NSPT for O(N) non-linear sigma model: the larger N the better

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2023 LATTICE

EuroPLEx H2020

Outline

1. Introduction

- Motivations of this work
- Basics of NSPT
- Why $O(N)$ non-linear sigma model?

2. NSPT on O(N) non-linear sigma model

- From the first to the fourth order computations
- Presenting large fluctuations at high-orders
- Fluctuations tamed at large N

3. Conclusions

- As expected, less problems for larger N
- Ongoing simulations and work in progress
- Renormalons

Motivations and sketch of ideas

- The problem in three sentence:

  1. Interested in calculating observables at high perturbative orders
  2. Applications in lattice QCD have been incredibly fruitful
  3. In low-dimensional systems: distributions are very difficult to explore

- We had already been aware of it for a long time

2000, R. Alfieri, F. Di Renzo, E. Onofri, L. Scorzato:
Understanding stochastic perturbation theory: toy models and statistical analysis

Also other groups know it:
A. Ramos, G. Catumba : private communication

- We revisited this topic after finding the same fluctuations in perturbative expansions around 1d QM non-trivial vacua

LATTICE22 / PoS - P.Baglioni, F. Di Renzo:
NSPT around instantons

The natural guess : less problems for more degrees of freedom

The lattice model

We consider the Euclidean O(N) non-linear sigma model in *2D*

$$S = \frac{1}{2g} \int d^2x \; (\partial_\mu \boldsymbol{s}) \cdot (\partial_\mu \boldsymbol{s}) \qquad\qquad \boldsymbol{s}(x) \cdot \boldsymbol{s}(x) = 1$$

We will study it on a *2D* lattice:

$$\boxed{S = -\frac{1}{g} \sum_{x,\mu} \boldsymbol{s}_x \cdot \boldsymbol{s}_{x+\mu}} \qquad\qquad \boxed{\boldsymbol{s}_x \cdot \boldsymbol{s}_x = 1}$$

N-component real scalar field — *A local constraint*

This is what we need:

- **We can tune the parameter N** *(modifying the number of degrees of freedom)*
- It is closely related to other very interesting models
- It shares some interesting features with QCD

Solving the constraints

- Identifying the correct degrees of freedom:

$$Z = \int \prod_x d\boldsymbol{s}_x \ \delta(\boldsymbol{s}^2 - 1) \ e^{\frac{1}{g} \sum_{x,\mu} \boldsymbol{s}_x \cdot \boldsymbol{s}_{x+\mu}}$$

$$\boldsymbol{s}_x = (\boldsymbol{\pi}_x, \sigma_x)$$

$$\sigma_x = \epsilon(x)\sqrt{1 - \boldsymbol{\pi}_x^2}$$

- The constraint disappears thanks to rescaling $\boldsymbol{\pi}_x^2 \to g\boldsymbol{\pi}_x^2$

- A theory with only π fields

$$Z = \int \prod_x d\boldsymbol{\pi}_x \ e^{-\frac{1}{2}\sum_{x,\mu}\left[(\Delta_\mu \boldsymbol{\pi}_x)^2 - \frac{1}{g}(\Delta_\mu \sqrt{1-g\boldsymbol{\pi}_x^2})^2\right] - \frac{1}{2}\sum_x \log(1-g\boldsymbol{\pi}_x^2)}$$

- Interaction terms in Taylor series:

Very complicated! at each order, new interaction vertices are generated
(not a problem for us...)

Basics of NSPT

Langevin equation for stochastic evolution

$$\frac{\partial \phi_j(\tau)}{\partial \tau} = -\frac{\partial S[\phi]}{\partial \phi_j(\tau)} + \eta_j(\tau)$$

$$\langle \eta_j(\tau) \rangle_\eta = 0 \qquad \langle \eta_j(\tau)\eta_k(\tau') \rangle_\eta = 2\delta_{jk}\delta(\tau - \tau')$$

Fokker-Planck equation

$$\frac{\partial P}{\partial \tau}(\phi, \tau) = \sum_j \frac{\partial}{\partial \phi_j}\Big(\frac{\partial S}{\partial \phi_j} + \frac{\partial}{\partial \phi_j}\Big) P(\phi, \tau)$$

$$P_{eq}(\phi) = \frac{e^{-S[\phi]}}{\mathcal{N}}$$

$$\lim_{\tau \to \infty} \langle \mathcal{O}[\phi_\eta(\tau)] \rangle_\eta = \langle \mathcal{O}[\phi] \rangle$$

We solve the Langevin equation numerically order-by-order!

1995, F. Di Renzo, G. Marchesini, P. Marenzoni, E. Onofri : Weak coupling perturbation theory by Langevin dynamics: Fourth loop and beyond

$$\phi_j(\tau) = \phi_j^{(0)}(\tau) + \sum_{n>0} \lambda^n \phi_j^{(n)}(\tau) \quad \forall j$$

+ *Euler integrator (simplest choice)*

$$\begin{cases} \phi_{j,i+1}^{(0)} = \phi_{j,i}^{(0)} - \Delta\tau \left[\frac{\partial S}{\partial \phi}\right]_i^{(0)} + \sqrt{2\Delta\tau}\ \eta_{j,i} \\ \phi_{j,i+1}^{(1)} = \phi_{j,i}^{(1)} - \Delta\tau \left[\frac{\partial S}{\partial \phi}\right]_i^{(1)} \\ \dots \end{cases}$$

- Set of hierarchical equations
- Exact at any order in perturbation theory
- Perturbative expansion of observables

$$\langle \mathcal{O} \rangle(\lambda) = \langle \mathcal{O}^{(0)}(\phi^{(0)}) \rangle + \lambda \langle \mathcal{O}^{(1)}(\phi^{(0)}, \phi^{(1)}) \rangle + \lambda^2 \langle \mathcal{O}^{(2)}(\phi^{(0)}, \phi^{(1)}, \phi^{(2)}) \rangle + \dots$$

NSPT on O(N) non-linear sigma model

Approaching the problem using NSPT

$$\pi^j_{y,i+1} = \pi^j_{y,i} + \Delta\tau \sum_\mu \left\{ \pi^j_{y+\mu,i} + \pi^j_{y-\mu,i} - \pi^j_{y,i} \left(\sqrt{\frac{1-g\boldsymbol{\pi}^2_{y+\mu,i}}{1-g\boldsymbol{\pi}^2_{y,i}}} + \sqrt{\frac{1-g\boldsymbol{\pi}^2_{y-\mu,i}}{1-g\boldsymbol{\pi}^2_{y,i}}} \right) \right\} + \Delta\tau \frac{g\pi^j_{y,i}}{1-g\boldsymbol{\pi}^2_{y,i}} + \sqrt{2\Delta\tau}\,\eta^j_{y,j}$$

Pay close attention to the observable to compute

$$\langle \pi^j_k \pi^i_z \rangle$$

Infrared-undefined propagator

$$E = \frac{1}{2N_{sites}} \sum_{x,\mu} \langle \boldsymbol{s}_x \cdot \boldsymbol{s}_{x+\mu} \rangle = \langle \boldsymbol{s}(0) \cdot \boldsymbol{s}(1) \rangle = g\langle \boldsymbol{\pi}(0) \cdot \boldsymbol{\pi}(1) \rangle + \langle \sqrt{1+g\boldsymbol{\pi}^2_0}\sqrt{1+g\boldsymbol{\pi}^2_1} \rangle$$

Well-defined propagator

$$= E^{(0)} + \sum_{n>0} g^n \, E^{(n)}$$

... and also at zero-mode

- Many different possible regularizations

1978, S. Elitzur : The applicability of perturbation expansion to two-dimensional Goldstone system

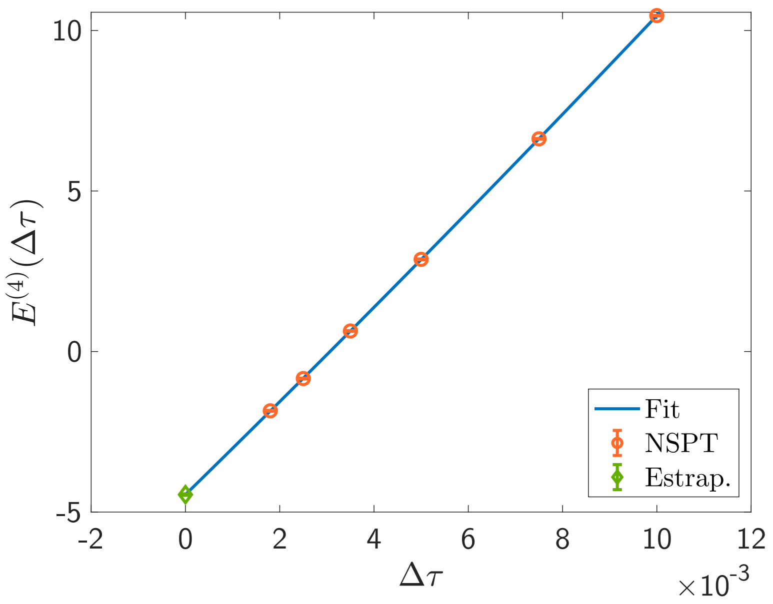
NSPT on O(N) non-linear sigma model

We run a variety of simulations :

2D, 20x20 lattice From O(3) to O(45) $n = 1 \; , \; .... \; , \; 14 \; , \; .... \; 23$

only for large N
(actually ongoing)

- Dealing with finite stochastic time step effects
- At the orders analytically known very small finite size corrections (only a few per mille)



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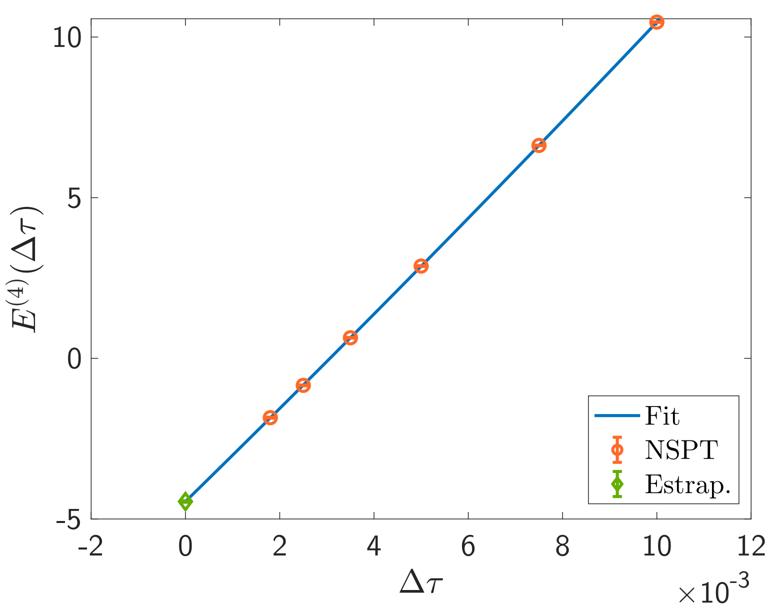
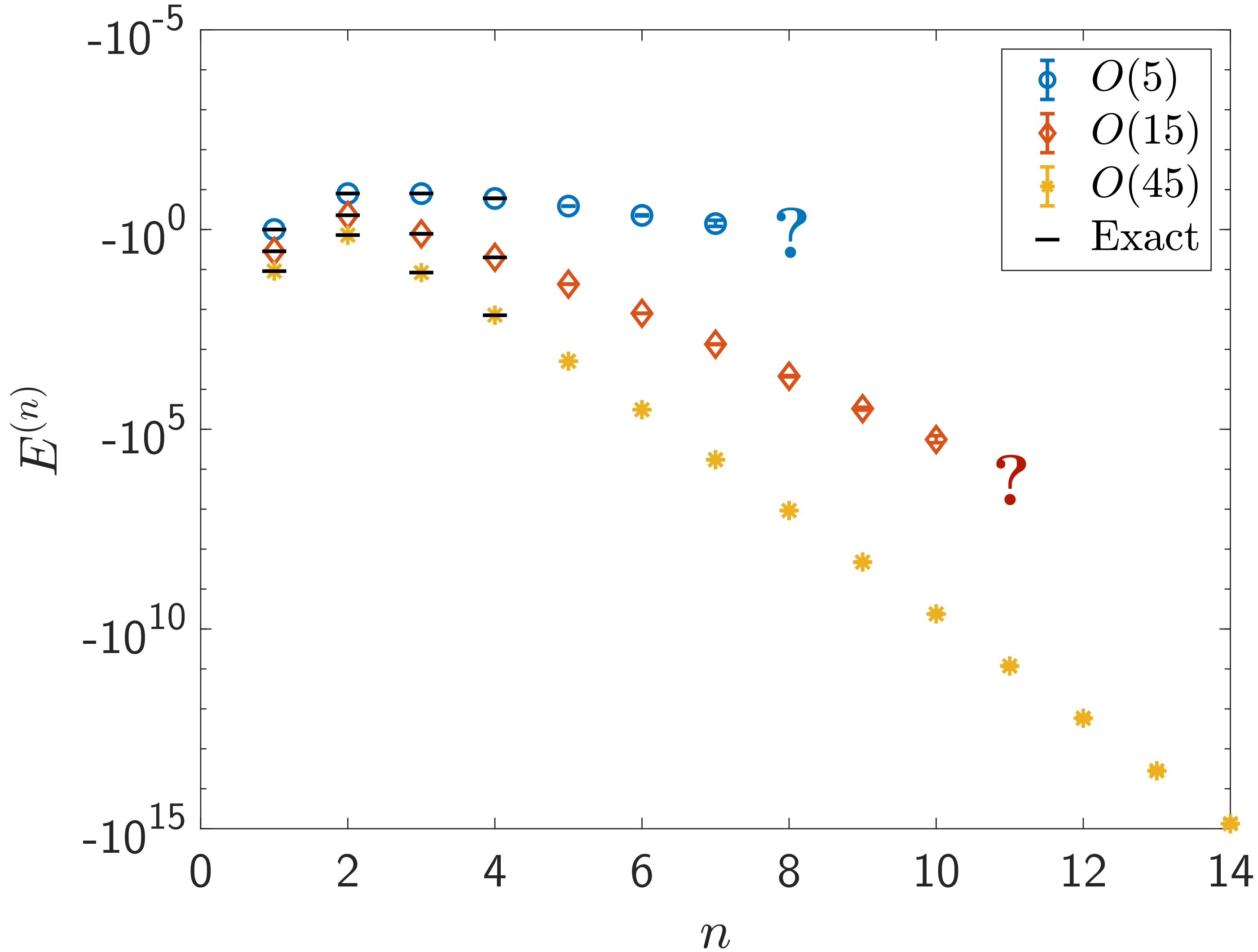


The big issue:



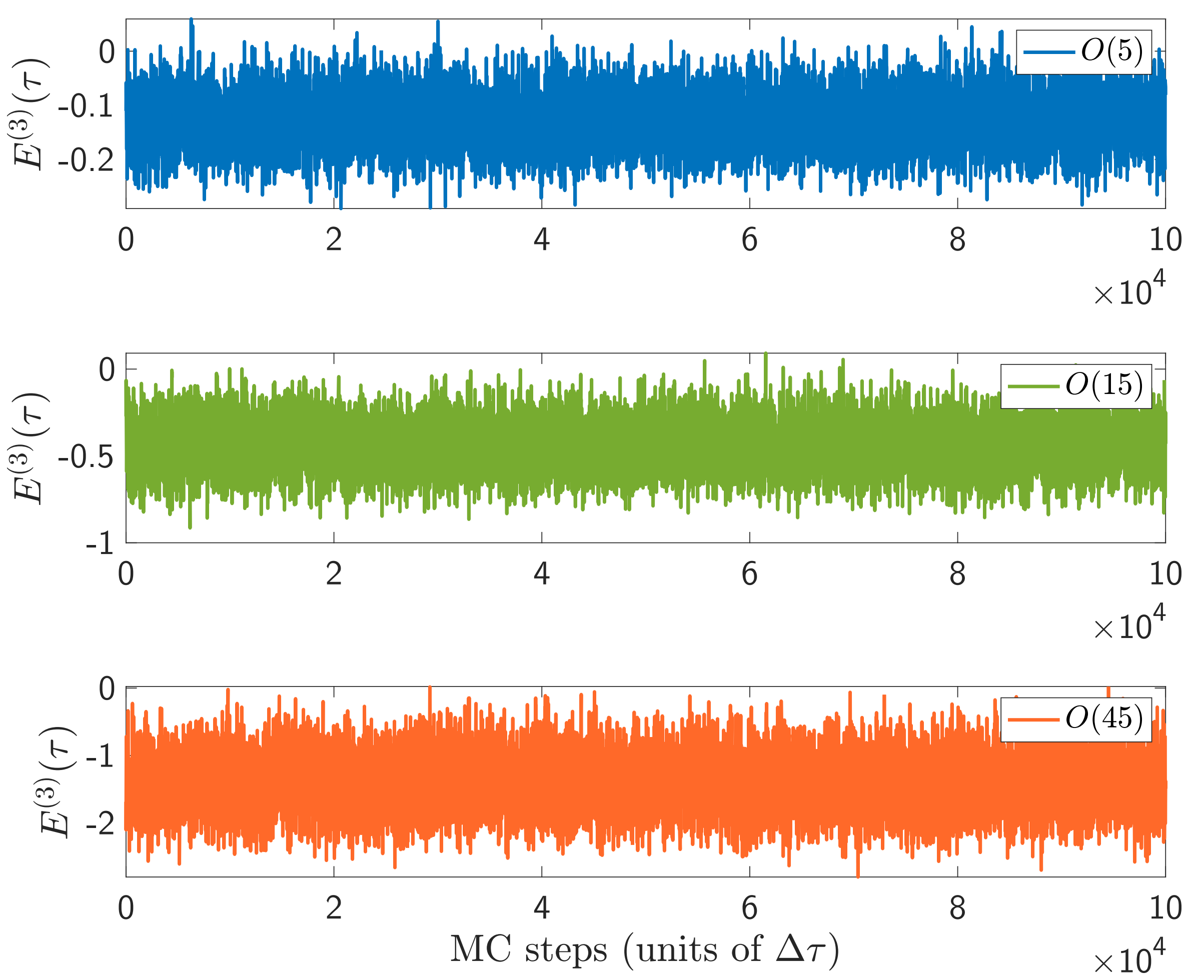
NSPT on O(N) non-linear sigma model

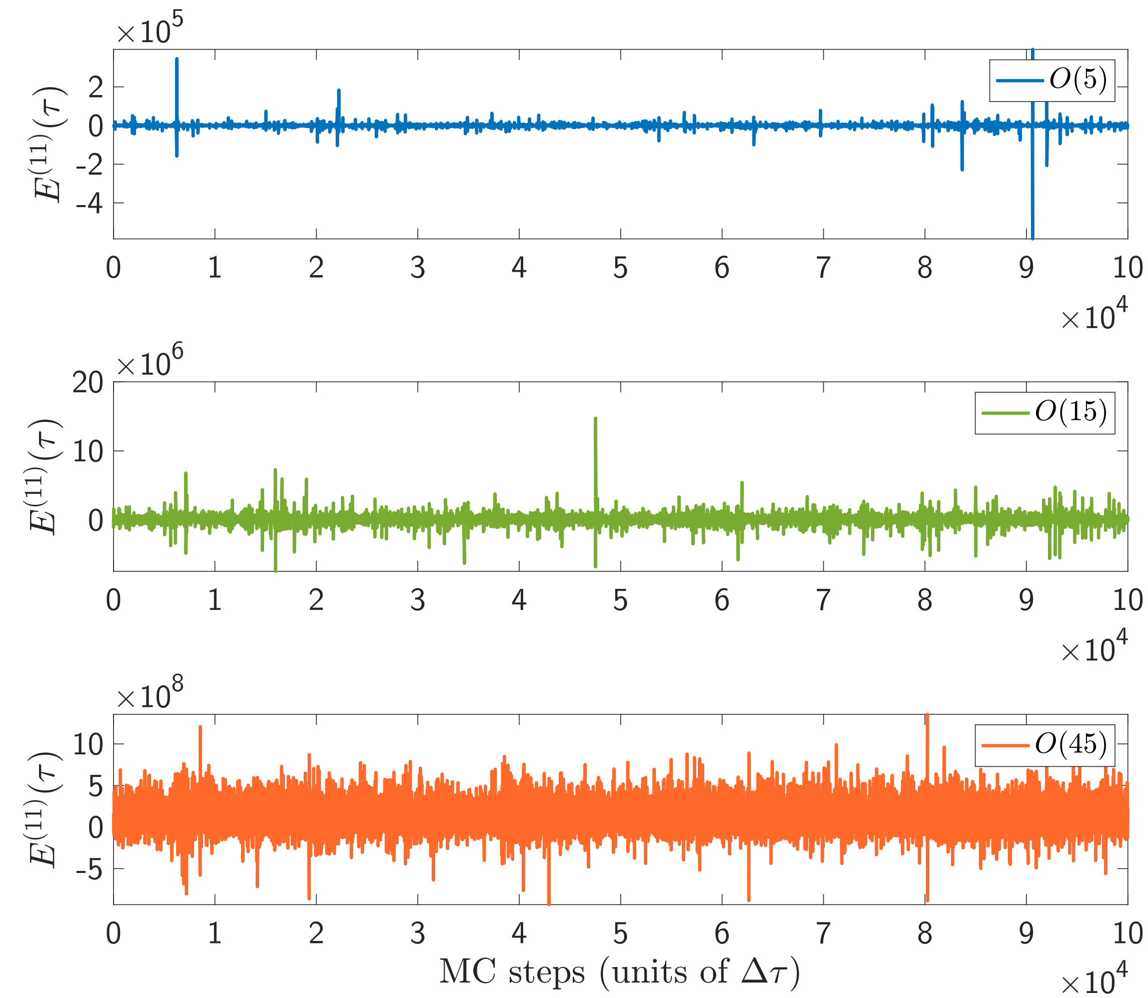
At low perturbative orders, we have good signals for every N

The situation changes when we reach high perturbative orders

We observe large deviations for small N

However, it seems that the situation remains under control for large N





NSPT on O(N) non-linear sigma model

Trying to be more precise... what happens to the mean during MC sampling?

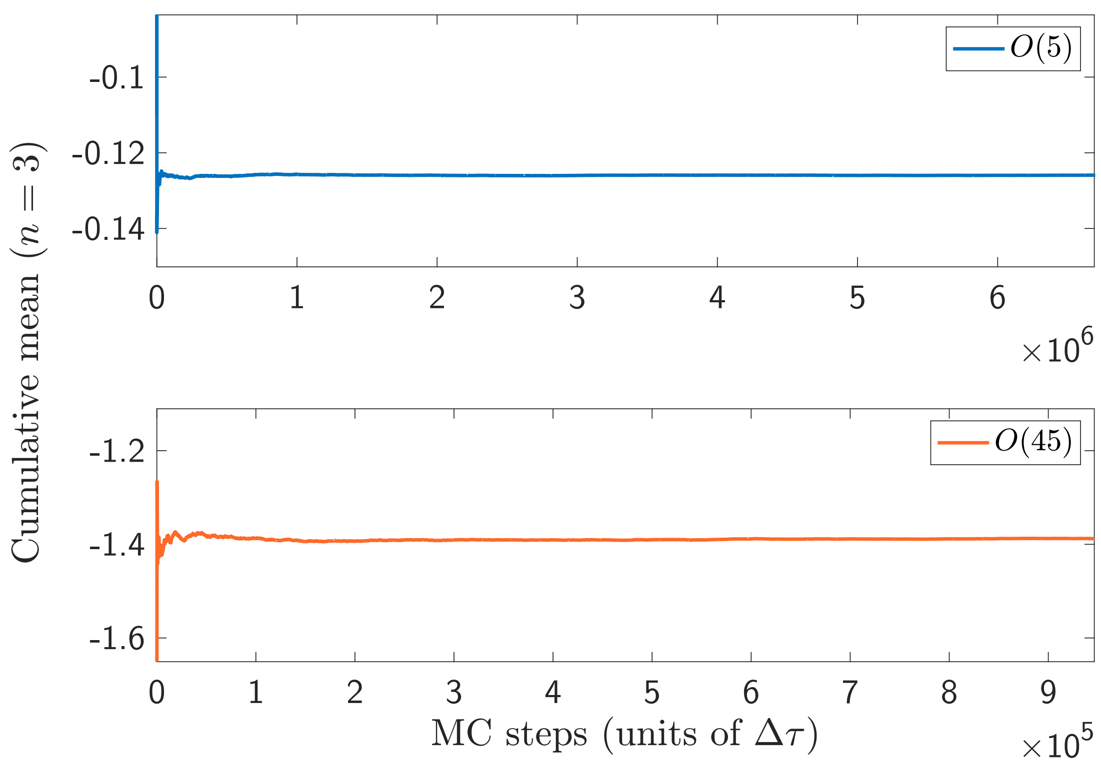
Difficulties in estimating the mean at high orders for small N

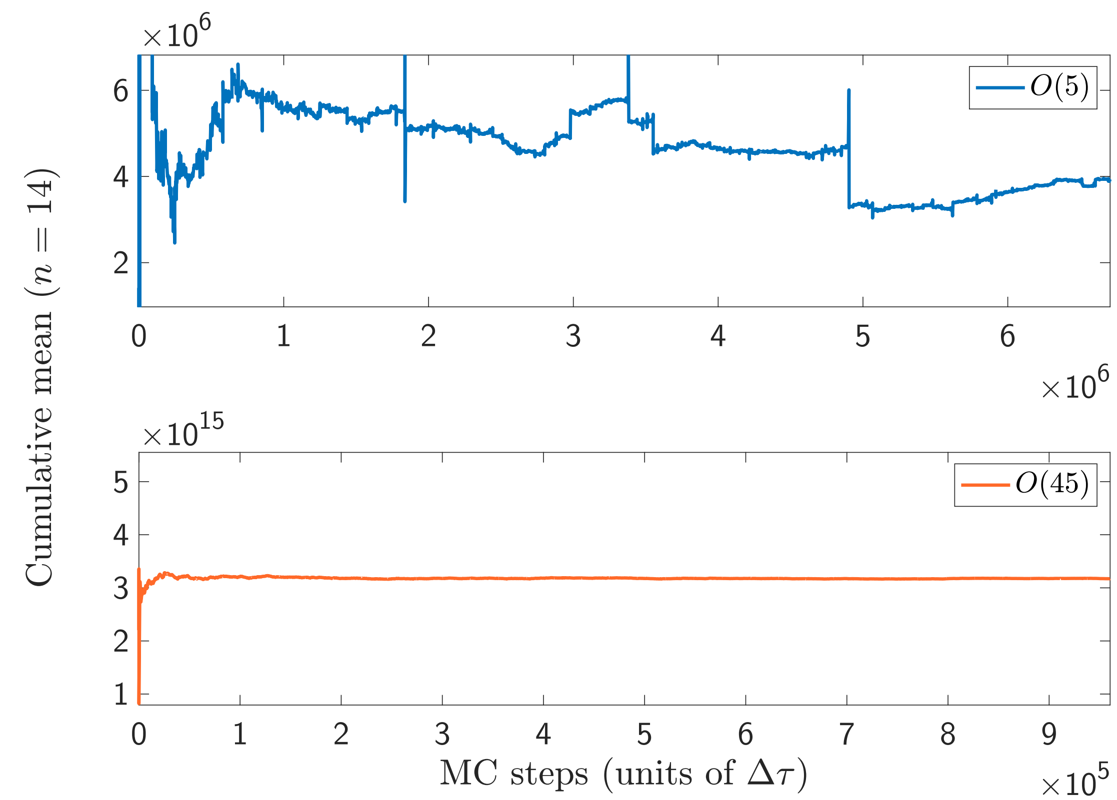
Cumulative mean:

$$\langle E^{(n)} \rangle_{i_{max}} = \frac{1}{N_{i_{max}}} \sum_{i=1}^{i_{max}} E_i^{(n)}$$

Our initial guess seems to hold true:

deviations under control as I increase the number of degrees of freedom





NSPT on O(N) non-linear sigma model (5)

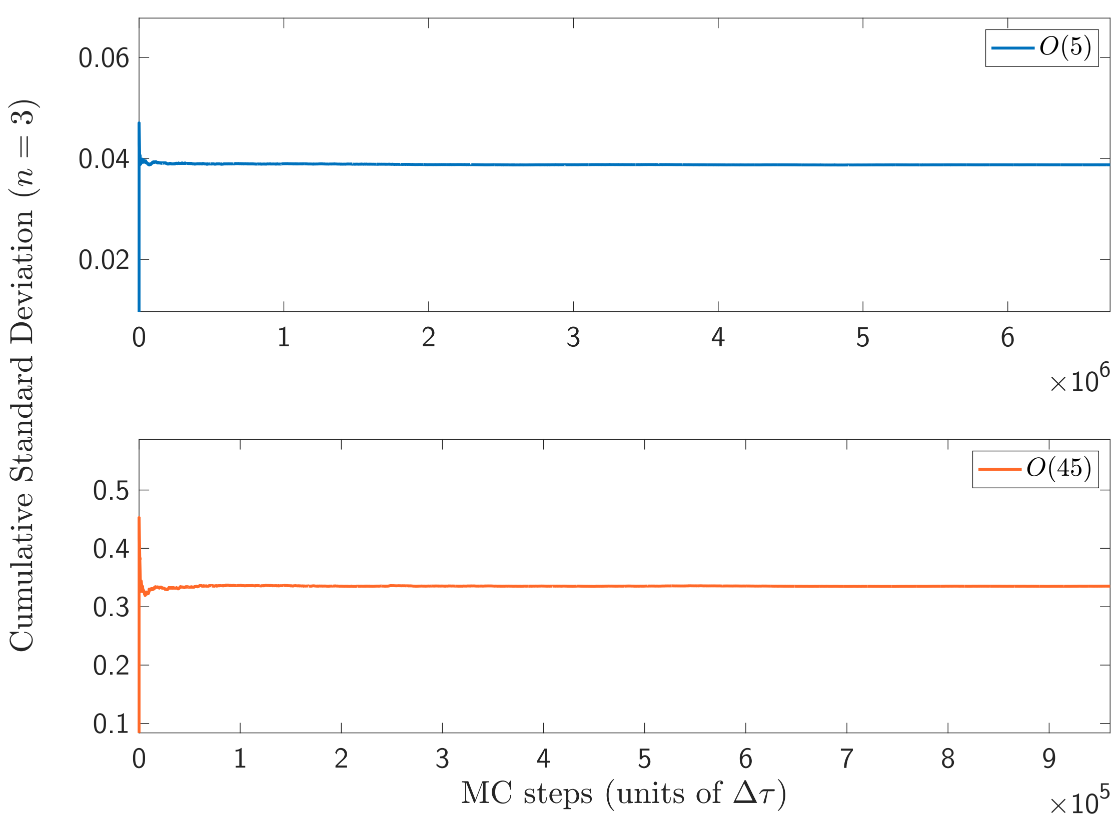
The situation is even worse when considering the estimation of the standard deviation of the distributions order-by-order

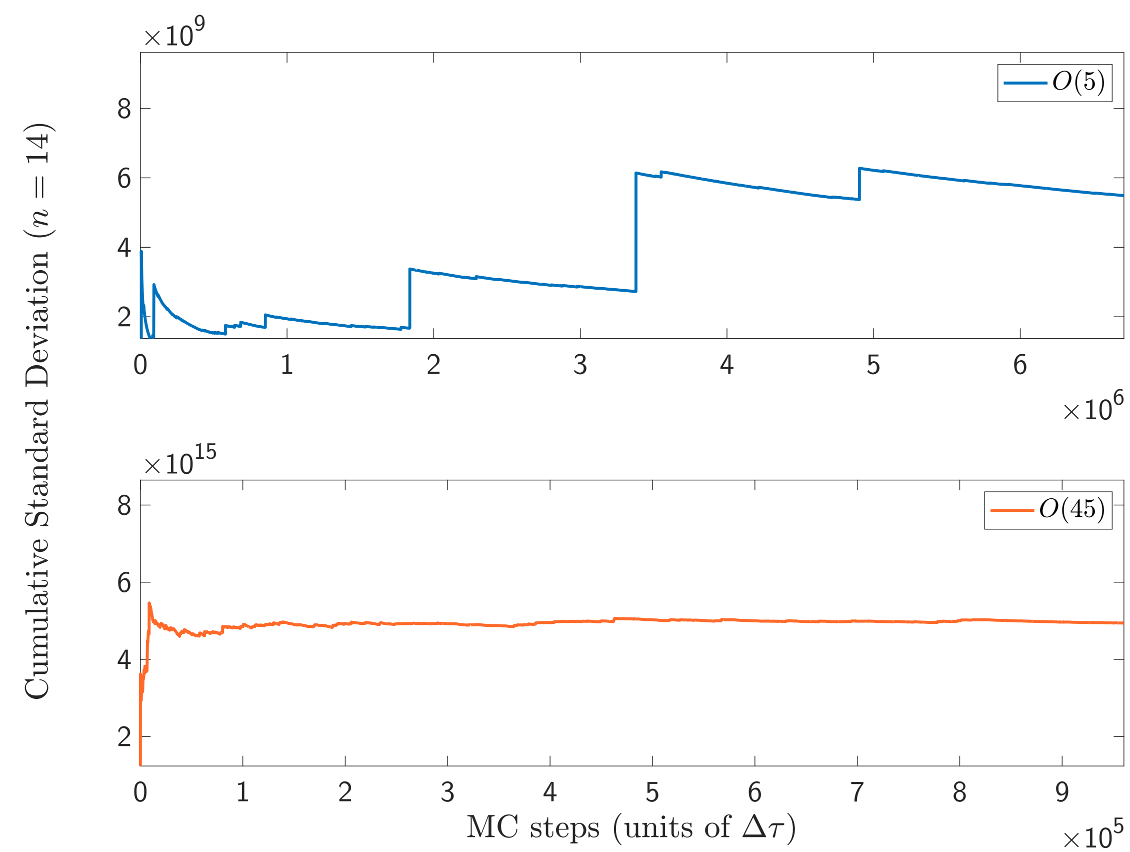
Cumulative Standard Deviation:

$$\sigma(E^{(n)})_{i_{max}} = \sqrt{\langle (E^{(n)})^2 \rangle_{i_{max}} - \langle E^{(n)} \rangle^2_{i_{max}}}$$

Still, performing simulations at larger and larger N, it seems that:

- We are dealing with distributions that are **easier** to explore (*the oscillations are mostly absorbed*)
- This is true even considering **the same amount of computational time**





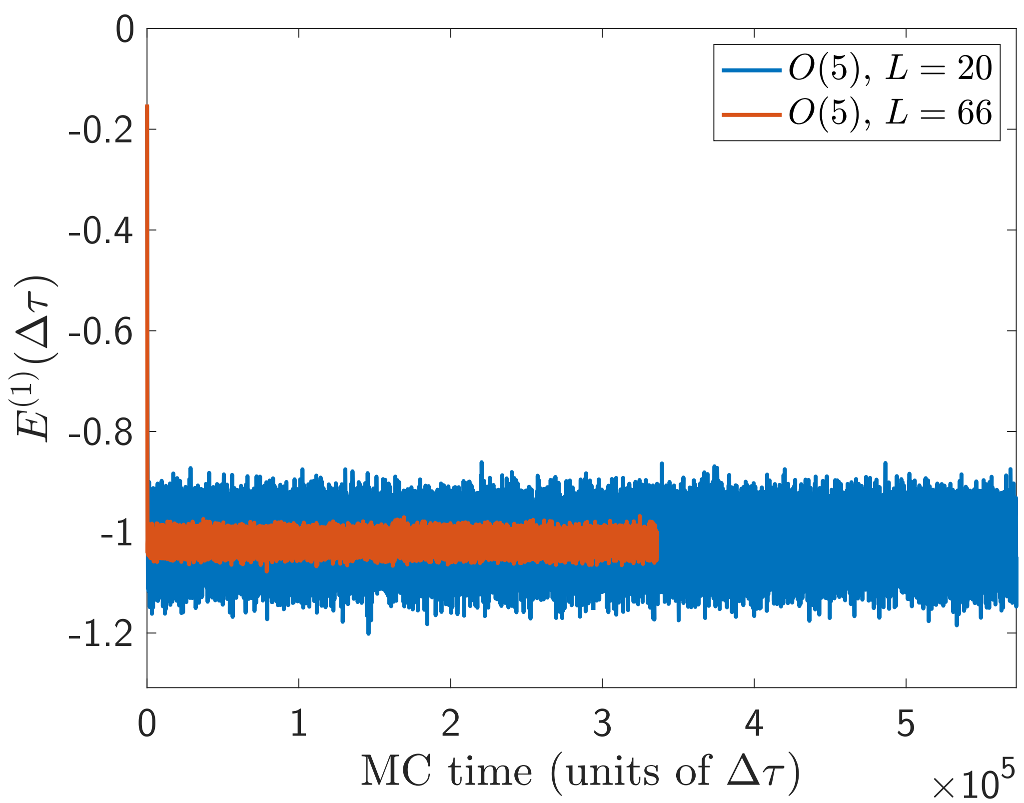
NSPT on O(N) non-linear sigma model

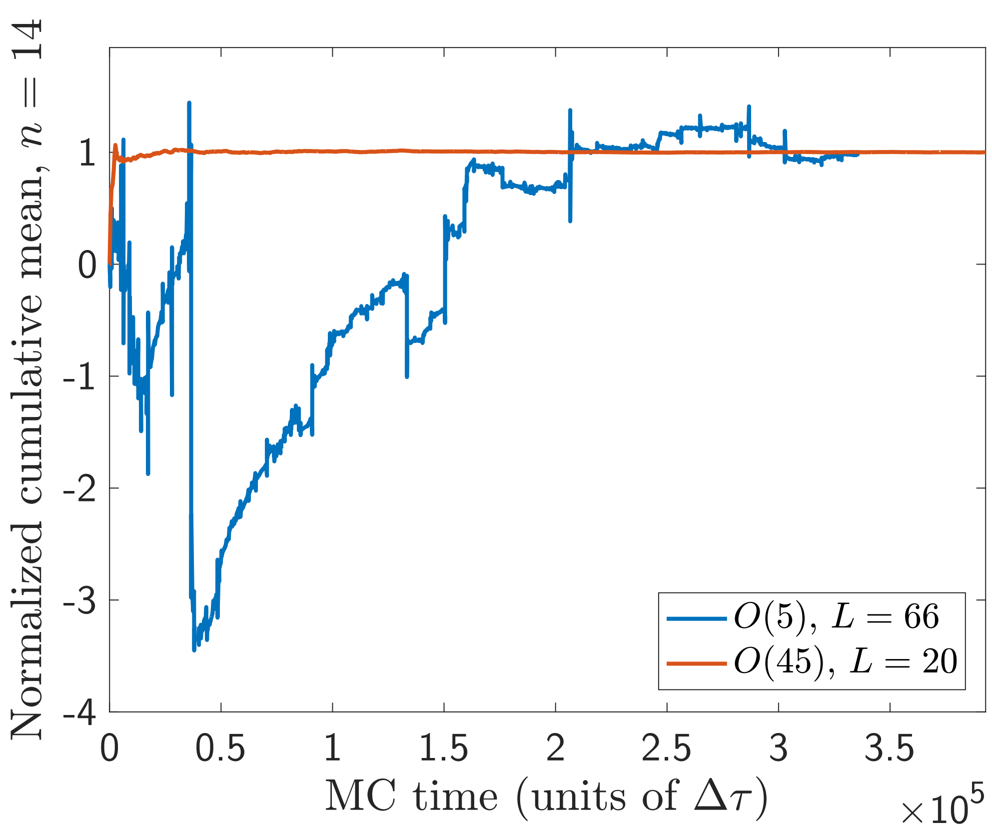
Is it just a matter of the number of degrees of freedom?

O(5) on a *2D* 66x66 lattice vs O(45) on a *2D* 20x20 lattice

We will certainly have effects of lattice self-averaging

However, **volume does not tame large n fluctuations!**



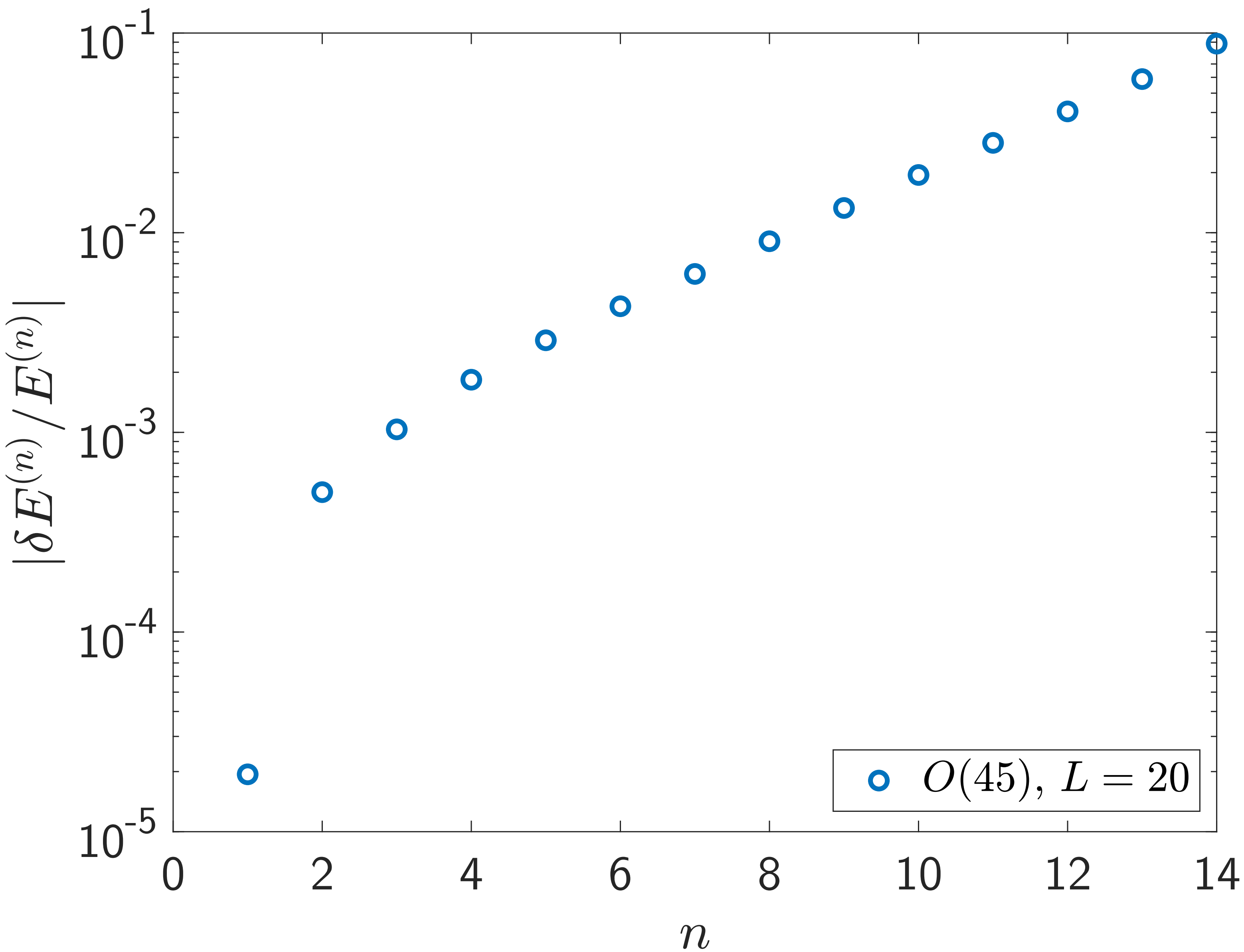


NSPT on O(N) non-linear sigma model

In the end: searching a quantitative description ...

Perhaps the relative errors can help us

From general considerations, we expect relative errors to increase with the order



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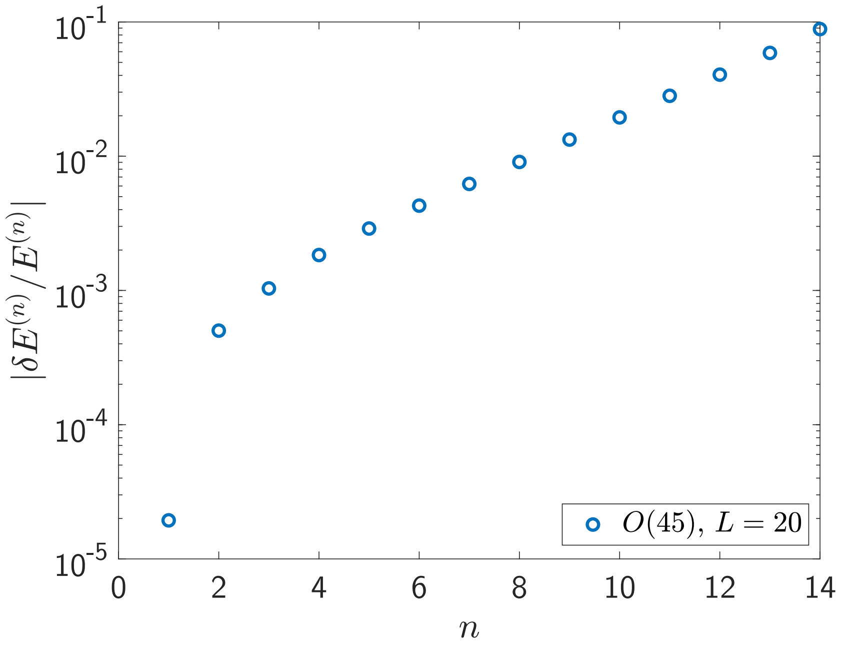
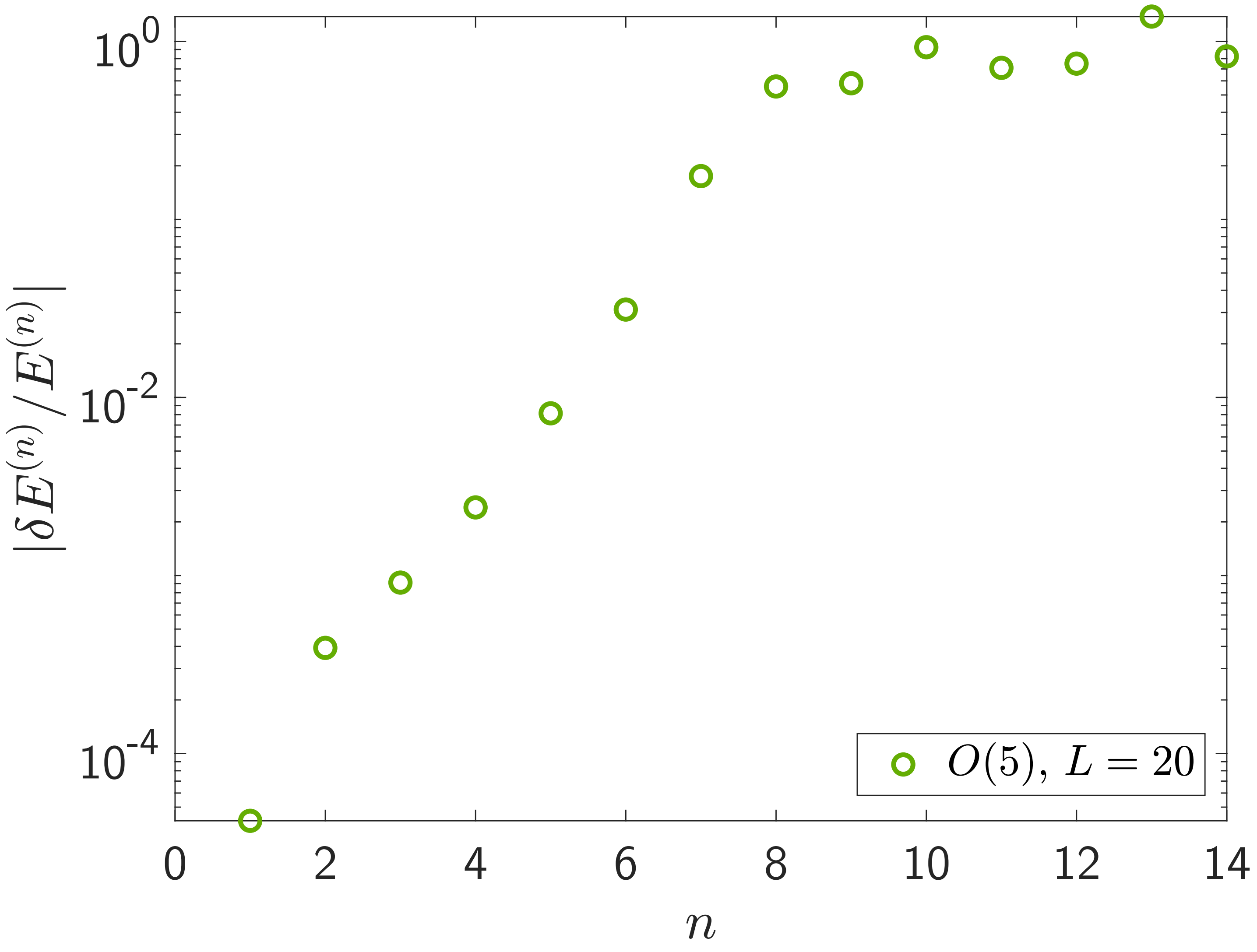


For smaller N, something must be going wrong!



NSPT on O(N) non-linear sigma model

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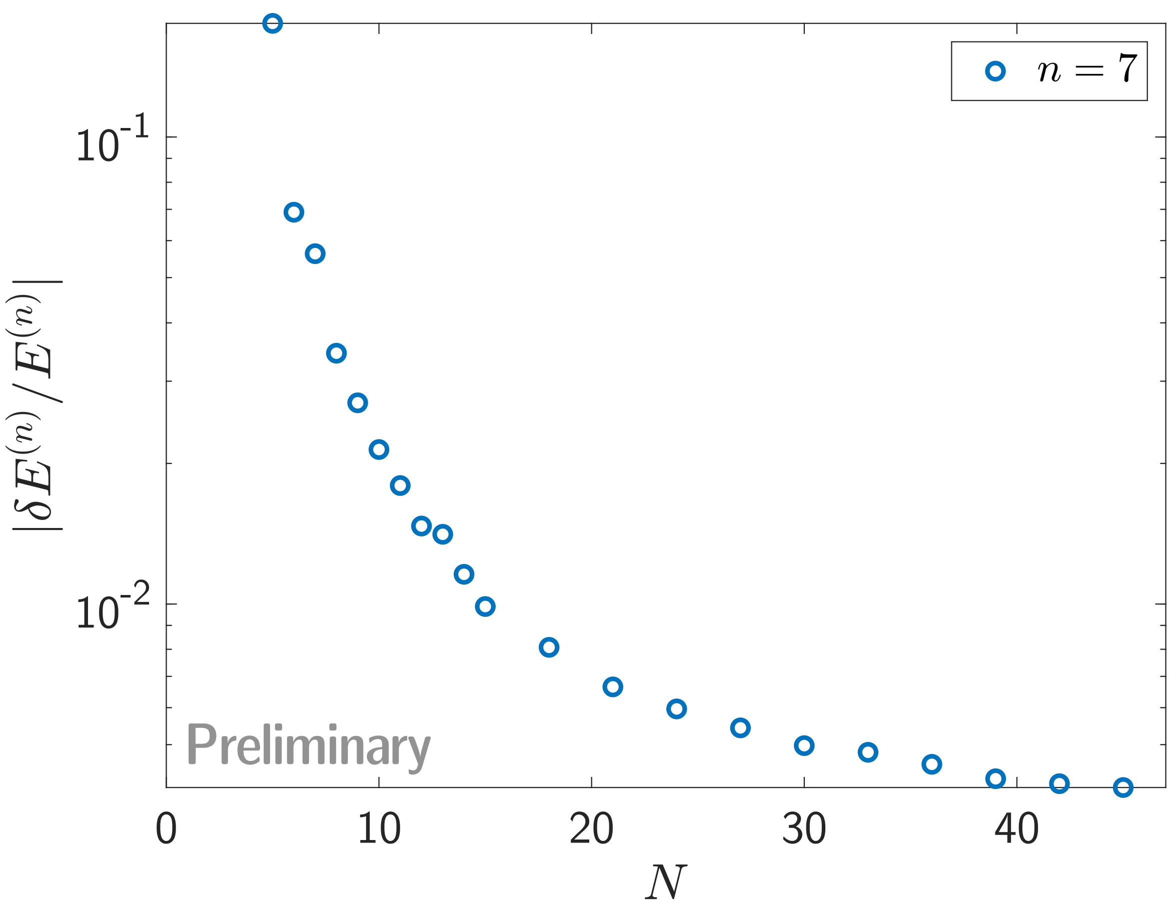

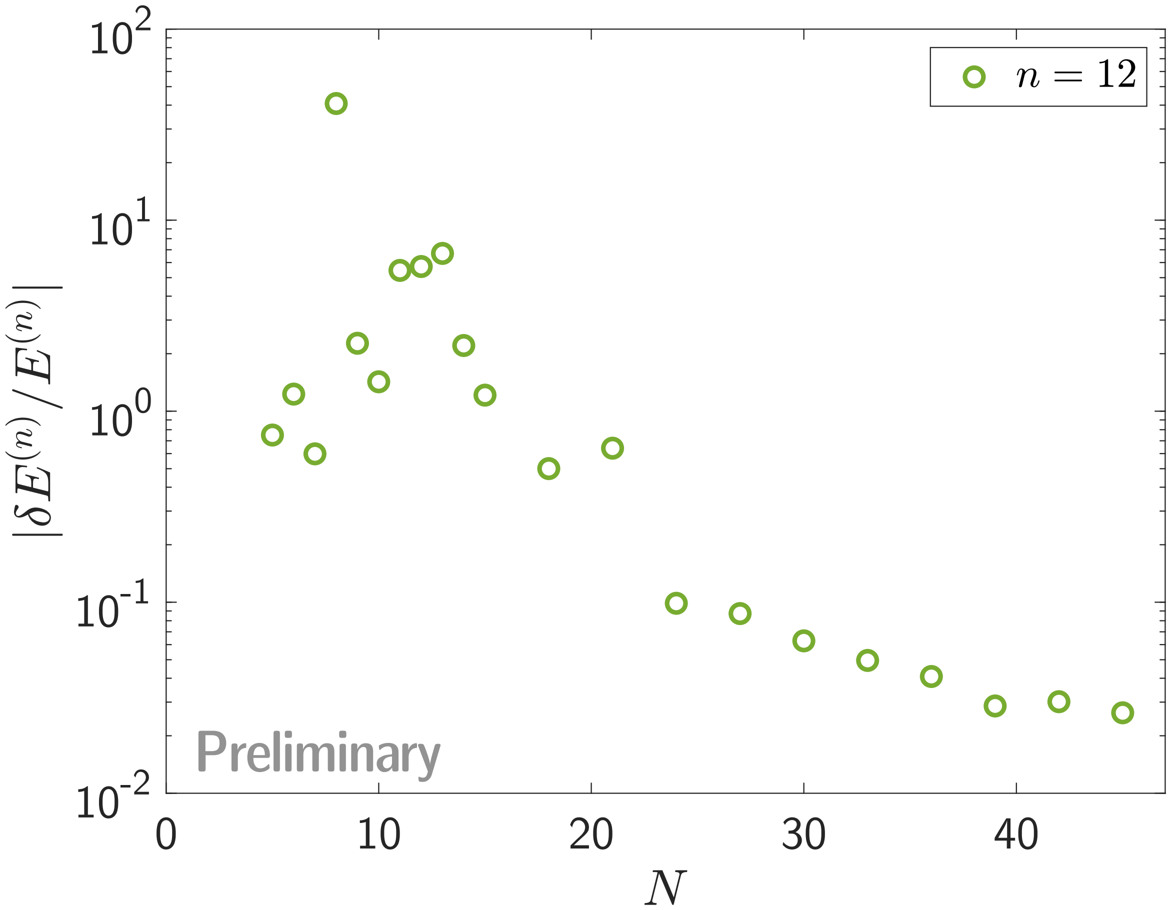

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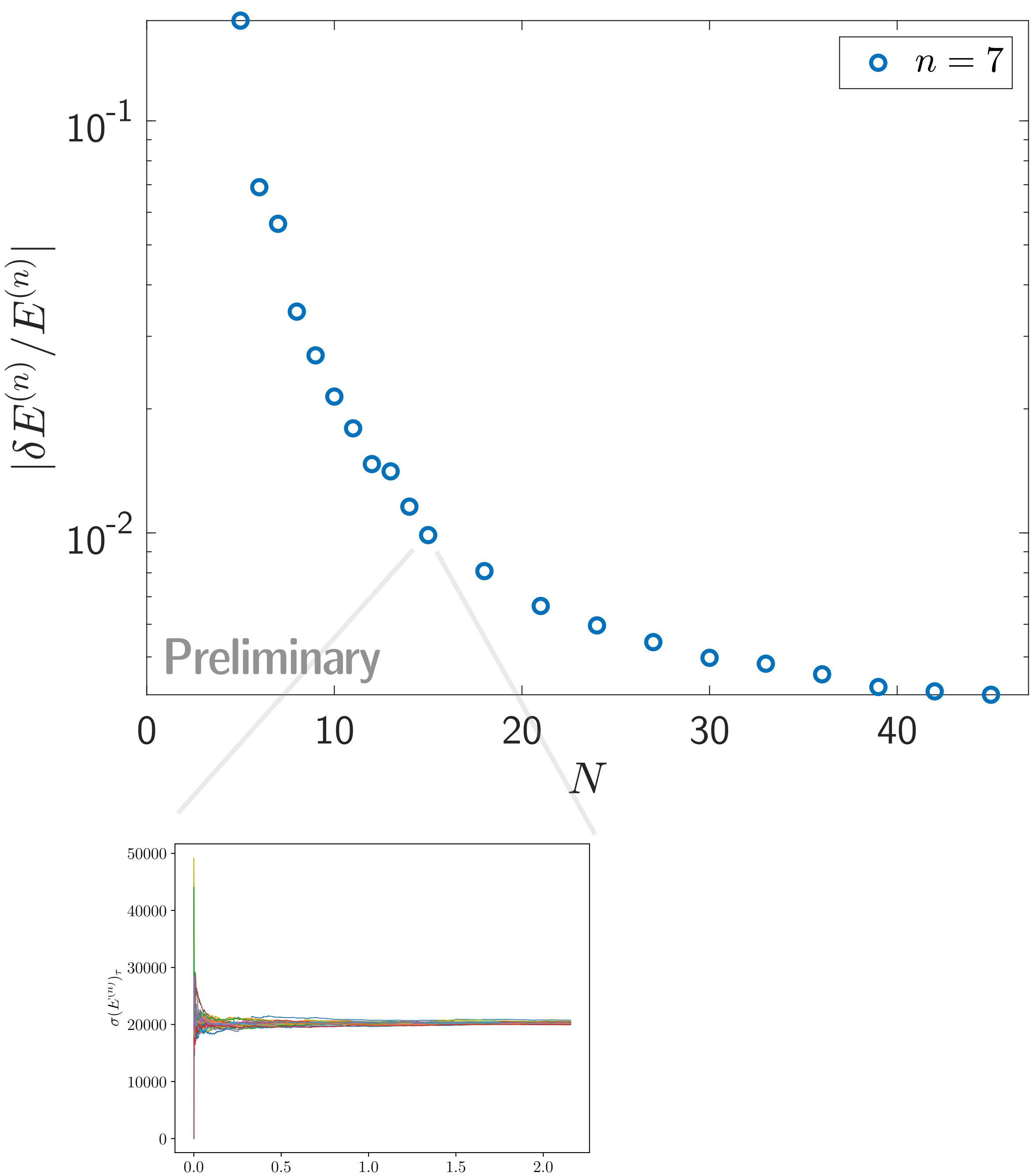
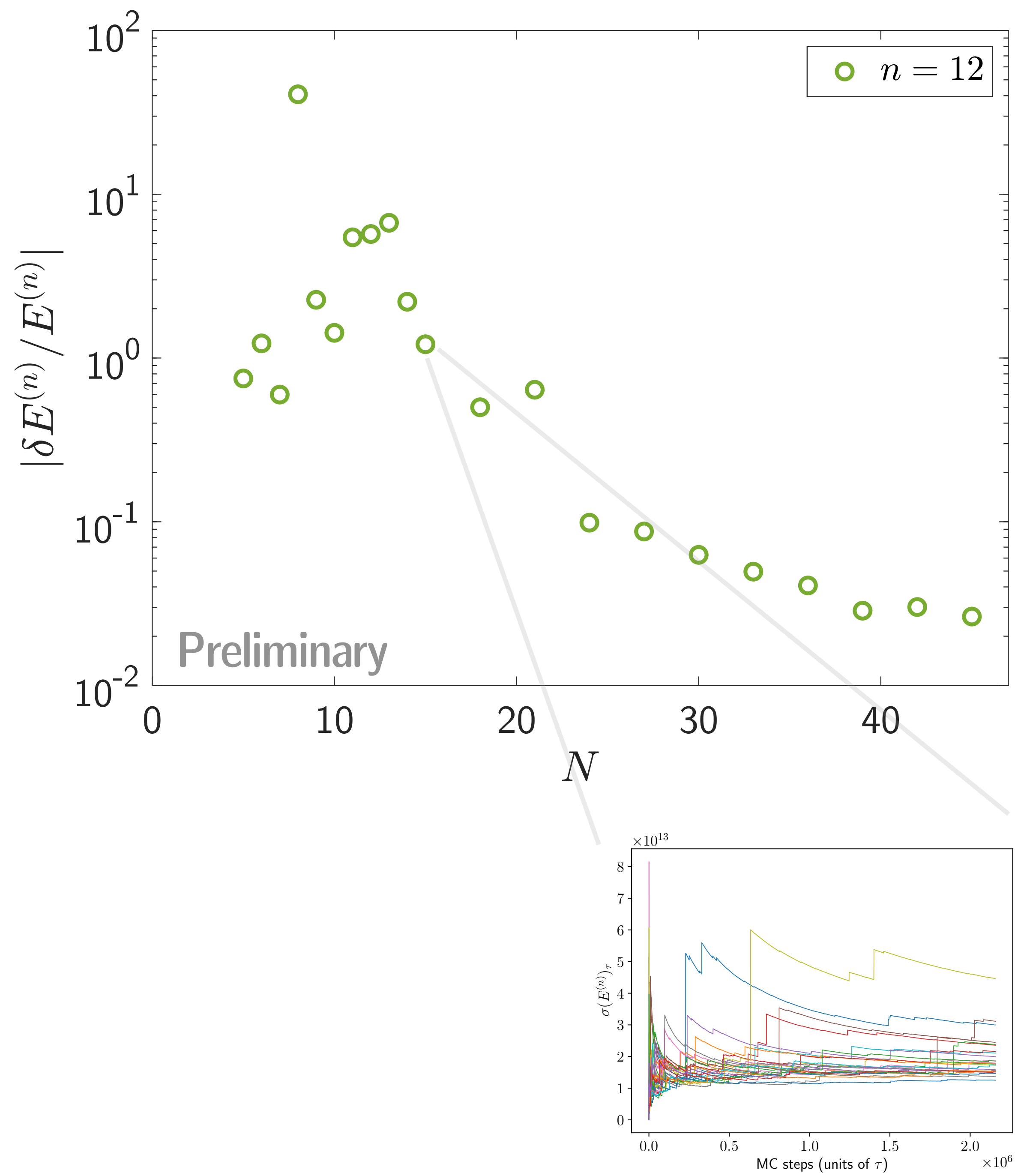


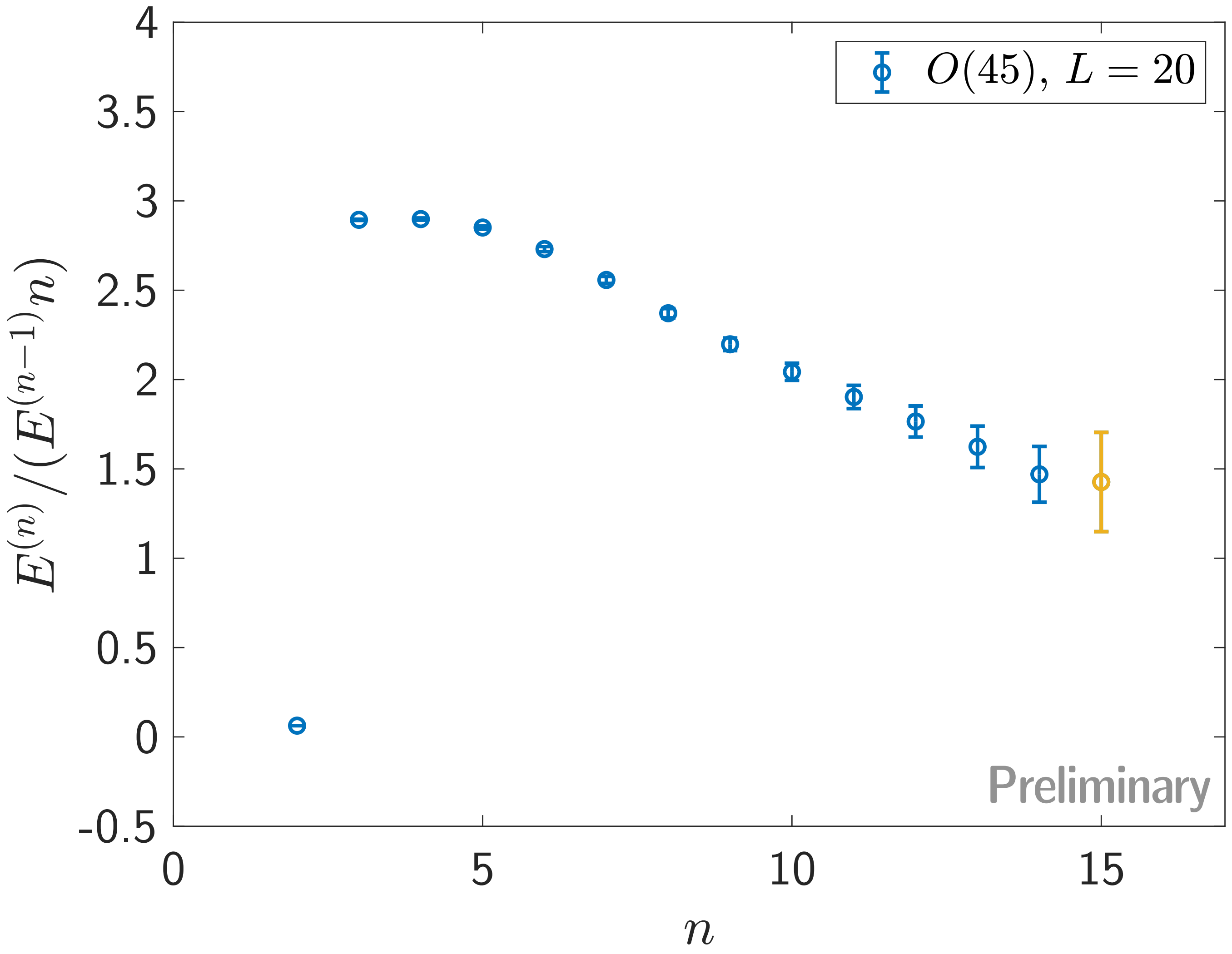


NSPT on O(N) non-linear sigma model

One possible following application:

performing asymptotic computations

... i.e. finding evidence of **renormalons** in non-linear sigma models



Still preliminary result:

currently ongoing simulations at higher-orders

Summary and perspectives

We numerically show that :

- Distributions (as expected) difficult to explore in small N $O(N)$ non-linear sigma model
- Increasing number of DoF per site (i.e. tuning N) implies more friendly distributions
- This is not a trivial effect of lattice (self)averaging (i.e. it is an N, not L effect)

Very near future works :

- More quantitative description, in particular relation between N and n
- Capturing asymptotic behaviour (renormalons)

Longer term plan :

- Tacking $CP^{(N-1)}$
- In this framework: perturbative expansion around non-trivial vacua

Thank you for your attention!