The $f_{PS}/m_V$ and $f_V/m_V$ ratios and the conformal window

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Context and motivation

Lower end of conformal window for $SU(3)$?

Important input for many BSM theories

No clear consensus, 8 – 13
Motivation

- Lattice would be ideal, but very costly: large finite volume effects, large systematic errors, need for large statistics, …

- All kinds of not “ab initio” approaches

- Our approach will also be speculative somewhat, but combine both perturbative and non-perturbative physics
Setup

- **Perturbative** calculations: reliable close to $N_f^{asympt} = 16.5$ (this work)

- **Non-perturbative** calculations: for low $2 \leq N_f \leq 10$ (past work)

- Combine both in a meaningful way
Setup

Define $f_{PS,V}$ and $m_V$ at finite fermion mass $m$

For all $N_f$: finite and scheme independent (physical)
Setup - below conformal window

Chiral limit - below conformal window

\[ f_{PS}, \ f_V, \ m_V \sim \Lambda \]

Ratio \[ \frac{f_{PS,V}}{m_V} = O(\Lambda)/O(\Lambda) = \text{const} \text{ finite} \]
Setup - inside conformal window

Chiral limit - inside conformal window

\[ f_{PS}, \ f_V, \ m_V \sim m^\alpha \]

With the same \( \alpha = \frac{1}{1+\gamma} \)

Ratio \( f_{PS,V}/m_V = O(m^\alpha)/O(m^\alpha) = \text{const} \) finite
Setup

The ratios are well-defined in the chiral limit for all $N_f \leq 16.5$

Just function of $N_f$
Past lattice work

Low $N_f$

- $f_{PS}/m_V$ in chiral, continuum limit for $2 \leq N_f \leq 10$

- Largely $N_f$-independent

- Some constant $\approx 1/8$

- $f_V$ from $f_{PS}$ using KSRF, $f_V = \sqrt{2} f_{PS}$
Setup

High $N_f$

- $N_f = 16.5$, free theory
- $m_V = 2m$
- $f_{PS,V} = 0$
- $f_{PS,V}/m_V = 0$

Something happens between $N_f = 10$ (non-zero ratio) and $N_f = 16.5$ (zero ratio)
Goals

Calculate $f_{PS,V}$ and $m_V$ in perturbation theory

See how far down we can go from $N_f = 16.5$

Hopefully match with highest $N_f = 10$ from the lattice studies
Bound states in perturbation theory (think of positronium)

Running scale $\mu = m$, $a(\mu) = \frac{g^2(\mu)}{16\pi^2}$

(p)NRQCD will give

$$f_{PS,V} = ma^{3/2}(m) (b_0 + b_1 a(m) + \ldots)$$

$$m_V = m(c_0 + c_1 a^2(m) + \ldots)$$

Here $\ldots$ contains $\log(a)$ too, coefficients depend on $N_f$
Perturbative calculation schematically

Ratio, $m$ drops out

Take chiral limit $m \to 0$, $a(m) \to a_*$ fixed point

$$\frac{f_{PS,V}}{m_V} = a_*^{3/2}(d_0 + d_1 a_* + d_2 a_*^2 + \ldots)$$

Here ... contains $\log(a_*)$ too, coefficients depend on $N_f$
Banks-Zaks expansion of $a_*$

$$\varepsilon = 16.5 - N_f$$ distance from upper end of conformal window

Use 5-loop $\beta$-function to expand

$$a_* = \varepsilon (e_0 + e_1 \varepsilon + e_2 \varepsilon^2 + e_3 \varepsilon^3 + \ldots)$$

Main results:

$$\frac{f_{PS,V}}{m_V} = \varepsilon^{3/2} (h_0 + h_1 \varepsilon + h_2 \varepsilon^2 + \ldots)$$

Here ... contains $\log(\varepsilon)$ too, coefficients are constants
Methodology

**NRQCD** – non-relativistic effective theory

**pNRQCD** – projection onto 2-body problem

$m_V$: energies from Schroedinger equation

$f_{PS,V}$: wave function at origin
Results, $m_V$

$$m_V = c_0 m \left( 1 + c_2 a^2(m) + c_{30} a^3(m) + c_{31} a^3(m) \log a(m) + O(a^4) \right)$$

$$c_0 = 2$$

$$c_2 = -2C_F^2 \pi^2$$

$$c_{30} = \frac{4}{9} \pi^2 C_A C_F^2 \left( 66 \log(4\pi C_F) - 97 \right)$$

$$c_{31} = \frac{88}{3} \pi^2 C_A C_F^2$$
Results, $f_V$ NNLO

$$f_V = b^V_0 m a^{3/2}(m) \left(1 + \sum_{n=1}^{3} \sum_{k=0}^{n} b^V_{nk} a^n(m) \log^k a(m) + O(a^4)\right)$$

$$b^V_0 = \sqrt{8N_c C^3_F \pi}, \quad b^V_{10} = \frac{161}{6} - \frac{11\pi^2}{3} + 33 \log \left(\frac{3}{16\pi}\right), \quad b^V_{11} = -33$$

$$b^V_{20} = \left(-\frac{64\pi^2}{27} + \frac{704}{27}\right) N_f + \frac{9781\zeta(3)}{9} - \frac{27\pi^4}{8} + \frac{1126\pi^2}{81} + \frac{9997}{72} +$$

$$+ \frac{1815 \log^2 \pi}{2} + \frac{1815}{2} \log^2 \left(\frac{16}{3}\right) + \log \left(\frac{16}{3}\right) \left(-\frac{2581}{2} + \frac{605\pi^2}{3} + 1815 \log(\pi)\right) +$$

$$+ \left(\frac{4325\pi^2}{27} - \frac{2581}{2}\right) \log(\pi) - \frac{256}{81} \pi^2 \log(8) - \frac{1120}{27} \pi^2 \log \left(\frac{8}{3}\right) - \frac{512}{9} \pi^2 \log(2)$$

$$b^V_{21} = \frac{4325\pi^2}{27} - \frac{2581}{2} + 1815 \log \left(\frac{16\pi}{3}\right), \quad b^V_{22} = \frac{1815}{2}.$$
Results, $f_V$ N$^3$LO

$$f_V = b_0^V m a^{3/2}(m) \left( 1 + \sum_{n=1}^{3} \sum_{k=0}^{n} b_{nk}^V a^n(m) \log^k a(m) + O(a^4) \right)$$

$$b_{30}^V = 0.8198 N_f^2 - 362.7 N_f - 1.0901(1) \times 10^6$$

$$b_{31}^V = -88.42 N_f - 7.7493 \times 10^5$$

$$b_{32}^V = -2.1651 \times 10^5$$

$$b_{33}^V = -2.3292 \times 10^4$$

Part of it numerical only
Results, $f_{PS}$ NNLO

$$f_{PS} = b_{PS}^0 m a^{3/2}(m) \left(1 + \sum_{n=1}^{2} \sum_{k=0}^{n} b_{nk}^{PS} a^n(m) \log^k a(m) + O(a^3)\right)$$

$$b_{PS}^0 = \sqrt{8 N_c C_F^3 \pi}, \quad b_{10}^{PS} = \frac{59}{2} - \frac{11 \pi^2}{3} + 33 \log \left(\frac{3}{16 \pi}\right), \quad b_{11}^{PS} = -33$$

$$b_{20}^{PS} = N_f \left(-\frac{32 \pi^2}{9} + \frac{344}{9}\right) + 961 \zeta(3) - \frac{27 \pi^4}{8} + \frac{1310 \pi^2}{27} + \frac{23053}{72} +$$

$$+ \frac{1815 \ln^2 \pi}{2} + \frac{1815}{2} \ln^2 \left(\frac{16}{3}\right) + \ln \left(\frac{16}{3}\right) \left(-\frac{2757}{2} + \frac{1271 \pi^2}{9} + 1815 \ln \pi\right) +$$

$$+ \left(\frac{1271 \pi^2}{9} - \frac{2757}{2}\right) \ln \pi - \frac{272}{9} \pi^2 \ln 2$$

$$b_{21}^{PS} = \frac{1271 \pi^2}{9} - \frac{2757}{2} + \frac{1815}{2} \ln \left(\frac{256 \pi^2}{9}\right), \quad b_{22}^{PS} = \frac{1815}{2}.$$
Main result, Banks-Zaks expansion of ratios

\[
\frac{f_V}{m_V} = \varepsilon^{3/2} C_0 \left( 1 + \sum_{n=1}^{3} \sum_{k=0}^{n} C_{nk} \varepsilon^n \log^k \varepsilon + O(\varepsilon^4) \right)
\]

\[
C_0 = 0.005826678
\]

\[
C_{10} = 0.4487893 \quad C_{11} = -0.2056075
\]

\[
C_{20} = 0.2444502 \quad C_{21} = -0.1624891 \quad C_{22} = 0.03522870
\]

\[
C_{30} = 0.10604(3) \quad C_{31} = -0.1128420 \quad C_{32} = 0.03695458 \quad C_{33} = -0.005633665
\]
Main result, Banks-Zaks expansion of ratios

\[ \frac{f_{PS}}{m_V} = \varepsilon^{3/2} C_0 \left( 1 + \sum_{n=1}^{2} \sum_{k=0}^{n} D_{nk} \varepsilon^n \log^k \varepsilon + O(\varepsilon^3) \right) \]

\[ D_{10} = 0.4654041 \quad D_{11} = -0.2056075 \]

\[ D_{20} = 0.2845697 \quad D_{21} = -0.1737620 \quad D_{22} = 0.03528692 \]
Notes

- Coefficients do not blow up (unlike $f_{V,PS}, m_V$ in terms of $a$)

- Coefficients are scheme independent
$f_V/m_V$

$N^3LO$ perturbative result

Direct lattice results only for $f_{PS}$

Use KSRF relation to extract $f_V$

$$f_V = \sqrt{2}f_{PS}$$

Conservatively assign 12% uncertainty
Main result - $f_V/m_V$
Main result - $f_V/m_V$
Main result - $f_V/m_V$
Main result - $f_V/m_V$
Main result - $f_V/m_V$ - speculation

![Graph showing $f_V/m_V$ vs $N_f$]
Main result - $f_V/m_V$

Important observations

- NNLO and $N^3$LO almost the same down to $N_f = 12$

- $N^3$LO matches at $N_f \approx 12$ last non-pertubative point $N_f = 10$
Main result - $f_{PS}/m_V$
Main result - $f_{PS}/m_V$
Main result - $f_{PS}/m_V$
Main result - $f_{PS}/m_V$ - speculation
Main result - $f_{PS}/m_V$

Important observations

- $N^3\text{LO}$ not available
- Assume similar to $f_V/m_V$ (speculation)
- Match seems to be around $N_f \approx 13$
Conclusions

- Perturbation theory perhaps reliable down to $N_f = 12$

- $N_f^* \approx 12$ and $N_f^* \approx 13$ from the two ratios

- In any case: abrupt change in ratios at these $N_f$

- Our method combines perturbative and non-perturbative input
Improvements for the future

- $N^3$LO calculation of $f_{PS}$ (difficult)

- Direct $f_V$ lattice calculation for $N_f \leq 10$ (doable)

- Perhaps $N_f = 11, 12$ lattice calculation (costly)

- $N^4$LO: 6-loop $\beta$-function would be needed (not any time soon)
Thank you for your attention!