Entanglement entropy from non-equilibrium lattice simulations

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Motivation

Entanglement in quantum systems is a broad subject, with applications in many different areas of physics, such as:

- Quantum information
- Condensed matter and CFT
- $AdS/CFT$ and quantum gravity
- Gauge theories (see talk by R. Amorosso, Thu 4.40 pm)

However analytical and numerical results are still limited to simple, highly symmetric systems.

Non-equilibrium techniques can provide an efficient tool to calculate entanglement-related quantities [Alba; 2016][D’Emidio; 2019][Zhao et al.; 2021][Song et al.; 2023].
Entanglement in QFT

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Non-equilibrium simulations

Ising 2D

Ising 3D

Conclusions and future prospects

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Entanglement in QFT

\[ S(A) = - \text{Tr}\{\rho_A \log \rho_A\} \]

\[ S_n = \frac{1}{1 - n} \log \text{Tr} \rho_A^n \]
A common way to calculate Rényi entropies and other entanglement measurements is to exploit the replica trick [Calabrese, Cardy; 2004]

\[ S_n = \frac{1}{1 - n} \log \frac{Z_n}{Z^n} \]

Image taken from [Cardy et. al.; 2007].

Image adapted from [Alba; 2016].
Entropic c-function

- Problem: Rényi entropies are always UV divergent.
- A common regularization consists in taking the derivative with respect to the length of the cut, that defines the so called entropic c-function, which is UV finite and encodes all the universal information contained in the Rényi entropies.

\[
C_n = \frac{l^{D-1}}{|\partial A|} \frac{\partial S_n}{\partial l}
\]

- Also \( \frac{\partial S_n}{\partial l} \) can be written in terms of a ratio of partition functions. Using a lattice regularization

\[
\frac{\partial S_n}{\partial l} \approx \frac{1}{1 - n} \frac{1}{a} \log \frac{Z_n(l + a)}{Z_n(l)}
\]

- In recent years the Turin group has exploited Jarzynski’s equality [Jarzynski; 1996] to perform high-precision lattice calculations of quantities involving ratios of partition functions [Caselle et. al.; 2016][Caselle et. al.; 2018][Francesconi et. al.; 2020][Caselle et. al.; 2022] (see talk by A. Nada, Mon 2.30 pm).
Jarzynski’s theorem

- Jarzynski’s theorem [Jarzynski; 1996] is an exact result that connects averages of out-of-equilibrium trajectories of a statistical system to equilibrium free energies.

- The theorem is valid both for real and Monte Carlo time evolution.

- Consider the one parameter evolution $H_{\lambda=0} \rightarrow H_{\lambda=1}$. Jarzynki’s theorem states that

$$\left\langle \exp \left( - \int \beta \delta W \right) \right\rangle = \frac{Z_{\lambda=1}}{Z_{\lambda=0}}$$
Our algorithm

\[ \frac{\partial S_n}{\partial l} \approx \frac{1}{1 - n a} \log \frac{Z_n(l + a)}{Z_n(l)} \]
Theoretical results for 2D CFTs

- The theoretical prediction for a CFT on a cylinder of spatial length $L$ is ($c = \frac{1}{2}$ for the Ising model)

$$C_2(x) = \frac{c}{8} \cos(\pi x) \quad x = \frac{l}{L}$$

- This result holds in the scaling limit $L, l \gg 1$, while for finite sizes scaling corrections can be relevant.

- The general theory of unusual corrections to scaling of the entanglement entropy was developed in [Calabrese, Cardy; 2010].

- In the case of the $D = 2$ Ising model one expects

$$C_2(x) = C_2^{\text{CFT}}(x) + \frac{k}{2L} \cot(\pi x)$$
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Benchmark: Ising 2D

Comparison of the three datasets

Entropic c-function, $\beta = 0.4406868$, $N = 2 \times 10^4$, $n_t = 128$

Entropic c-function as a function of $\beta$
Some models in $D = 3$

- For the $D = 3$ Ising model no analytical solution is known and only few numerical studies are present in literature [Inglis, Melko; 2013] [Kulchytskyy et. al.; 2019].
- We compared our numerical results at the critical point with three different models:
  - the $2D$ function
  - a function proposed in a study of resonance-valence-bond (RVB) dimers [Stéphan et. al.; 2012]
  - a function derived in [Chen et. al.; 2015] in a holographic setup using the Ryu-Takayanagi formula [Ryu, Takayanagi; 2006]

$$S_{2;2D}(x; c, k) = c \log(\sin(\pi x)) + k$$

$$S_{2;RVB}(x; c, k) = -2c \log \left\{ \frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)} \frac{\theta_3(2x\tau)\theta_3(2(1-x)\tau)}{\eta(2x\tau)\eta(2(1-x)\tau)} \right\} + k$$

$$S_{2;AdS}(x; c, k) = c\chi(x)^{-\frac{1}{3}} \left\{ \int_0^1 \frac{dy}{y^2} \left( \frac{1}{\sqrt{P(\chi(x),y)}} - 1 \right) - 1 \right\} + k$$
Results for Ising 3D

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Entropic c-function, $\beta = 0.2216$

Entropic c-function as a function of $\beta$

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- Our data for the $2D$ Ising model are in perfect agreement with the CFT prediction.
- In both cases we obtained precise results in a small amount of time ($< 800$ CPU hours for the largest lattice size both in $D = 2, 3$).
- This algorithm can be generalized to arbitrary spin models and gauge theories.

Future work:
- Exploit the duality properties of the $3D$ Ising model to study the entanglement content of the $\mathbb{Z}_2$ gauge theory.
- Extension to non-Abelian gauge theories? [Buividovich, Polikarpov; 2008][Itou et. al; 2015][Rabenstein et. al; 2018]
Appendix
Some numerical details

- For our simulations we adapted the code found in [Komura, Okabe; 2014], implementing the replica space and Jarzynski’s algorithm.

- The code is written in CUDA C to achieve high parallelization.

- We obtained precise results in a small amount of time: data for $L = 128$ required approximately 750 hours on the CINECA Marconi100 accelerated cluster, based on IBM Power9 architecture and Volta NVIDIA GPUs.

- Data for $L = 24, 48$ required respectively $\sim 270, 620$ hours on the CINECA Marconi100 accelerated cluster.
Benchmarks of the algorithm: 2D

Direct and reverse, $L/a = 32, \beta = 0.4406868, x = 0.421875, n_t = 128$

Entropic c-function, $L/a = 32, \beta = 0.4406868, N = 2 \times 10^4, n_t = 128$

$L/a = 32, \beta = 0.4406868, N = 2 \times 10^4, n_t = 512$
Benchmarks of the algorithm: $3D$

Direct and reverse, $L/a = 24, \beta = 0.2216, x = 0.104167, n_t = 50$

Entropic c-function, $L/a = 24, \beta = 0.2216$

$L/a = 24, \beta = 0.2216, N = 2.5 \times 10^4, n_t = 50$

$x = 0.104167$

$x = 0.895833$
Duality transformation in 2D