The f_{PS}/m_V and f_V/m_V ratios and the conformal window

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Context and motivation

Lower end of conformal window for $SU(3)$?

Important input for many BSM theories

No clear consensus, $8-13$

Motivation

- Lattice would be ideal, but very costly: large finite volume effects, large systematic errors, need for large statistics, ...

- All kinds of not “ab initio” approaches

- Our approach will also be speculative somewhat, but combine both **perturbative** and **non-perturbative** physics

Setup

- **Perturbative** calculations: reliable close to $N_f^{asympt} = 16.5$ (this work)

- **Non-perturbative** calculations: for low $2 \le N_f \le 10$ (past work)

- Combine both in a meaningful way

Setup

Define $f_{PS,V}$ and m_V at finite fermion mass m

For all N_f: finite and scheme independent (physical)

Setup - below conformal window

Chiral limit - below conformal window

$$f_{PS}, \;\; f_V, \;\; m_V \sim \Lambda$$

Ratio $f_{PS,V}/m_V = O(\Lambda)/O(\Lambda) = const$ finite

Setup - inside conformal window

Chiral limit - inside conformal window

$$f_{PS},\;\; f_V,\;\; m_V \sim m^{\alpha}$$

With the *same* $\alpha = \frac{1}{1+\gamma}$

Ratio $f_{PS,V}/m_V = O(m^{\alpha})/O(m^{\alpha}) = const$ finite

Setup

The ratios are well-defined in the chiral limit for all $N_f \leq 16.5$

Just function of N_f

Past lattice work

Low N_f

JHEP 05 (2019) 197, [arXiv: 1905.01909]
JHEP 07 (2021) 202, [arXiv: 2107.05996]

- f_{PS}/m_V in chiral, continuum limit for $2 \leq N_f \leq 10$
- Largely N_f-independent
- Some constant $\approx 1/8$
- f_V from f_{PS} using KSRF, $f_V = \sqrt{2}\, f_{PS}$

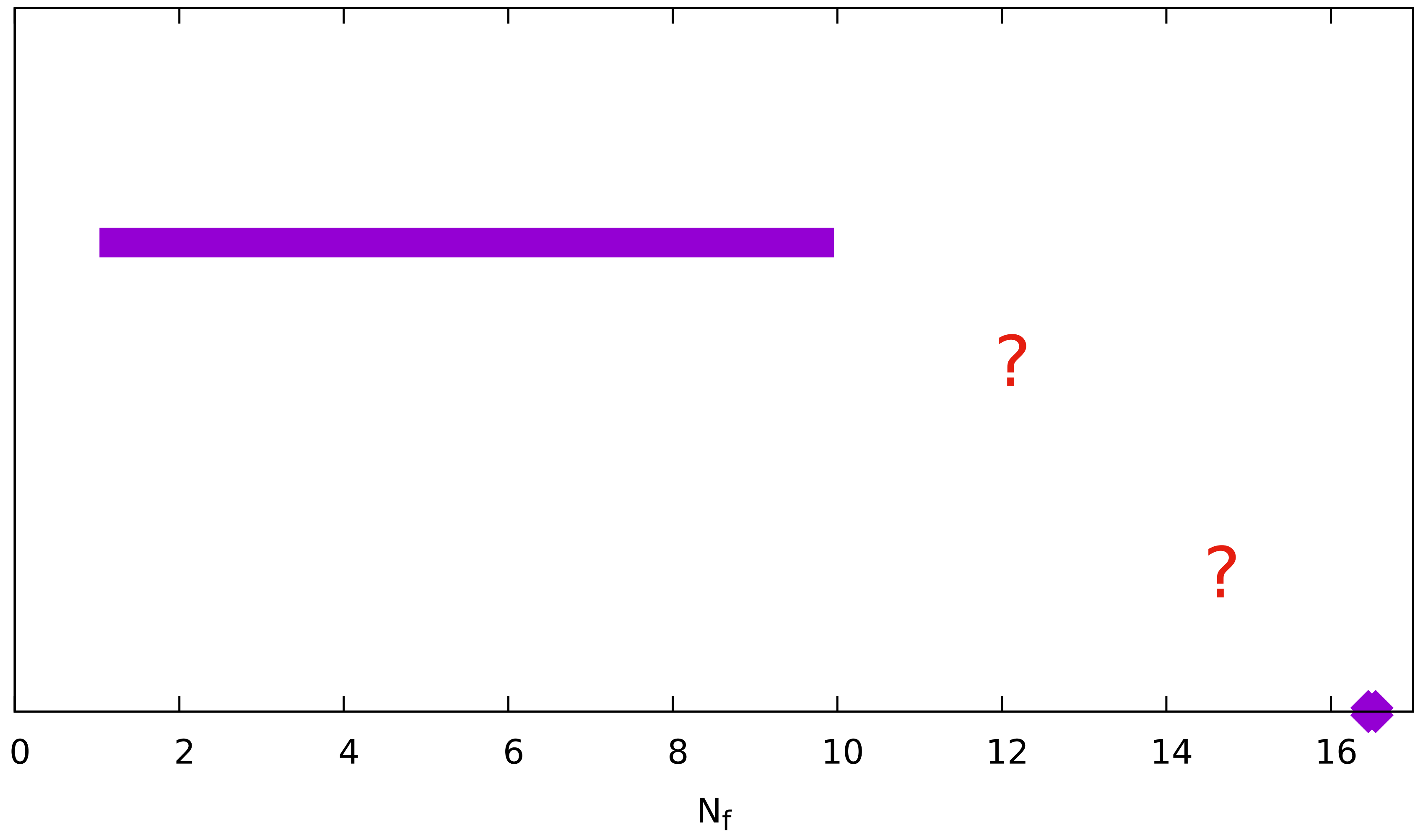
Setup

High N_f

- $N_f = 16.5$, free theory
- $m_V = 2m$
- $f_{PS,V} = 0$
- $f_{PS,V}/m_V = 0$

Something happens between $N_f = 10$ (non-zero ratio) and $N_f = 16.5$ (zero ratio)

Cartoon



Goals

Calculate $f_{PS,V}$ and m_V in perturbation theory

See how far down we can go from $N_f = 16.5$

Hopefully match with highest $N_f = 10$ from the lattice studies

Bound states in perturbation theory (think of positronium)

Running scale $\mu = m$, $a(\mu) = \frac{g^2(\mu)}{16\pi^2}$

(p)NRQCD will give

$$f_{PS,V} = m\, a^{3/2}(m)\, (b_0 + b_1\, a(m) + \ldots)$$

$$m_V = m(c_0 + c_1\, a^2(m) + \ldots)$$

Here ... contains $\log(a)$ too, coefficients depend on N_f

Perturbative calculation schematically

Ratio, m drops out

Take chiral limit $m \to 0$, $a(m) \to a_*$ fixed point

$$\frac{f_{PS,V}}{m_V} = a_*^{3/2}(d_0 + d_1\, a_* + d_2\, a_*^2 + \ldots)$$

Here ... contains $\log(a_*)$ too, coefficients depend on N_f

Banks-Zaks expansion of a_*

$\varepsilon = 16.5 - N_f$ distance from upper end of conformal window

Use 5-loop β-function to expand

$$a_* = \varepsilon\,(e_0 + e_1\,\varepsilon + e_2\,\varepsilon^2 + e_3\,\varepsilon^3 + \ldots)$$

Main results:

$$\frac{f_{PS,V}}{m_V} = \varepsilon^{3/2}(h_0 + h_1\,\varepsilon + h_2\,\varepsilon^2 + \ldots)$$

Here ... contains $\log(\varepsilon)$ too, coefficients are constants

Methodology

NRQCD – non-relativistic effective theory

pNRQCD – projection onto 2-body problem

m_V: energies from Schroedinger equation

$f_{PS,V}$: wave function at origin

Results, m_V

$$m_V = c_0 m \left(1 + c_2 a^2(m) + c_{30} a^3(m) + c_{31} a^3(m) \log a(m) + O(a^4)\right)$$

$$c_0 = 2$$

$$c_2 = -2 C_F^2 \pi^2$$

$$c_{30} = \frac{4}{9} \pi^2 C_A C_F^2 \left(66 \log(4\pi C_F) - 97\right)$$

$$c_{31} = \frac{88}{3} \pi^2 C_A C_F^2$$

Results, f_V NNLO

$$f_V = b_0^V\, m\, a^{3/2}(m) \left(1 + \sum_{n=1}^{3} \sum_{k=0}^{n} b_{nk}^V\, a^n(m) \log^k a(m) + O(a^4)\right)$$

$$b_0^V = \sqrt{8 N_c C_F^3 \pi}, \qquad b_{10}^V = \frac{161}{6} - \frac{11\pi^2}{3} + 33 \log\left(\frac{3}{16\pi}\right), \qquad b_{11}^V = -33$$

$$b_{20}^V = \left(-\frac{64\pi^2}{27} + \frac{704}{27}\right) N_f + \frac{9781\zeta(3)}{9} - \frac{27\pi^4}{8} + \frac{1126\pi^2}{81} + \frac{9997}{72} +$$

$$+ \frac{1815 \log^2 \pi}{2} + \frac{1815}{2} \log^2\left(\frac{16}{3}\right) + \log\left(\frac{16}{3}\right)\left(-\frac{2581}{2} + \frac{605\pi^2}{3} + 1815 \log(\pi)\right) +$$

$$+ \left(\frac{4325\pi^2}{27} - \frac{2581}{2}\right) \log(\pi) - \frac{256}{81}\pi^2 \log(8) - \frac{1120}{27}\pi^2 \log\left(\frac{8}{3}\right) - \frac{512}{9}\pi^2 \log(2)$$

$$b_{21}^V = \frac{4325\pi^2}{27} - \frac{2581}{2} + 1815 \log\left(\frac{16\pi}{3}\right), \qquad b_{22}^V = \frac{1815}{2}.$$

Results, f_V N^{3}LO

$$f_V = b_0^V\, m\, a^{3/2}(m) \left(1 + \sum_{n=1}^{3} \sum_{k=0}^{n} b_{nk}^V a^n(m) \log^k a(m) + O(a^4)\right)$$

$$b_{30}^V = 0.8198 N_f^2 - 362.7 N_f - 1.0901(1) \times 10^6$$

$$b_{31}^V = -88.42 N_f - 7.7493 \times 10^5$$

$$b_{32}^V = -2.1651 \times 10^5$$

$$b_{33}^V = -2.3292 \times 10^4$$

Part of it numerical only

Results, f_{PS} NNLO

$$f_{PS} = b_0^{PS} m a^{3/2}(m) \left(1 + \sum_{n=1}^{2} \sum_{k=0}^{n} b_{nk}^{PS} a^n(m) \log^k a(m) + O(a^3)\right)$$

$$b_0^{PS} = \sqrt{8 N_c C_F^3} \pi, \quad b_{10}^{PS} = \frac{59}{2} - \frac{11\pi^2}{3} + 33 \log\left(\frac{3}{16\pi}\right), \quad b_{11}^{PS} = -33$$

$$b_{20}^{PS} = N_f \left(-\frac{32\pi^2}{9} + \frac{344}{9}\right) + 961\zeta(3) - \frac{27\pi^4}{8} + \frac{1310\pi^2}{27} + \frac{23053}{72} +$$

$$+ \frac{1815 \log^2 \pi}{2} + \frac{1815}{2} \log^2\left(\frac{16}{3}\right) + \log\left(\frac{16}{3}\right) \left(-\frac{2757}{2} + \frac{1271\pi^2}{9} + 1815 \log \pi\right) +$$

$$+ \left(\frac{1271\pi^2}{9} - \frac{2757}{2}\right) \log \pi - \frac{272}{9} \pi^2 \log 2$$

$$b_{21}^{PS} = \frac{1271\pi^2}{9} - \frac{2757}{2} + \frac{1815}{2} \log\left(\frac{256\pi^2}{9}\right), \qquad b_{22}^{PS} = \frac{1815}{2}.$$

Main result, Banks-Zaks expansion of ratios

$$\frac{f_V}{m_V} = \varepsilon^{3/2} C_0 \left(1 + \sum_{n=1}^{3} \sum_{k=0}^{n} C_{nk}\, \varepsilon^n \log^k \varepsilon + O(\varepsilon^4) \right)$$

$$C_0 = 0.005826678$$

$C_{10} = 0.4487893 \quad C_{11} = -0.2056075$

$C_{20} = 0.2444502 \quad C_{21} = -0.1624891 \quad C_{22} = 0.03522870$

$C_{30} = 0.10604(3) \quad C_{31} = -0.1128420 \quad C_{32} = 0.03695458 \quad C_{33} = -0.005633665$

Main result, Banks-Zaks expansion of ratios

$$\frac{f_{PS}}{m_V} = \varepsilon^{3/2} C_0 \left(1 + \sum_{n=1}^{2} \sum_{k=0}^{n} D_{nk}\, \varepsilon^n \log^k \varepsilon + O(\varepsilon^3) \right)$$

$D_{10} = 0.4654041$ $D_{11} = -0.2056075$

$D_{20} = 0.2845697$ $D_{21} = -0.1737620$ $D_{22} = 0.03528692$

Notes

- Coefficients do not blow up (unlike $f_{V,PS}, m_V$ in terms of a)
- Coefficients are scheme independent

f_V/m_V

N^{3}LO perturbative result

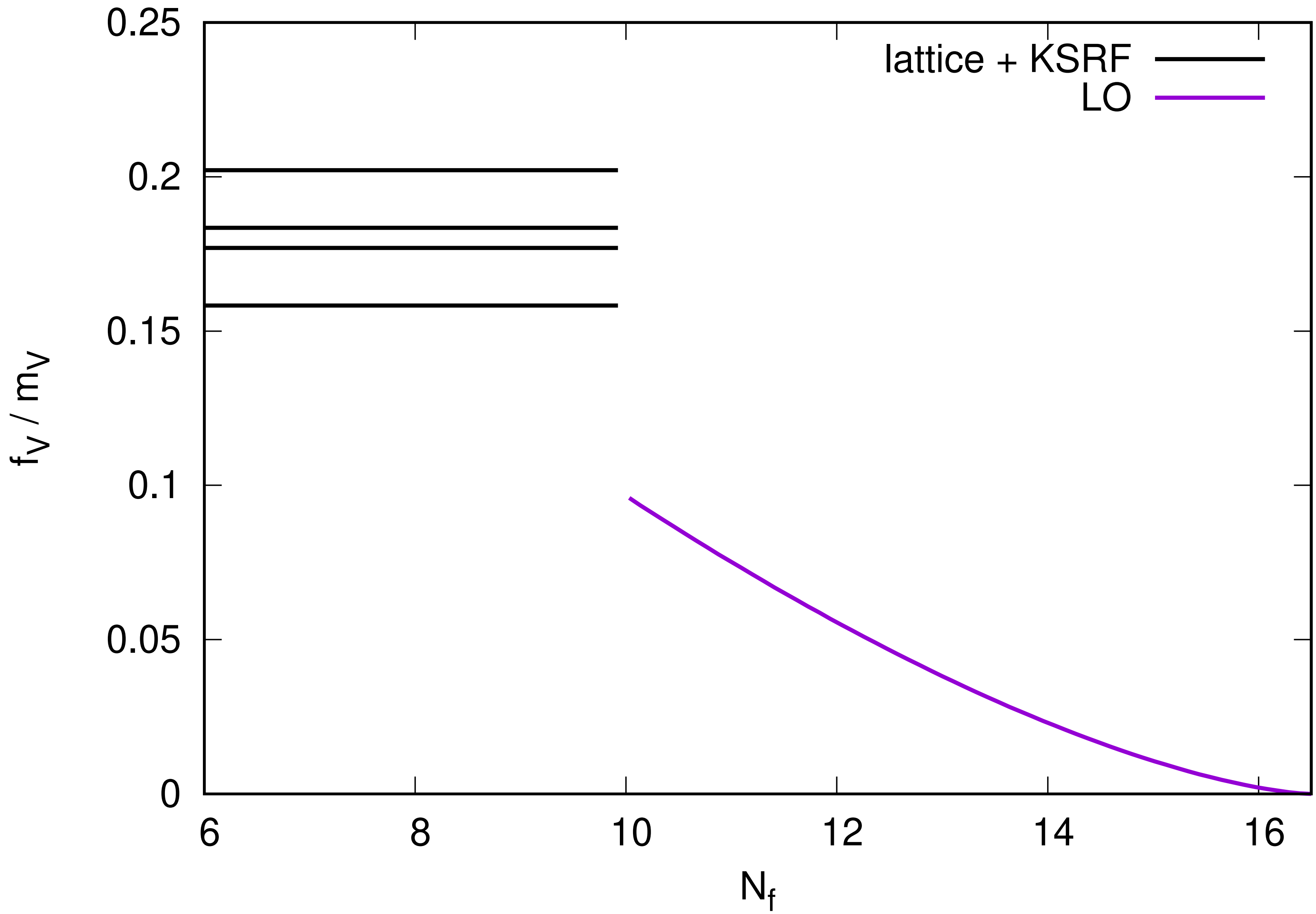
Direct lattice results only for f_{PS}

Use KSRF relation to extract f_V

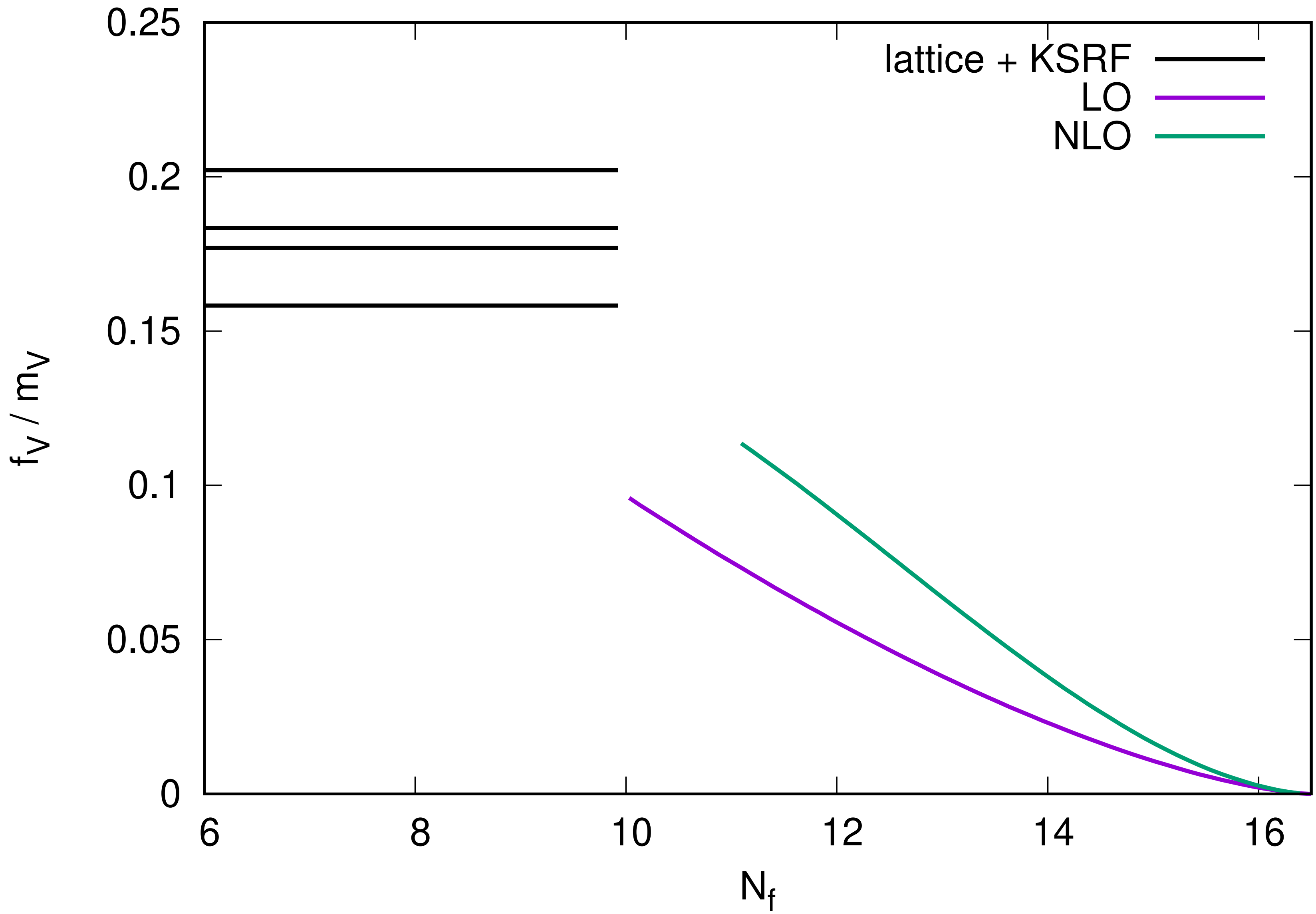
$$f_V = \sqrt{2} f_{PS}$$

Conservatively assign 12% uncertainty

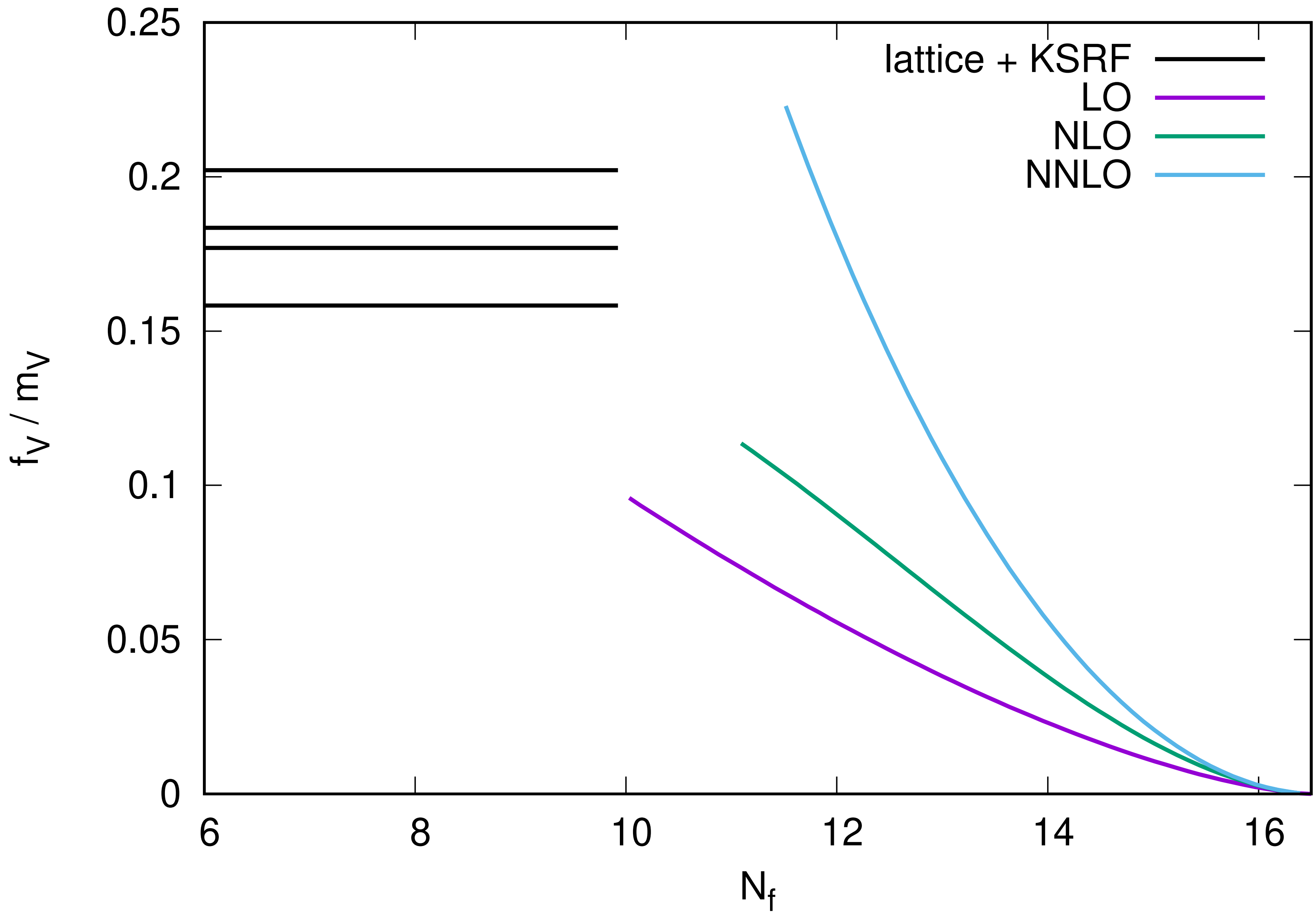
Main result - f_V/m_V



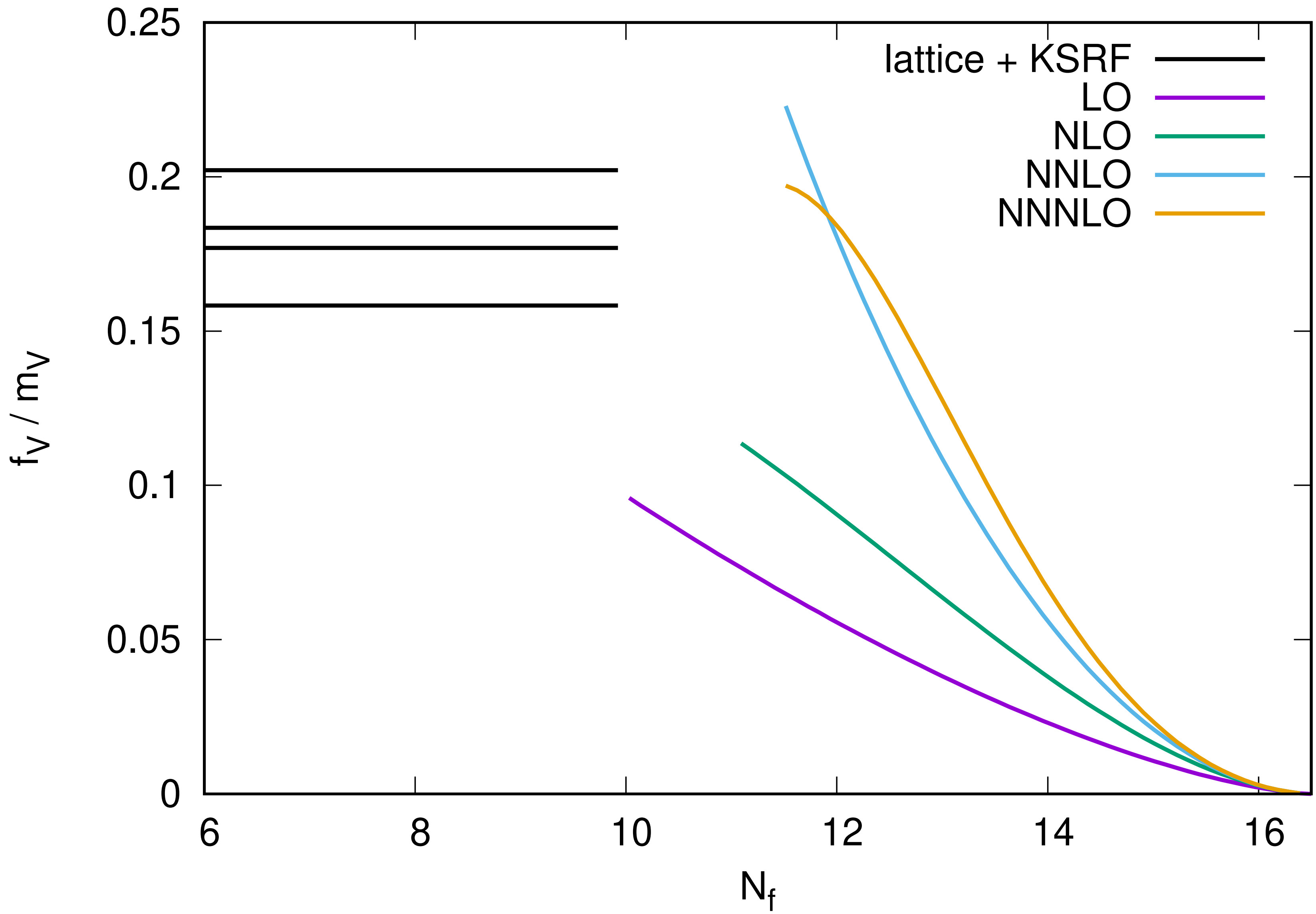
Main result - f_V/m_V



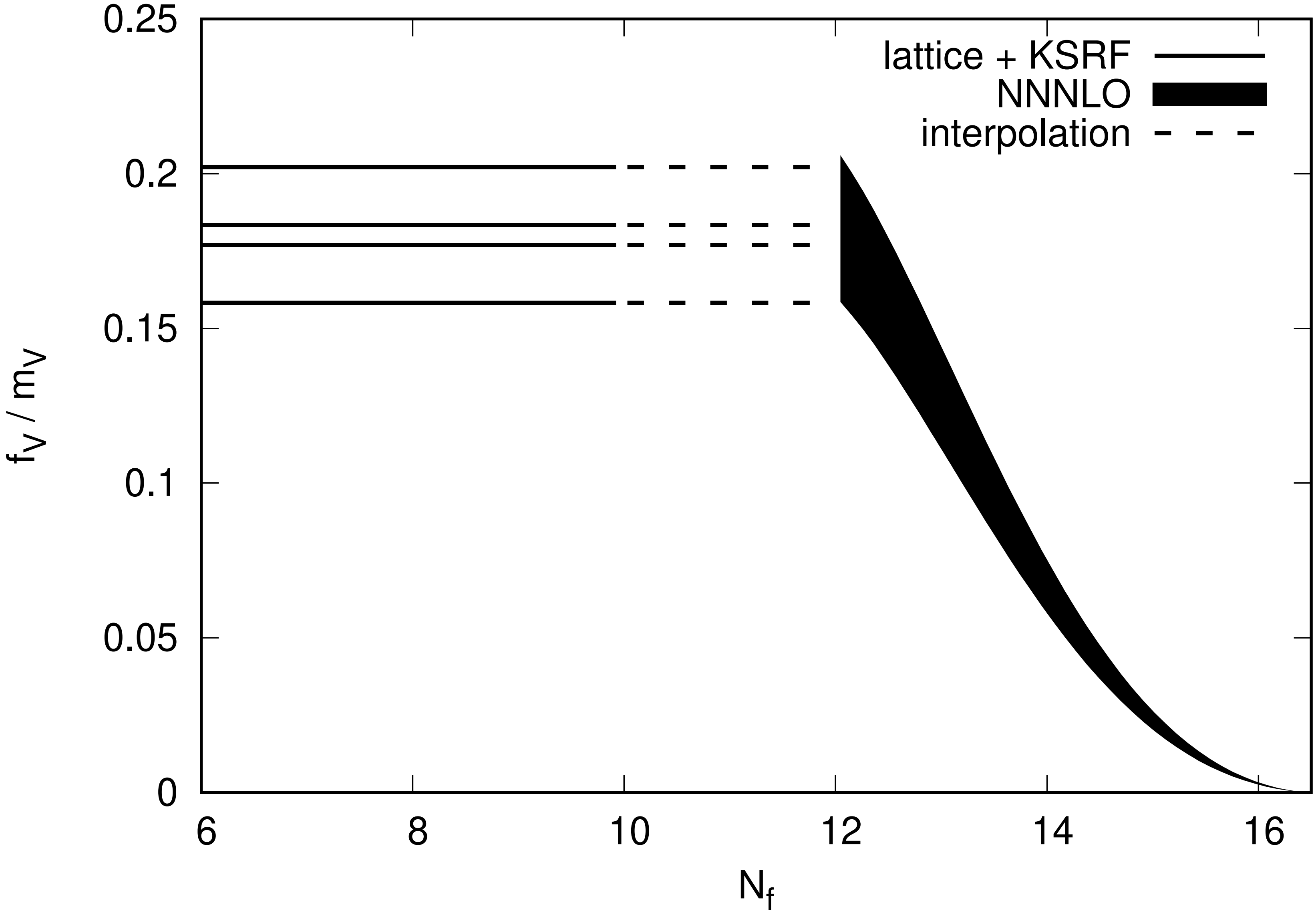
Main result - f_V/m_V



Main result - f_V/m_V



Main result - f_V/m_V - speculation

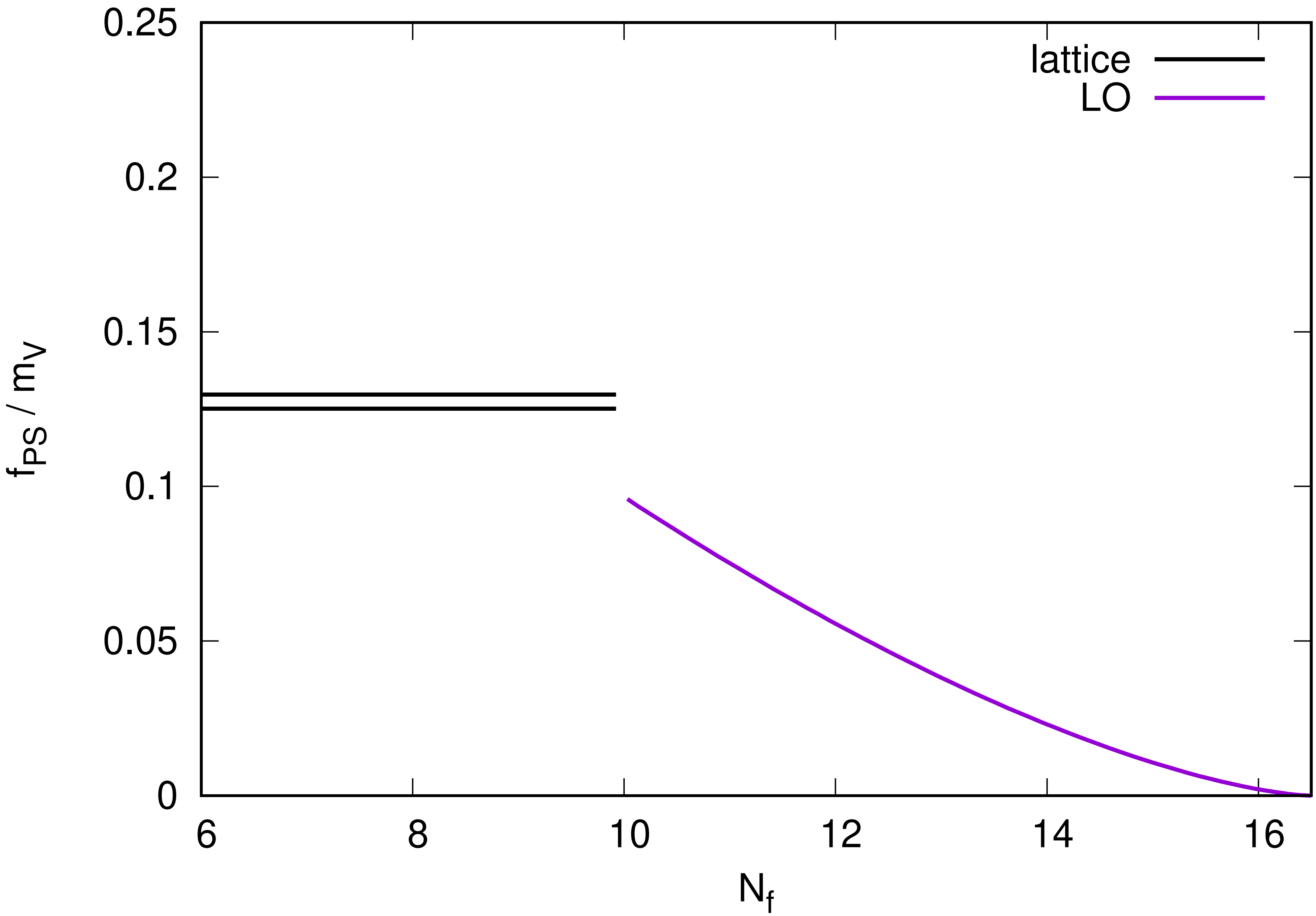


Main result - f_V/m_V

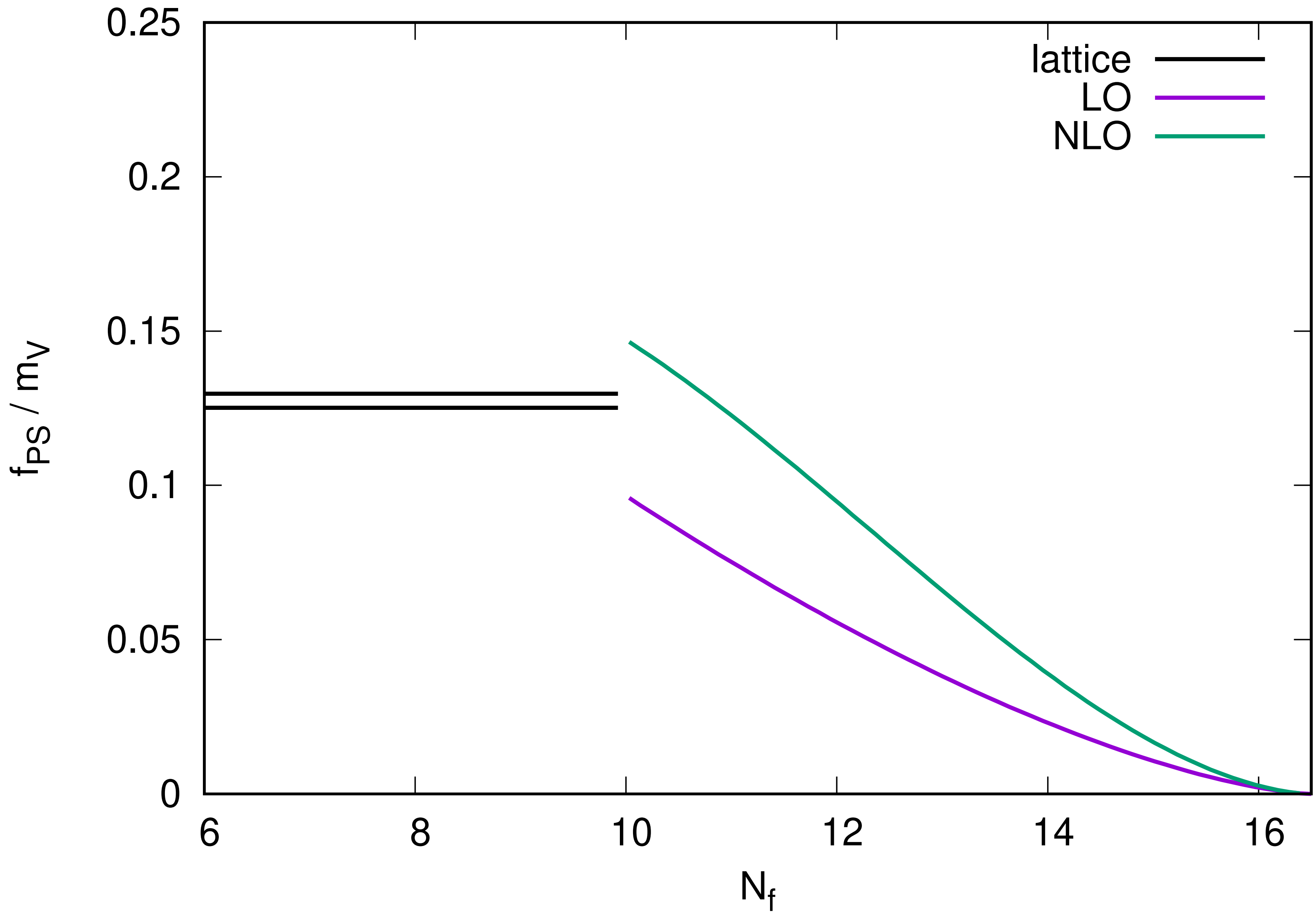
Important observations

- NNLO and N^3LO almost the same down to $N_f = 12$
- N^3LO matches at $N_f \approx 12$ last non-pertubative point $N_f = 10$

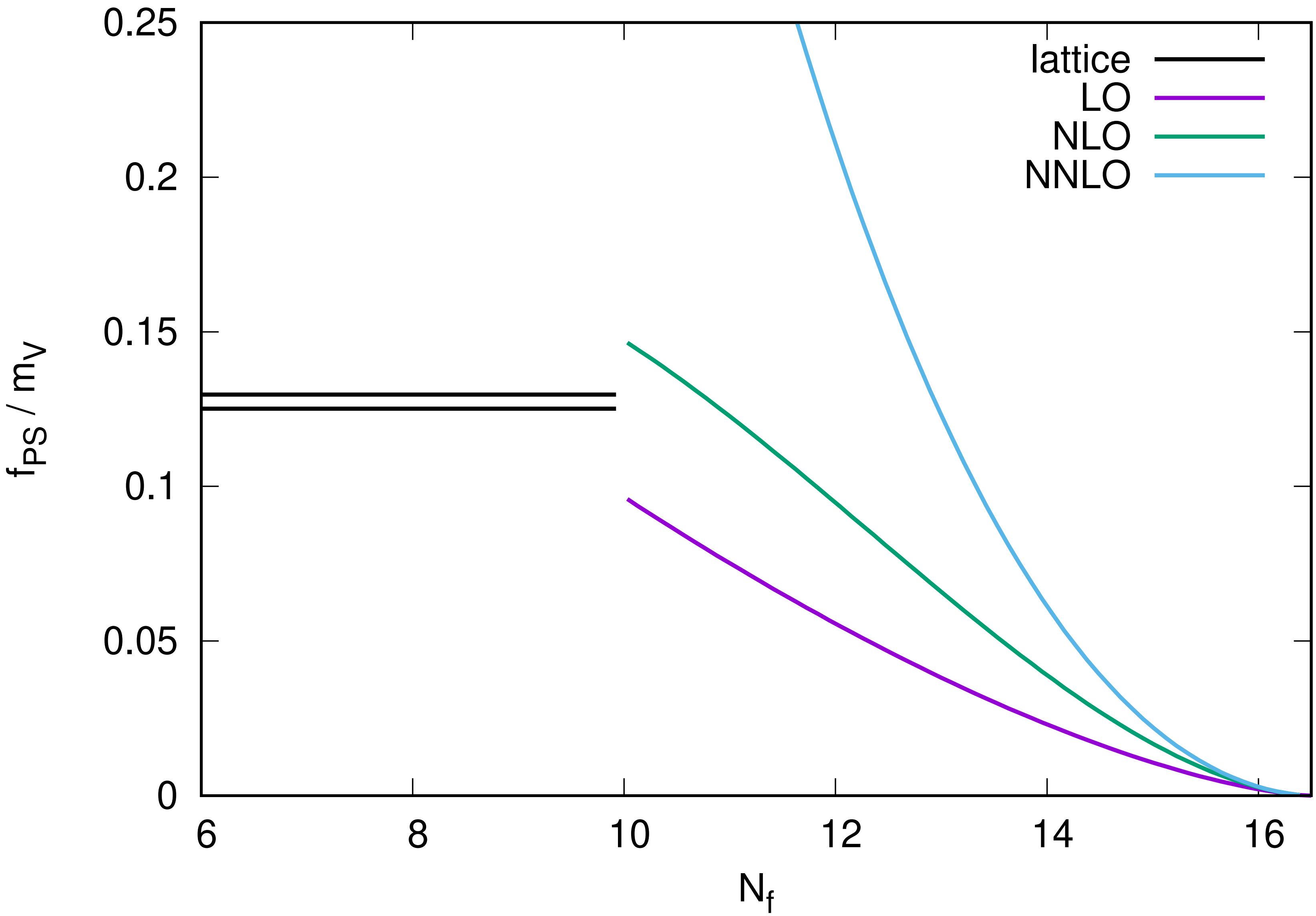
Main result - f_{PS}/m_V



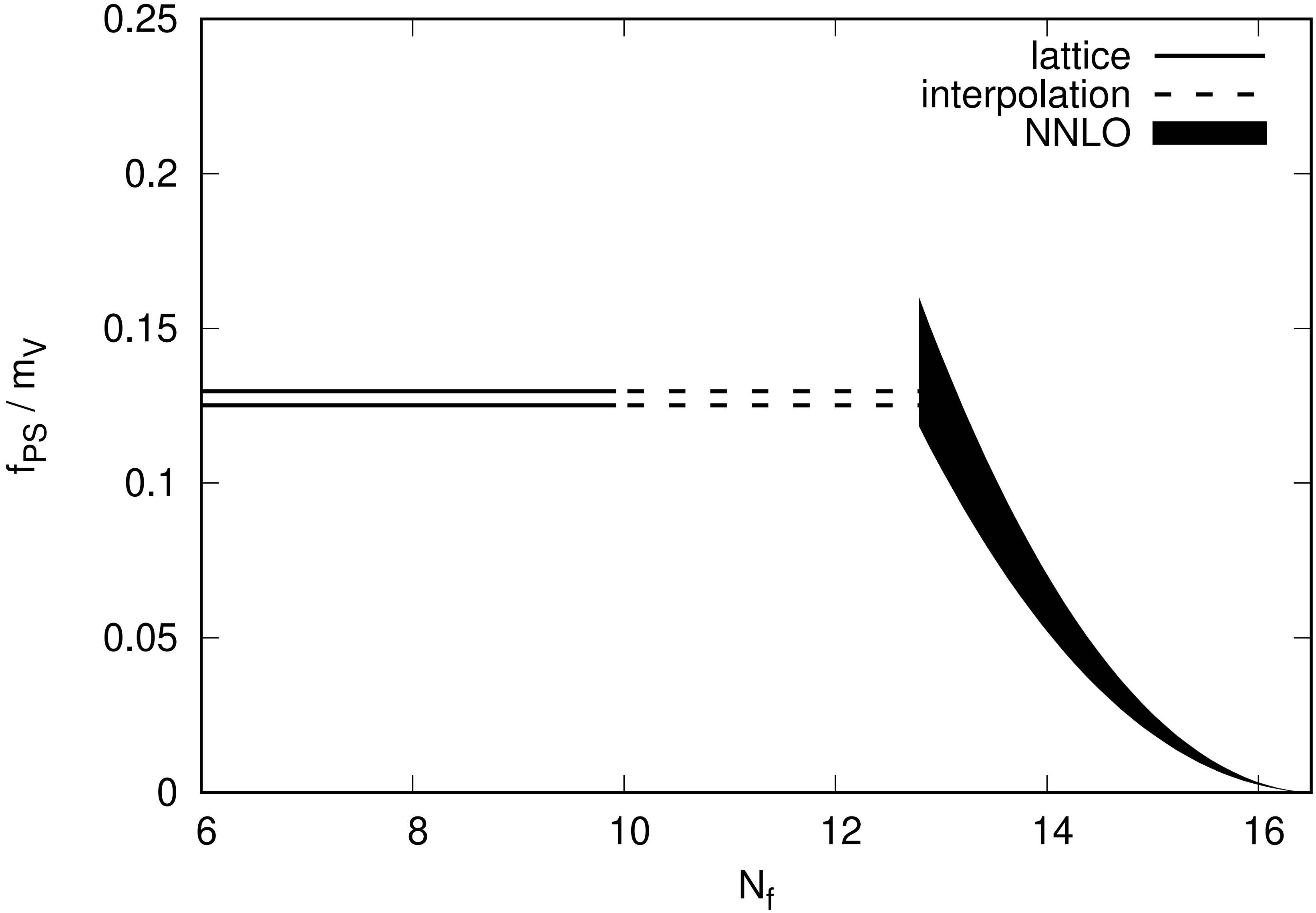
Main result - f_{PS}/m_V



Main result - f_{PS}/m_V



Main result - f_{PS}/m_V - speculation



Main result - f_{PS}/m_V

Important observations

- N^{3}LO not available
- Assume similar to f_V/m_V (speculation)
- Match seems to be around $N_f \approx 13$

Conclusions

- Perturbation theory perhaps reliable down to $N_f = 12$
- $N_f^* \approx 12$ **and** $N_f^* \approx 13$ **from the two ratios**
- In any case: abrupt change in ratios at these N_f
- Our method combines perturbative and non-perturbative input

Improvements for the future

- N^3LO calculation of f_{PS} (difficult)
- Direct f_V lattice calculation for $N_f \leq 10$ (doable)
- Perhaps $N_f = 11,\ 12$ lattice calculation (costly)
- N^4LO: 6-loop β-function would be needed (not any time soon)

Thank you for your attention!