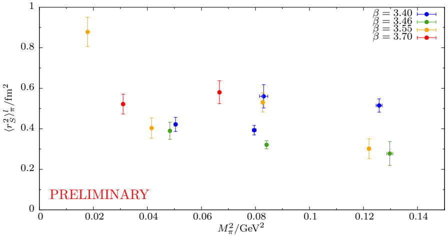




- Chiral behavior consistent with leading log M_π behavior predicted by NLO $SU(2)$ χPT.
- No clear / large lattice artifacts.
- A few outliers in $r_S^{\pi,l}$ and $r_S^{\pi,0}$ on open BC boxes with small T and / or insufficient momentum resolution:

Expect improvement from increased two-point function statistics.

$\rightarrow$ Further analysis required...



Pheno value from $\pi\pi$-scattering:

$\langle r_S^2 \rangle_\pi^l = 0.61(04)\,\mathrm{fm}^2$ *Nucl. Phys. B603, 125 (2001)*

Summary and outlook

- **Preliminary study of the pion scalar form factor on 13 CLS $N_f = 2+1$ ensembles:**
  - Promising first results for $F_S^{\pi}(Q^2)$ and scalar radii.
  - Signal quality already competitive with older lattice calculations.
  - **Unprecedented momentum resolution on large and fine lattices at physical M_π.**
- **Future plans:**
  - Add more ensembles (also on $m_s = \mathrm{const}$ trajectory) and double gauge statistics on e.g. D200, E250 and E300.
  - Increase two-point function statistics by factor ~ 4 to further improve signal for 2+1 diagrams
  - Refine excited state analysis and carry out physical extrapolation to extract $\bar{l}_4$ from $\langle r_S^2 \rangle_\pi^l$.
  - Data also available for kaon and all 16 local + one-link displaced operator insertions.
    $\rightarrow$ can compute e.g. $F_V^{\pi,K}(Q^2)$.
  - (Combined) physical extrapolations using $SU(3)$-χPT.