Entanglement entropy from non-equilibrium lattice simulations

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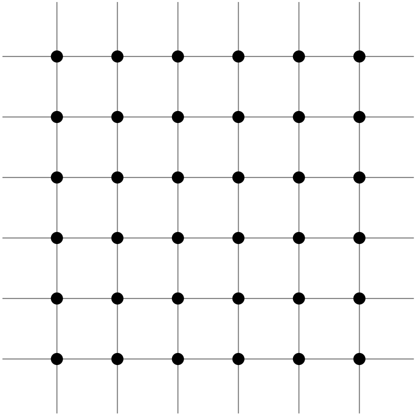
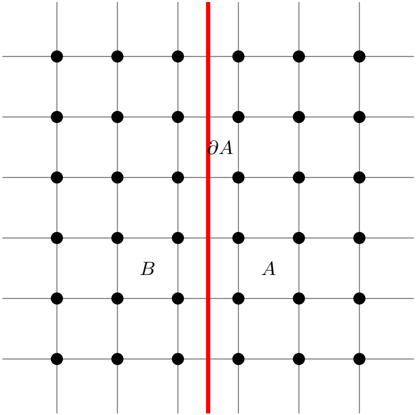
Motivation

- Entanglement in quantum systems is a broad subject, with applications in many different areas of physics, such as:
  - Quantum information
  - Condensed matter and CFT
  - AdS/CFT and quantum gravity
  - Gauge theories (see talk by R. Amorosso, Thu 4.40 pm)
- However analytical and numerical results are still limited to simple, highly symmetric systems.
- Non-equilibrium techniques can provide an efficient tool to calculate entanglement-related quantities [Alba; 2016][D'Emidio; 2019][Zhao *et al.*; 2021][Song *et al.*; 2023].

Entanglement in QFT



Entanglement in QFT



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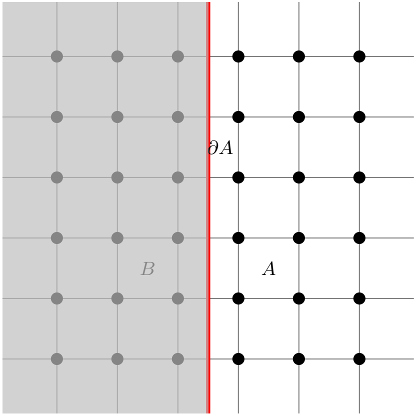
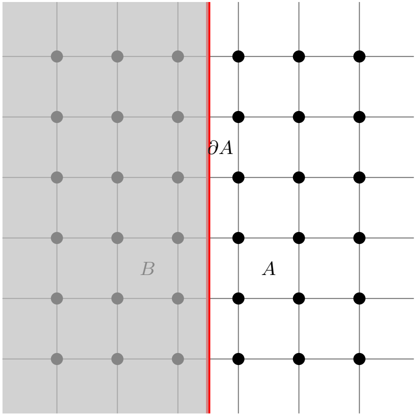


Entanglement in QFT



$$S(A) = -\operatorname{Tr}\{\rho_A \log \rho_A\} \qquad S_n = \frac{1}{1-n} \log \operatorname{Tr} \rho_A^n$$

Replica trick

- A common way to calculate Rényi entropies and other entanglement measurements is to exploit the replica trick [Calabrese, Cardy; 2004]

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{Z^n}$$

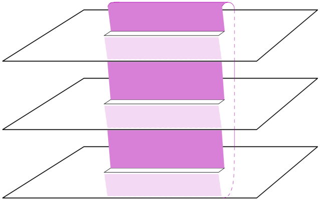


Image taken from [Cardy *et. al.*; 2007].

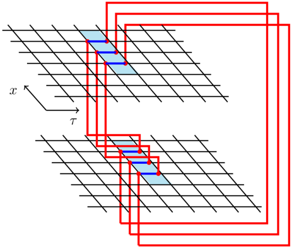


Image adapted from [Alba; 2016].

Entropic c-function

- Problem: Rényi entropies are always UV divergent.
- A common regularization consists in taking the derivative with respect to the length of the cut, that defines the so called entropic c-function, which is UV finite and encodes all the universal information contained in the Rényi entropies.

$$C_n = \frac{l^{D-1}}{|\partial A|}\frac{\partial S_n}{\partial l}$$

- Also $\partial S_n/\partial l$ can be written in terms of a ratio of partition functions. Using a lattice regularization

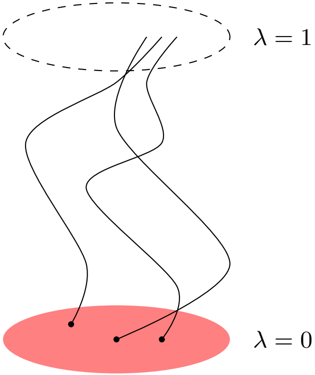
$$\frac{\partial S_n}{\partial l} \simeq \frac{1}{1-n}\frac{1}{a}\log\frac{Z_n(l+a)}{Z_n(l)}$$

- In recent years the Turin group has exploited Jarzynski's equality [Jarzynski; 1996] to perform high-precision lattice calculations of quantities involving ratios of partition functions [Caselle *et. al.*; 2016][Caselle *et. al.*; 2018][Francesconi *et. al.*; 2020][Caselle *et. al.*; 2022] (see talk by A. Nada, Mon 2.30 pm).

Jarzinski's theorem

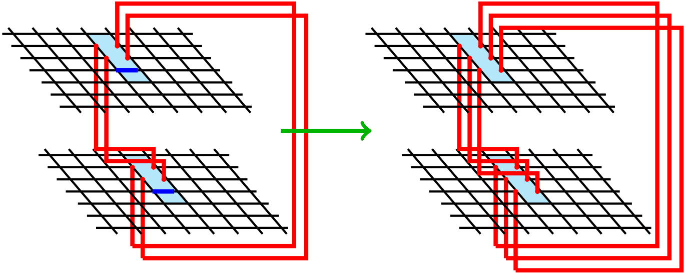
- Jarzynski's theorem [Jarzynski; 1996] is an exact result that connects averages of out-of-equilibrium trajectories of a statistical system to equilibrium free energies.
- The theorem is valid both for real and Monte Carlo time evolution.
- Consider the one parameter evolution $H_{\lambda=0} \to H_{\lambda=1}$. Jarzynki's theorem states that

$$\left\langle \exp\left(-\int \beta\delta W\right)\right\rangle = \frac{Z_{\lambda=1}}{Z_{\lambda=0}}$$



Our algorithm

$$\frac{\partial S_n}{\partial l} \simeq \frac{1}{1-n}\frac{1}{a}\log\frac{Z_n(l+a)}{Z_n(l)}$$



Theoretical results for $2D$ CFTs

- The theoretical prediction for a CFT on a cylinder of spatial length L is ($c = \frac{1}{2}$ for the Ising model)

$$C_2(x) = \frac{c}{8}\cos(\pi x) \qquad\qquad x = \frac{l}{L}$$

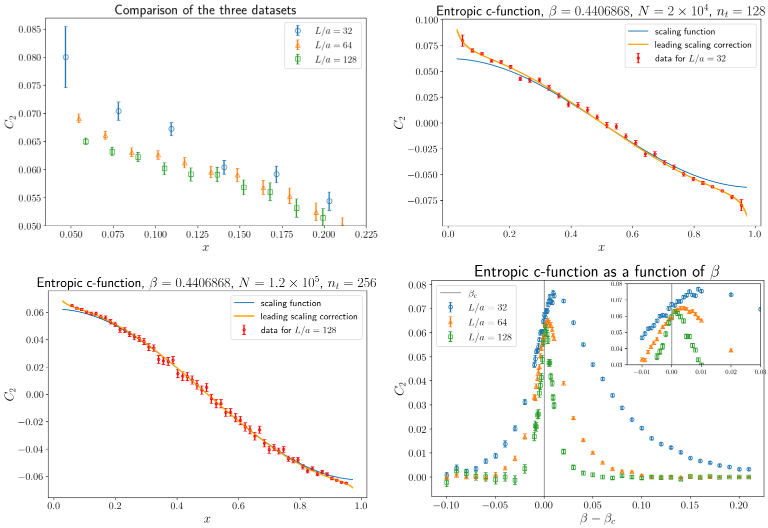
- This result holds in the scaling limit $L, l \gg 1$, while for finite sizes scaling corrections can be relevant.

- The general theory of unusual corrections to scaling of the entanglement entropy was developed in [Calabrese, Cardy; 2010].

- In the case of the $D = 2$ Ising model one expects

$$C_2(x) = C_2^{\text{CFT}}(x) + \frac{k}{2L}\cot(\pi x)$$

Benchmark: Ising $2D$



Some models in $D = 3$

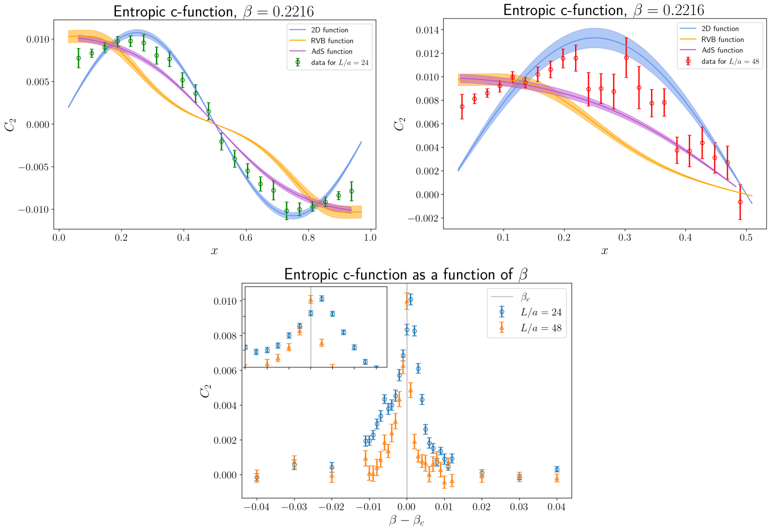
- For the $D = 3$ Ising model no analytical solution is known and only few numerical studies are present in literature [Inglis, Melko; 2013] [Kulchytskyy *et. al.*; 2019].
- We compared our numerical results at the critical point with three different models:
  - the $2D$ function
  - a function proposed in a study of resonance-valence-bond (RVB) dimers [Stéphan *et. al.*; 2012]
  - a function derived in [Chen *et. al.*; 2015] in a holographic setup using the Ryu-Takayanagi formula [Ryu, Takayanagi; 2006]

$$S_{2;2D}(x; c, k) = c \log(\sin(\pi x)) + k$$

$$S_{2;RVB}(x; c, k) = -2c \log \left\{ \frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)} \frac{\theta_3(2x\tau)\theta_3(2(1-x)\tau)}{\eta(2x\tau)\eta(2(1-x)\tau)} \right\} + k$$

$$S_{2;AdS}(x; c, k) = c\chi(x)^{-\frac{1}{3}} \left\{ \int_0^1 \frac{\mathrm{d}y}{y^2} \left(\frac{1}{\sqrt{P(\chi(x), y)}} - 1 \right) - 1 \right\} + k$$

Results for Ising $3D$



Conclusions and future prospects

- Our data for the $2D$ Ising model are in perfect agreement with the CFT prediction.
- In both cases we obtained precise results in a small amount of time (< 800 CPU hours for the largest lattice size both in $D = 2, 3$).
- This algorithm can be generalized to arbitrary spin models and gauge theories.

Future work:

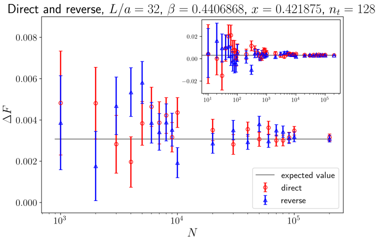
- Exploit the duality properties of the $3D$ Ising model to study the entanglement content of the $\mathbb{Z}_2$ gauge theory.
- Extension to non-Abelian gauge theories? [Buividovich, Polikarpov; 2008][Itou *et. al*; 2015][Rabenstein *et. al*; 2018]

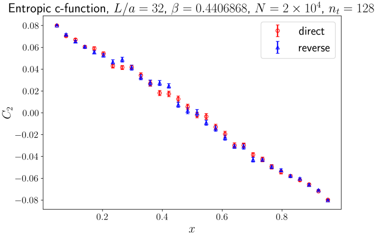
Appendix

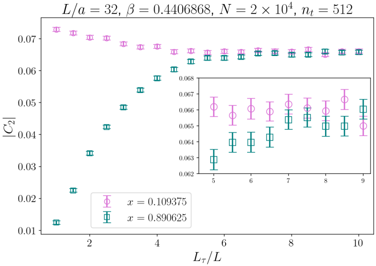
Some numerical details

- For our simulations we adapted the code found in [Komura, Okabe; 2014], implementing the replica space and Jarzynski's algorithm.
- The code is written in CUDA C to achieve high parallelization.
- We obtained precise results in a small amount of time: data for $L = 128$ required approximatively 750 hours on on the CINECA Marconi100 accelerated cluster, based on IBM Power9 architecture and Volta NVIDIA GPUs.
- Data for $L = 24, 48$ required respectively $\sim 270, 620$ hours on on the CINECA Marconi100 accelerated cluster.

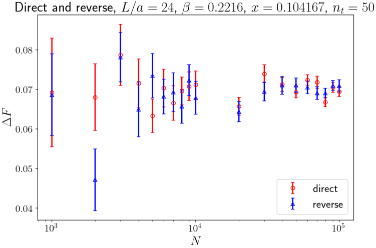
Benchmarks of the algorithm: $2D$

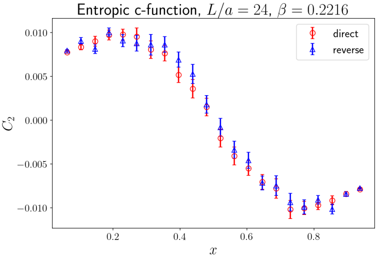


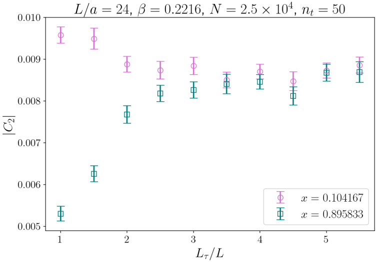




Benchmarks of the algorithm: $3D$







Duality transformation in $2D$