Real-time dynamics of the Schwinger model via variational quantum algorithms

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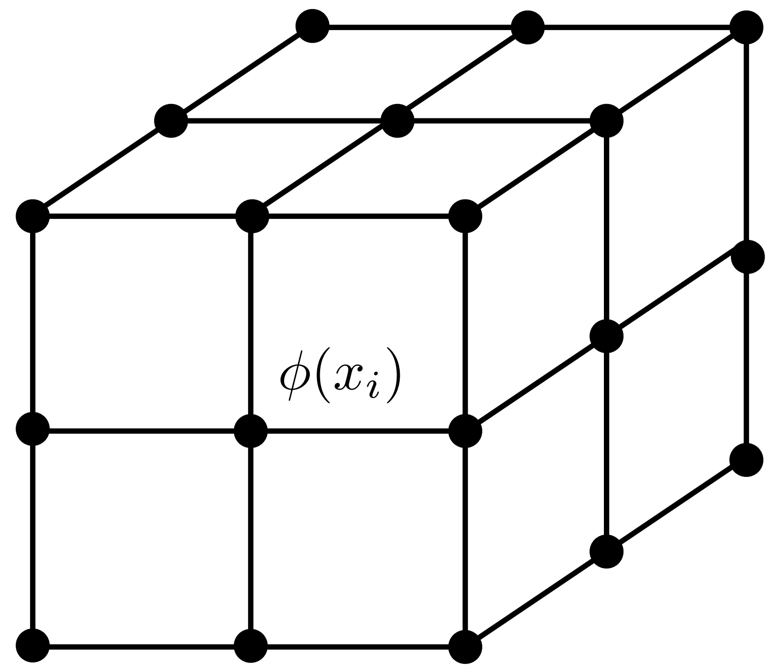
Based on:
LN, A. Bapat and C. W. Bauer (LBNL) [arXiv:2302.10933] (to be published in PRD)

Lattice gauge theory

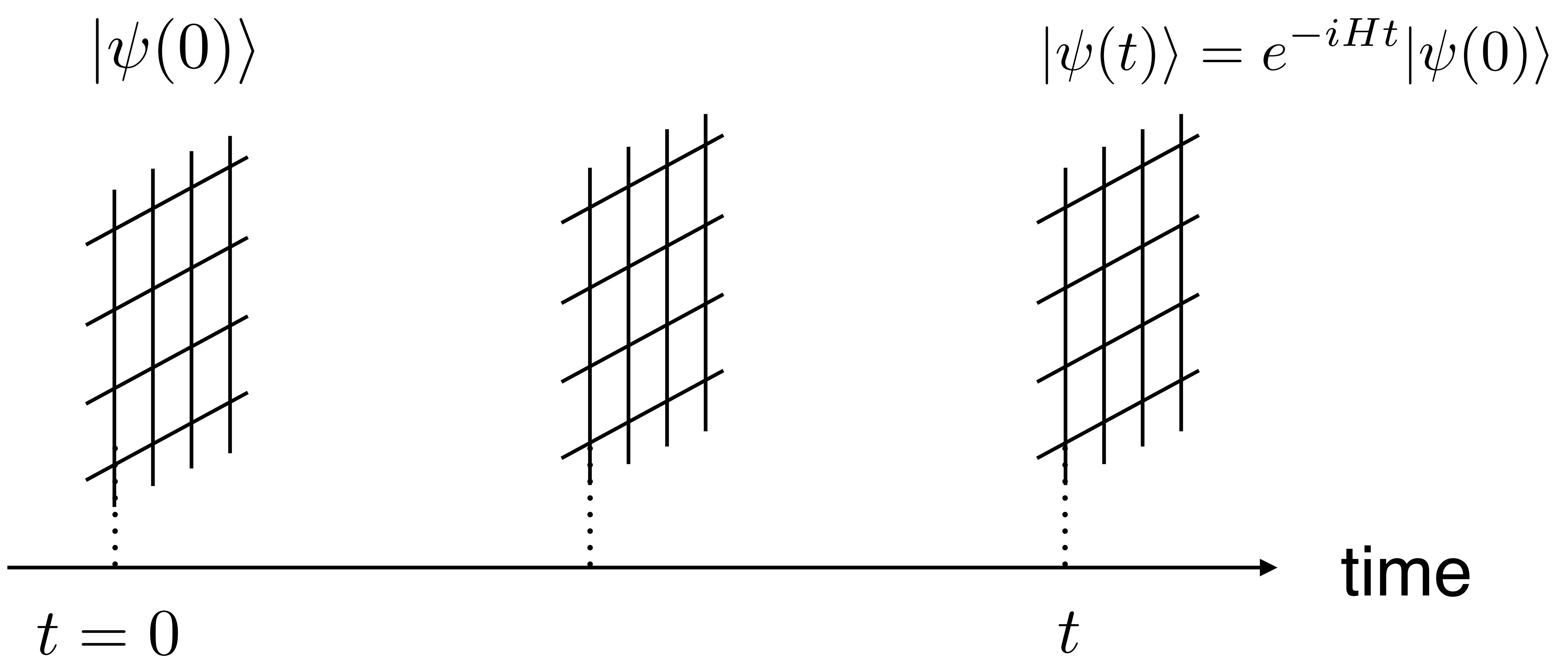
- (conventional) lattice gauge theory
  - discretize **spacetime**
    →using Monte Carlo method

$$Z = \int [d\phi] e^{-S[\phi]} \to \sum_{\{\phi_i\}} e^{-S(\phi_i)}$$

  - infamous sign problem
    - topological term
    - real-time dynamics, etc.



- Hamiltonian simulation
  - discretize **space**
  - no sign problem!
  - need exponential resources…
    - quantum simulation
    - tensor network, etc.

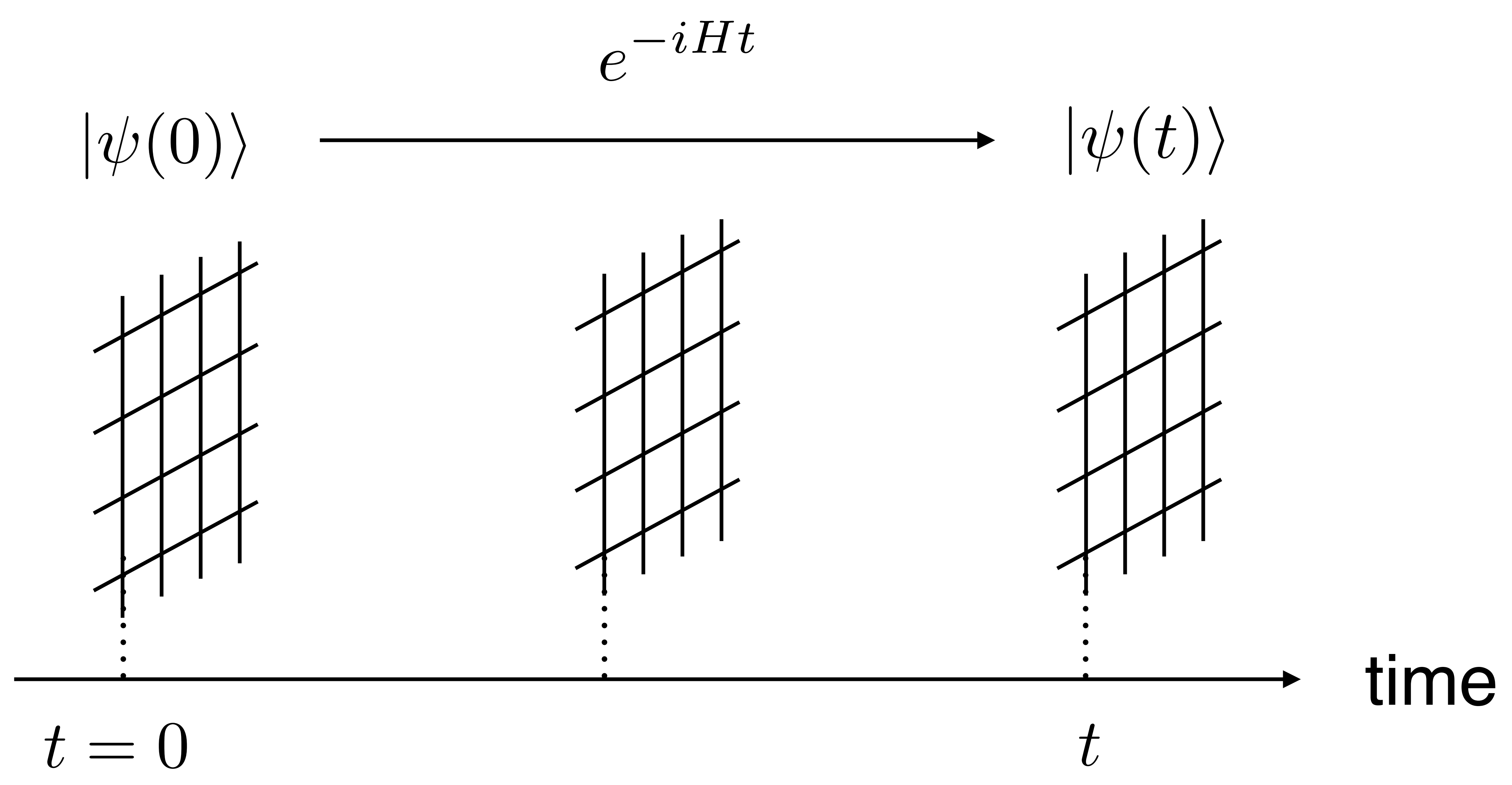


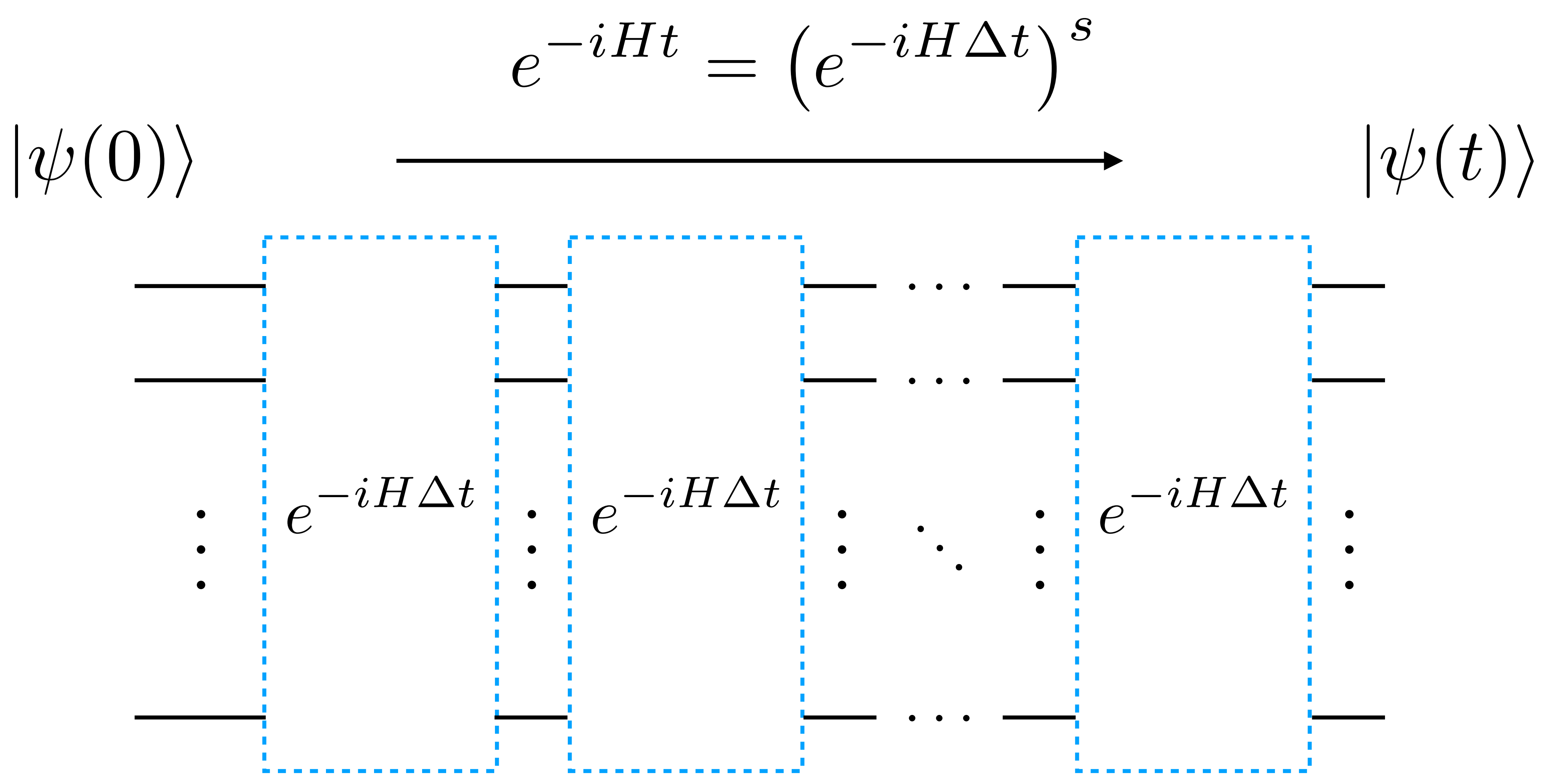
Quantum simulation

- quantum simulation:
  simulation using a quantum computer
  - real-time evolution $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$
  - adiabatic time evolution

$$|\psi_{\rm GS}\rangle = e^{-i\int dt H_A(t)t}|\psi_{\rm GS}^{(0)}\rangle$$

- applications to HEP: e.g. scattering problem
  [Jordan, Lee, Preskill, Science 336, 1130-1133 (2012)]
  - good: exponential advantage
  - bad:
    - still need many resources
      (#gates, #depths, #qubits)
    - near-term (NISQ) applications?





Schwinger model

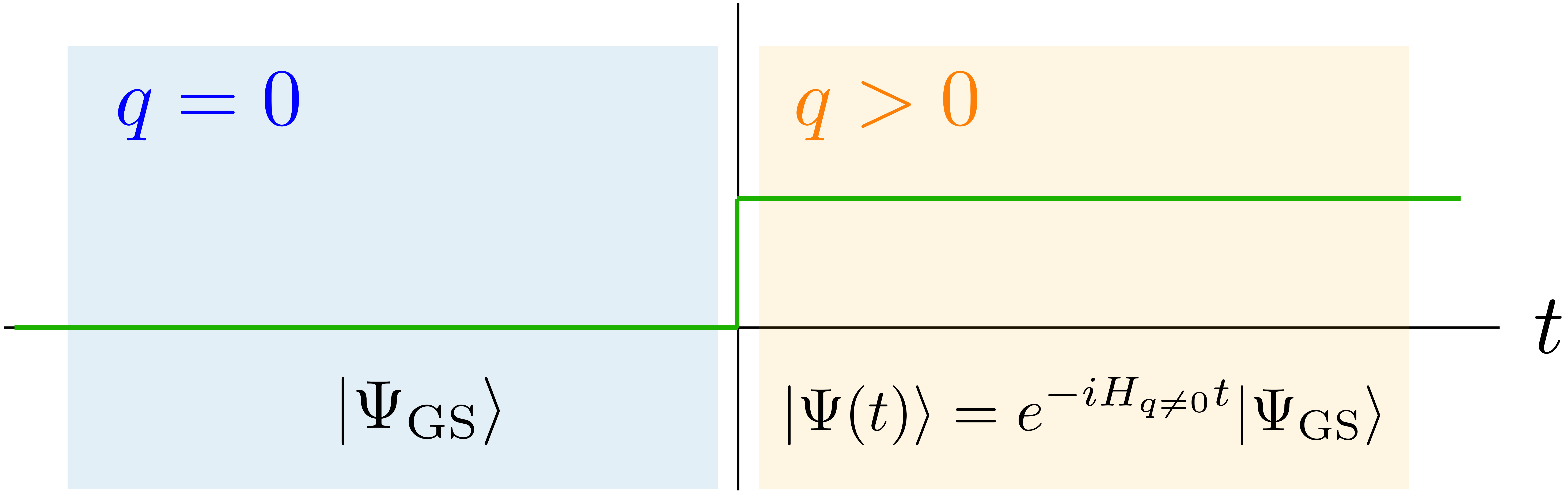
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu} + \mathrm{i}\bar{\psi}\gamma^{\mu}\left(\partial_{\mu} + \mathrm{i}gA_{\mu}\right)\psi - m\bar{\psi}\psi$$

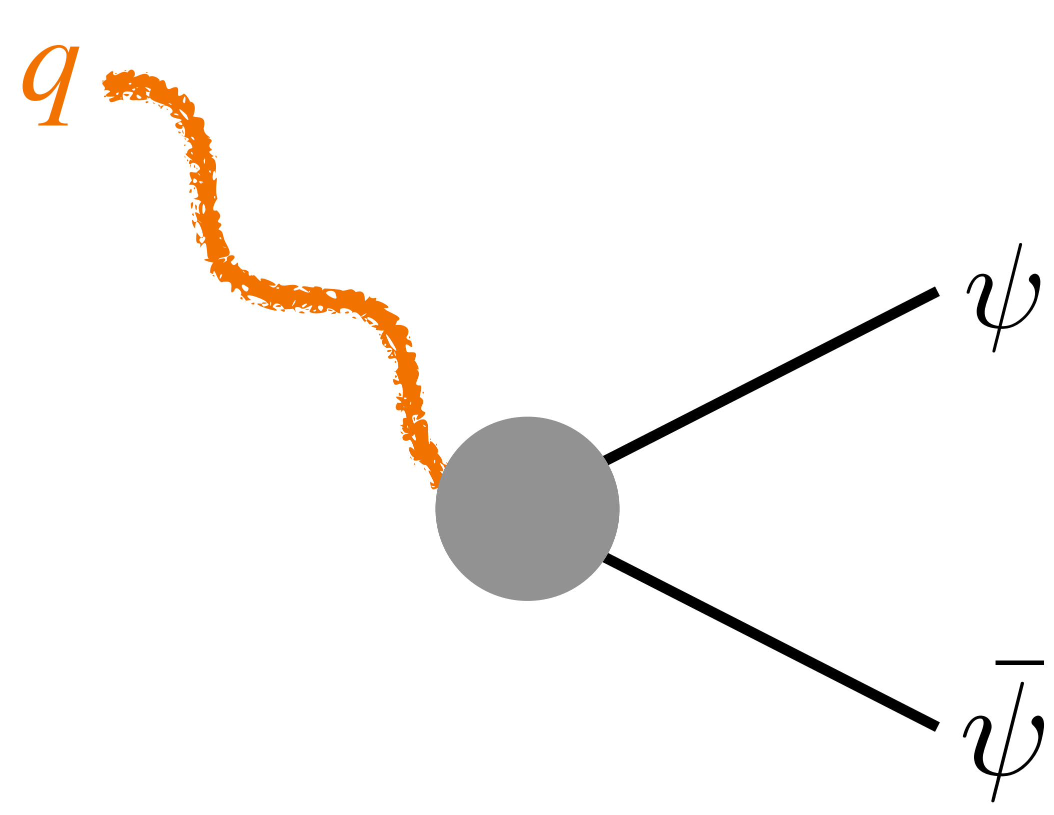
- simple toy model: 1+1d U(1) gauge theory = **Schwinger model** [Schwinger, **Phys. Rev.** 128, 2425]
  - exactly solvable for $m = 0$
  - mass perturbation is available for small mass regime
- simple but still non-trivial
  - screening/confinement phenomena
  - we can include the topological term (cannot be treated in the MC method)
    - the effects of the external field (constant θ)
    - the effects of probe charges (position dependent θ)

Quench dynamics in the Schwinger model

- Schwinger effect: particle pair creation due to strong external electric field [Schwinger, Phys. Rev. 82, 664, (1951)]
- Method: variational quantum algorithm (variational quantum eigensolver/simulation)

$q = \theta/(2\pi)$: external electric field





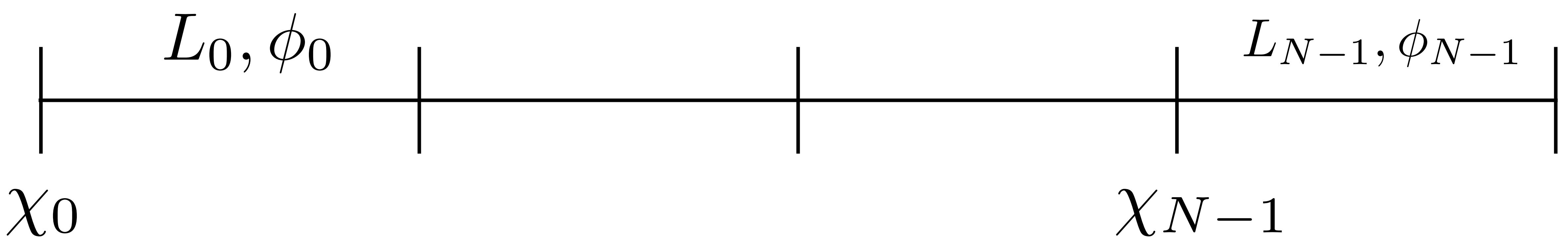
Ground state without external field

time evolution with external field
→pair-creation?

Method

Lattice Hamiltonian of Schwinger model

- χ_n: staggered fermion [Kogut, Susskind, Phys. Rev. D **11**, 395]
- L_n, ϕ_n: link variables (gauge field)



$$H_{\mathrm{lat}} = J \sum_{n=0}^{N-2} \left(L_n + \frac{\theta}{2\pi} \right)^2 - \mathrm{i}w \sum_{n=0}^{N-2} \left(\chi_n^\dagger e^{\mathrm{i}\phi_n} \chi_{n+1} - \mathrm{c.c.} \right) + m \sum_{n=0}^{N-1} (-)^n \chi_n^\dagger \chi_n$$

- gauge invariance: Gauss's law constraint

$$L_n - L_{n-1} = \chi_n^\dagger \chi_n - \frac{1-(-)^n}{2}$$

- we can **eliminate** gauge fields (open boundary condition)!
  - automatically gauge invariant, no boson fields
  - cannot be used in higher dimension

Spin description of the Schwinger model

- continuum Hamiltonian→lattice Hamiltonian (Kogut-Susskind formulation)
- lattice Hamiltonian + Gauss's law→fermionic representation of lattice Hamiltonian
- fermionic lattice Hamiltonian→spin system (Jordan-Wigner transformation)

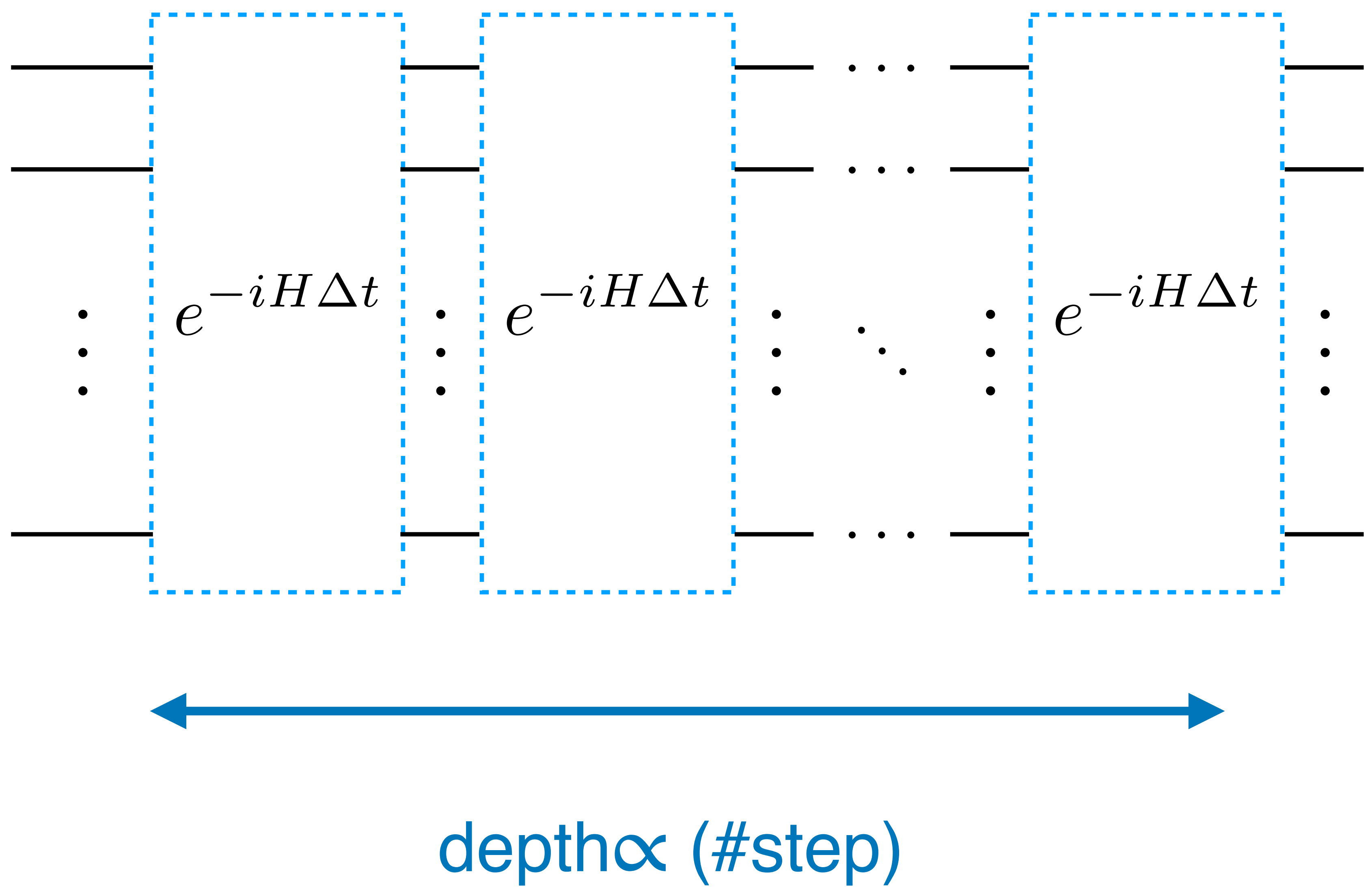
$$H_{\rm spin} = J \sum_{n=0}^{N-2} \left(\sum_{k=0}^{n} \frac{Z_k + (-)^k}{2} + q \right)^2 + \frac{w}{2} \sum_{n=0}^{N-2} (X_n X_{n+1} + Y_n Y_{n+1}) + \frac{m}{2} \sum_{n=0}^{N-1} (-)^n Z_n$$

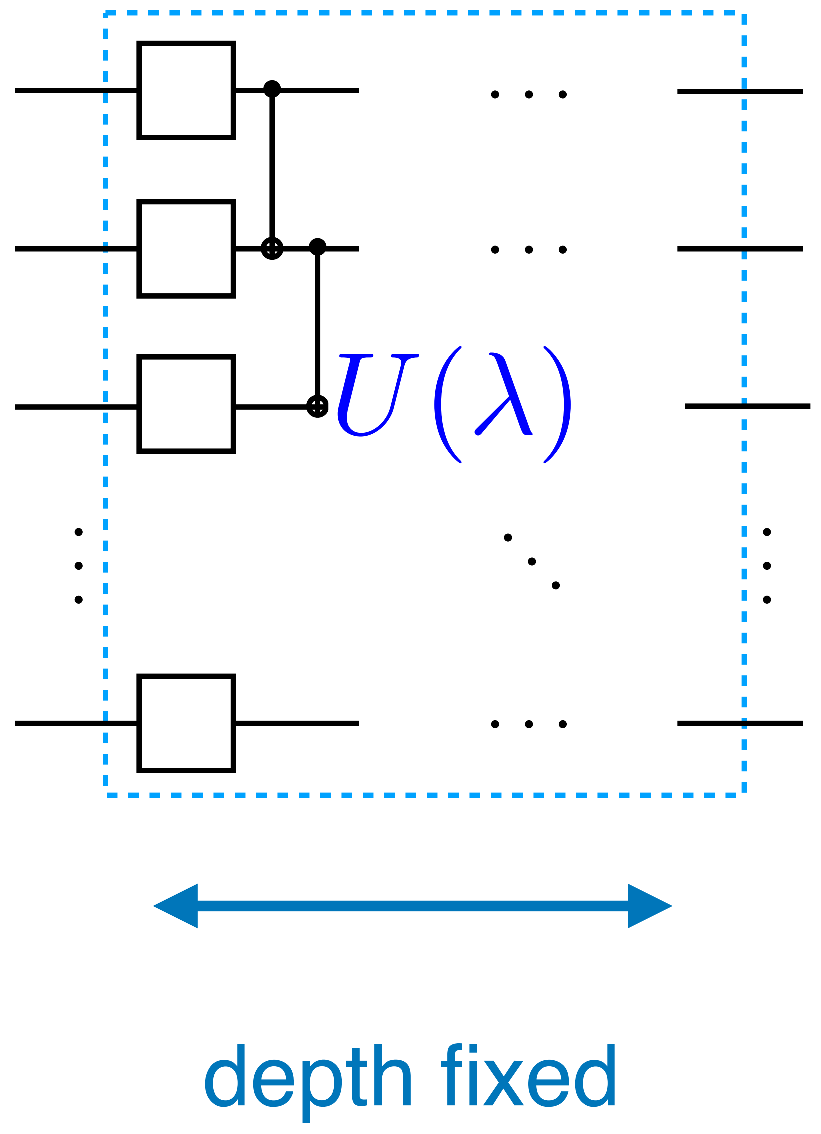
electric field term | fermion kinetic term | fermion mass term

(X, Y, Z: Pauli matrices)

Suzuki-Trotter vs variational method

- Suzuki-Trotter method
  - #depth grows with #steps
  - decoherence problem on NISQ devices
- variational quantum algorithm (VQA)
  - approximate states by ansatz with fixed depth
  - state preparation: variational quantum eigensolver (VQE)
  - time-evolution: variational quantum simulation (VQS)





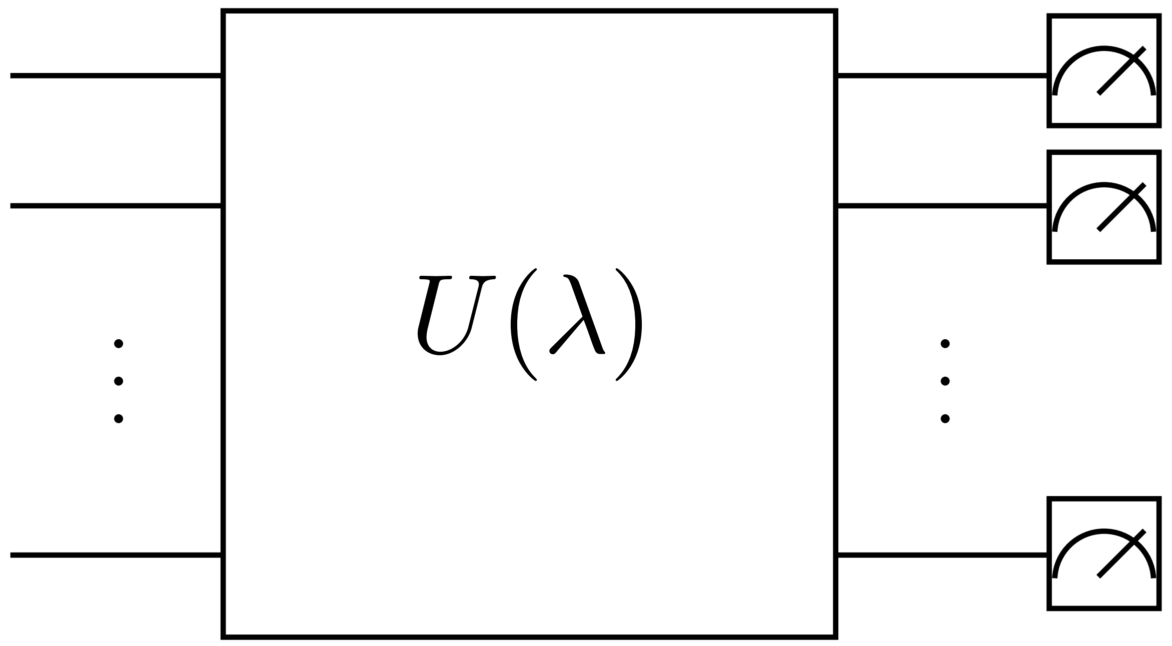
$$|\psi(\lambda)\rangle \approx \begin{cases} |\Psi_{\rm GS}\rangle & \text{(VQE)} \\ e^{-iHt}|\psi(0)\rangle & \text{(VQS)} \end{cases}$$

Variational quantum eigensolver

[Peruzzo, A. *et al.* Nat. Commun. 5:4213 (2014)]

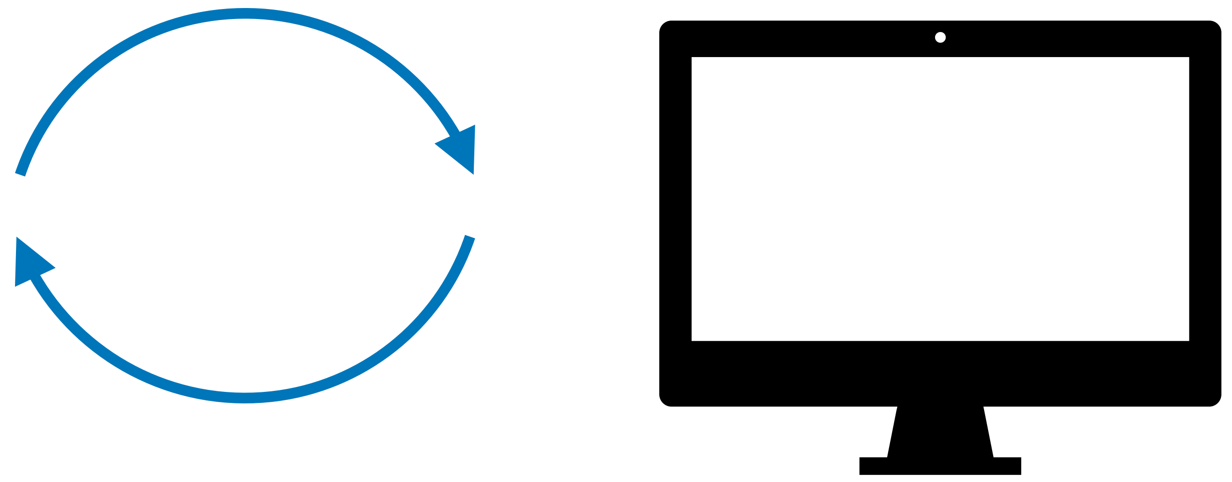
- goal: obtain the ground state
- approximate the ground state by ansatz $|\psi(\lambda)\rangle$
- optimize cost function $C(\lambda) = \langle\psi(\lambda)|H|\psi(\lambda)\rangle$ via classical computer
  $\rightarrow$ground state is given by $|\psi(\lambda_*)\rangle$

$$|\psi(\lambda)\rangle = U(\lambda)|0\rangle$$



$$C(\lambda) = \langle\psi(\lambda)|H|\psi(\lambda)\rangle$$

classical computer



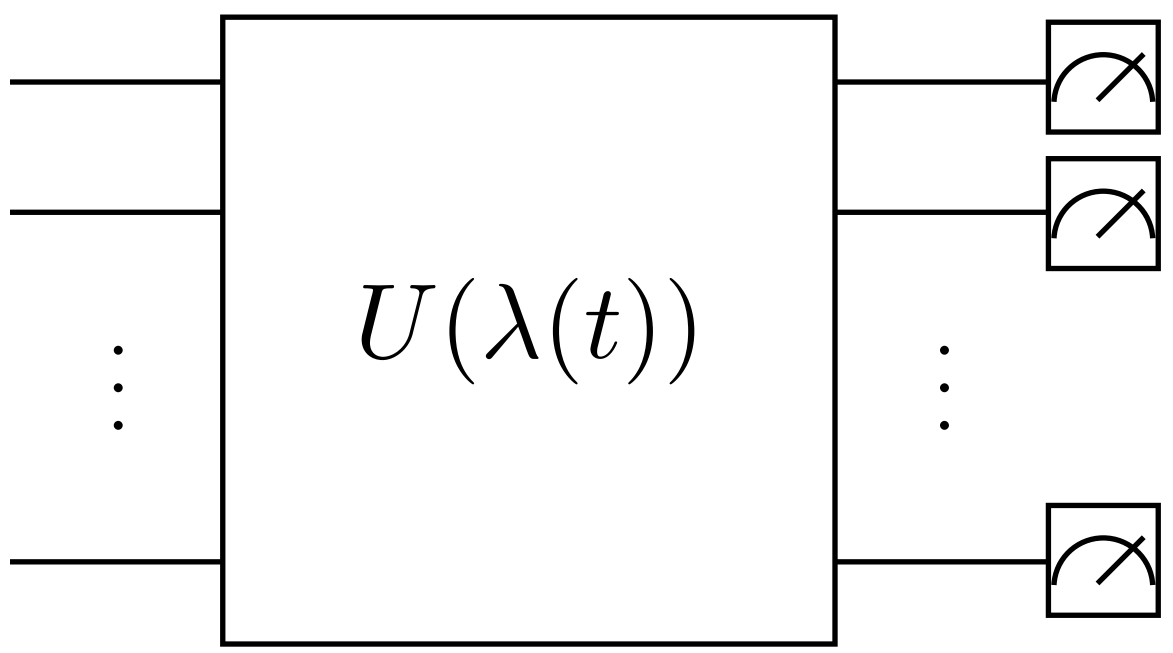
$$\lambda_* = \arg\min_{\lambda} C(\lambda)$$

Variational quantum simulation

[Li, Benjamin, Phys. Rev. X 7, 021050, (2017)]

- goal: obtain time-evolved state $|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$
- approximate $|\Psi(t)\rangle$ by ansatz $|\psi(\lambda(t))\rangle$ with time-dependent parameters
- evolution of states → evolution of parameters $\lambda(t)$ via McLacran's variational principle
- we use the same ansatz (Hamiltonian variational ansatz) for both VQE and VQS
  →quench dynamics: set $\lambda(0) = \lambda_*$ (obtained by VQE)

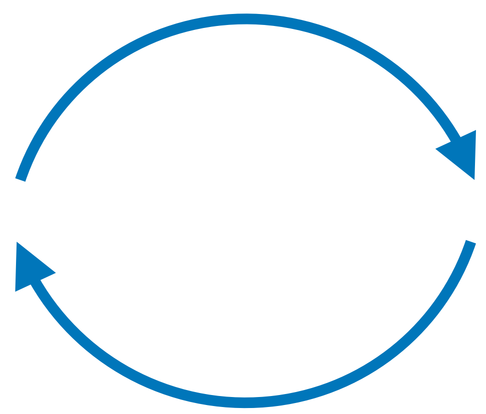
$$|\psi(\lambda(t))\rangle = U(\lambda(t))|0\rangle$$



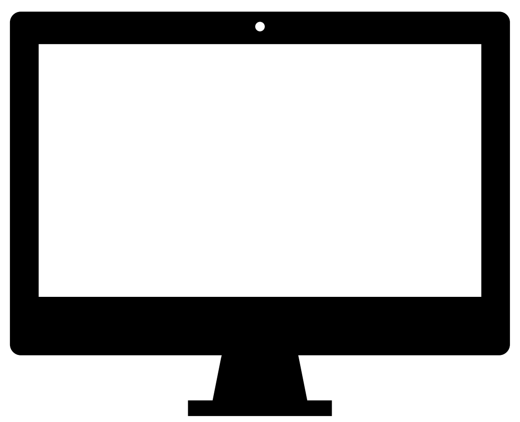
$$M_{ij} = \mathrm{Re}\frac{\partial\langle\psi(\lambda)|}{\partial\lambda_i}\frac{\partial|\psi(\lambda)\rangle}{\partial\lambda_j}$$

$$V_i = \mathrm{Im}\frac{\partial\langle\psi(\lambda)|}{\partial\lambda_i}H|\psi(\lambda)\rangle$$

(+correction terms)



classical computer



$$\sum_j M_{ij}\dot{\lambda}_j = V_i$$

Hamiltonian variational ansatz (HVA)

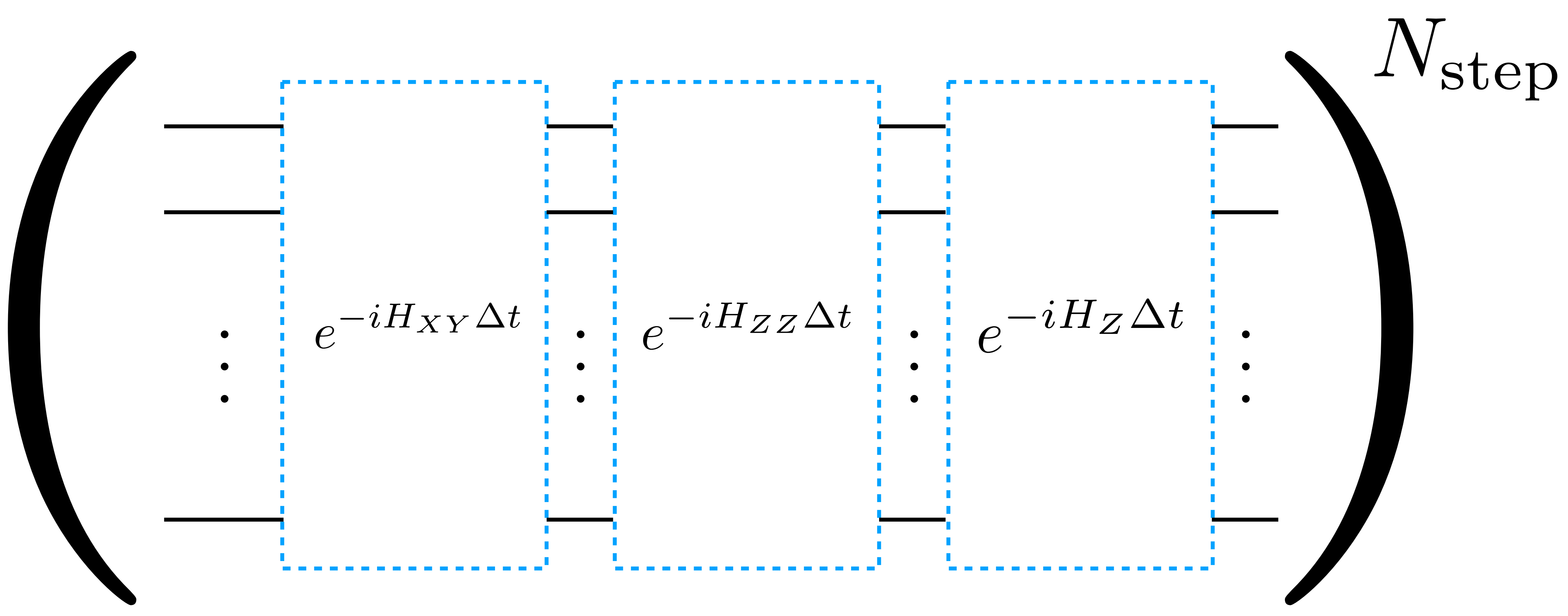
[Wecker, Hastings, Troyer, Phys. Rev. A, 92, 042303 (2015)]
[Ho, Hsieh, SciPost Phys. 6, 029 (2019)]
[Wiersema, *et. al.* PRX Quantum 1, 020319 (2020)]

- motivation: imitating Suzuki-Trotter decomposition of adiabatic or real-time evolution
- we use U(1) preserving decomposition

$$H = H_{XY}^{(\text{even})} + H_{XY}^{(\text{odd})} + H_{ZZ} + H_Z$$

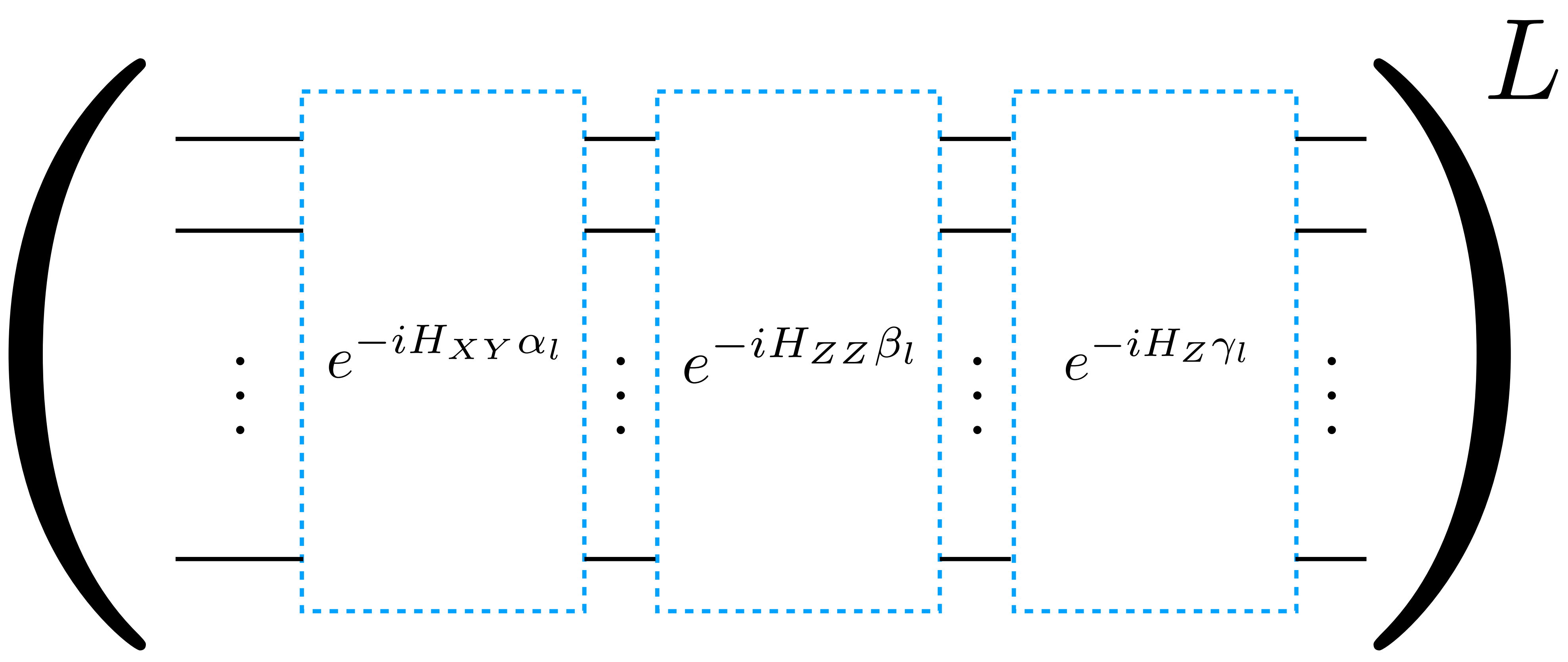
- replace Δt with variational parameters $(\alpha_l, \beta_l, \gamma_l)$
- parameters (α, β, γ) can depend on sites

Suzuki-Trotter evolution



$$\Delta t = T_{\max}/N_{\text{step}}$$

HVA



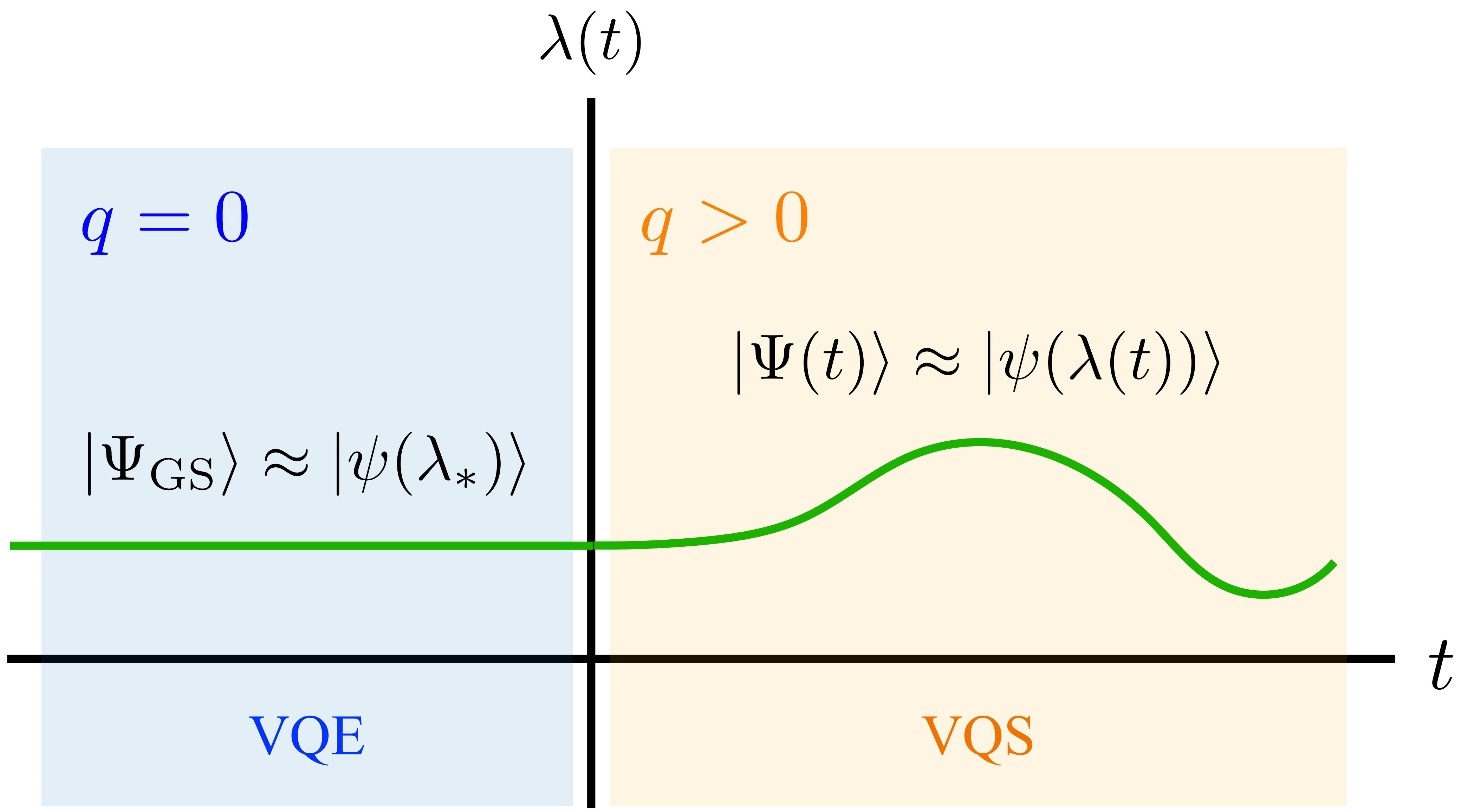
$(\alpha_l, \beta_l, \gamma_l)$: variational parameters

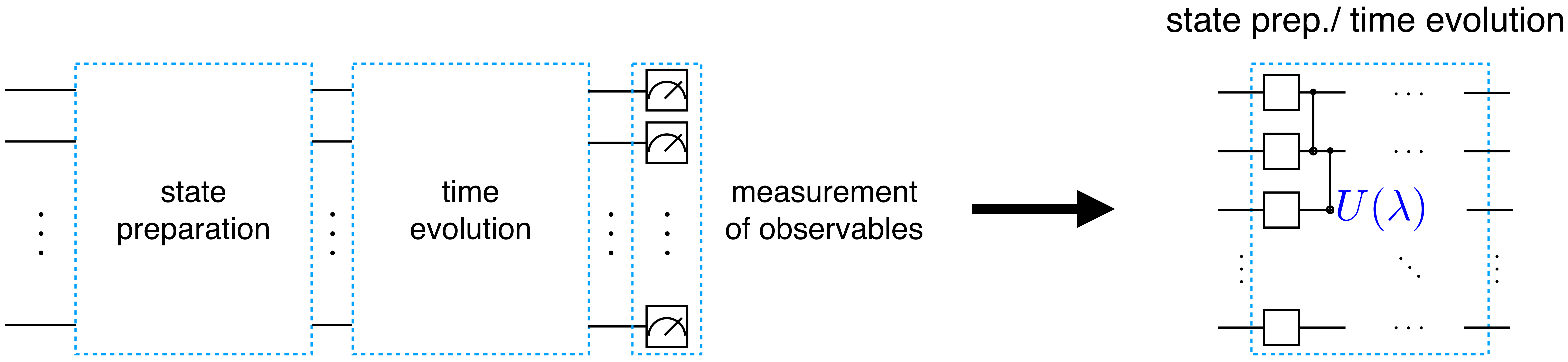
Summary of our protocol

- Quench dynamics in the Schwinger model
  - ground state without external field q: $|\Psi_{\rm GS}\rangle$
  - time evolution via Hamiltonian with external field q:

$$|\Psi(t)\rangle = e^{-iH_{q\neq 0}t}|\Psi_{\rm GS}\rangle$$

- perform VQE and VQS using the same ansatz $|\psi(\lambda)\rangle$
  - reduce overall circuit depth
  - simulation with fixed depth

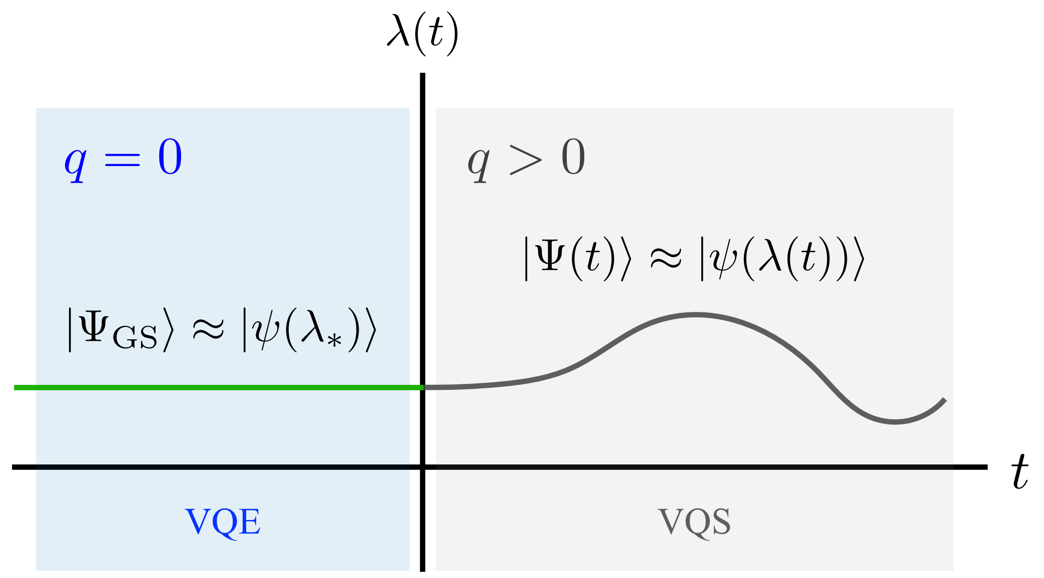




Results

(classical) statevector simulation

Ground state preparation via VQE

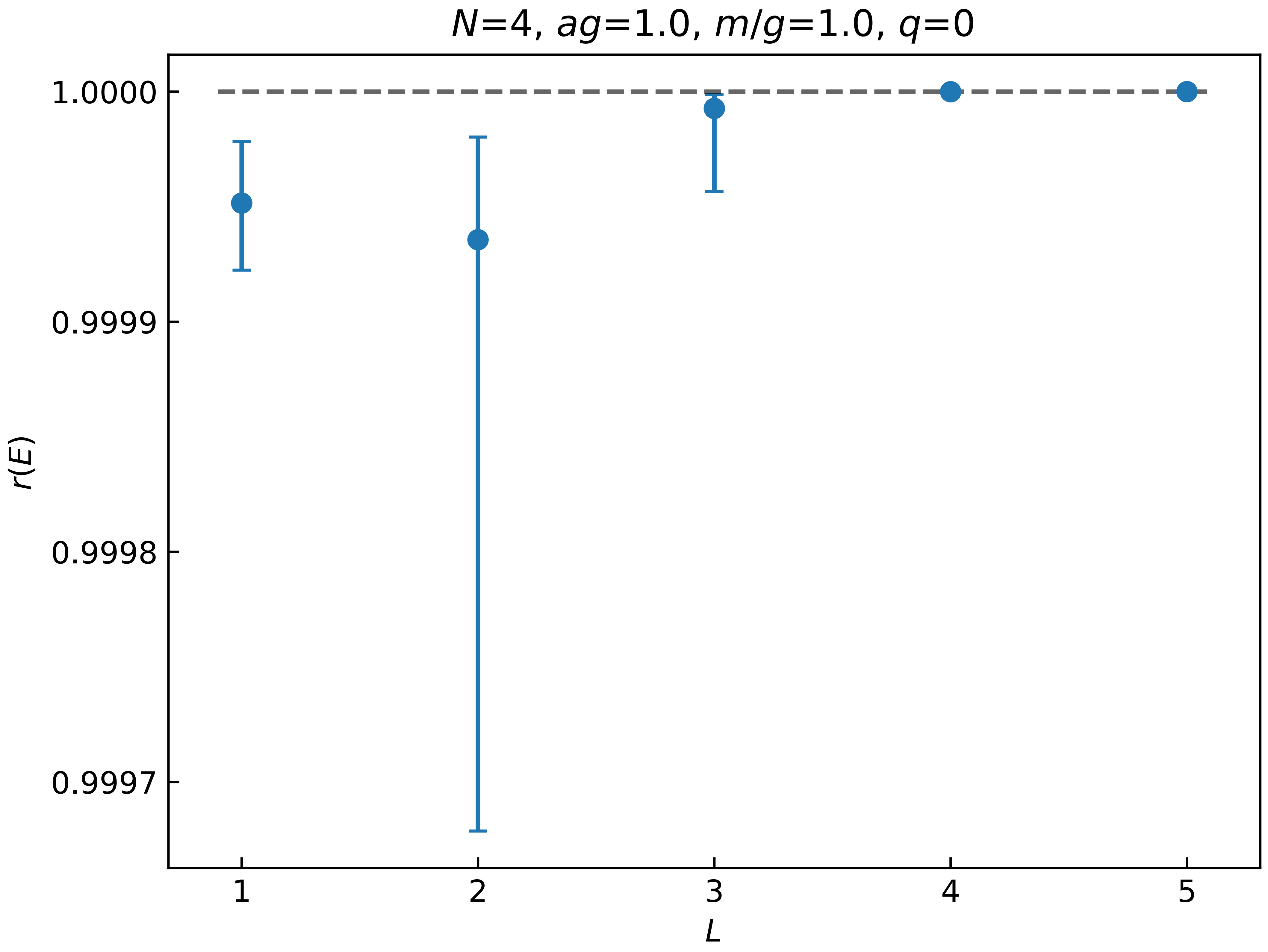


- compare VQE results with exact diagonalization (ED)

- a metric of accuracy: $r(E) = \dfrac{E_{\max} - E_{\rm VQE}}{E_{\max} - E_{\min}}$

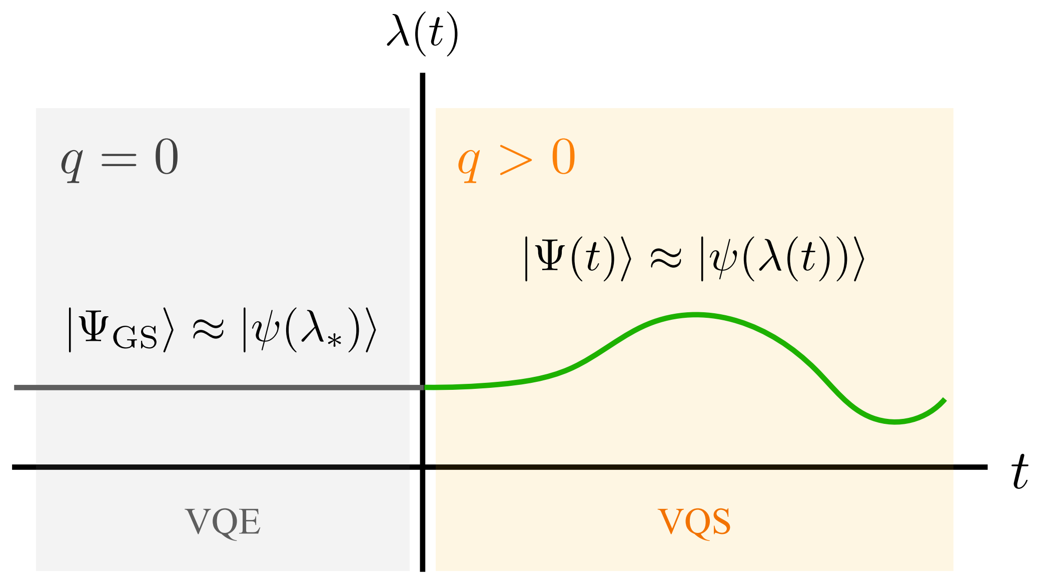
  - $E_{\max}, E_{\min}$: max/min energy obtained by ED
  - $r(E) = 1$ for the best case
  - $r(E) = 0$ for the worst case

- L : depth of ansatz
- quality drastically improves for $L \geq 4$



- 20 samples with different initialization
- dots/bars represent medians and 25-75 percentiles

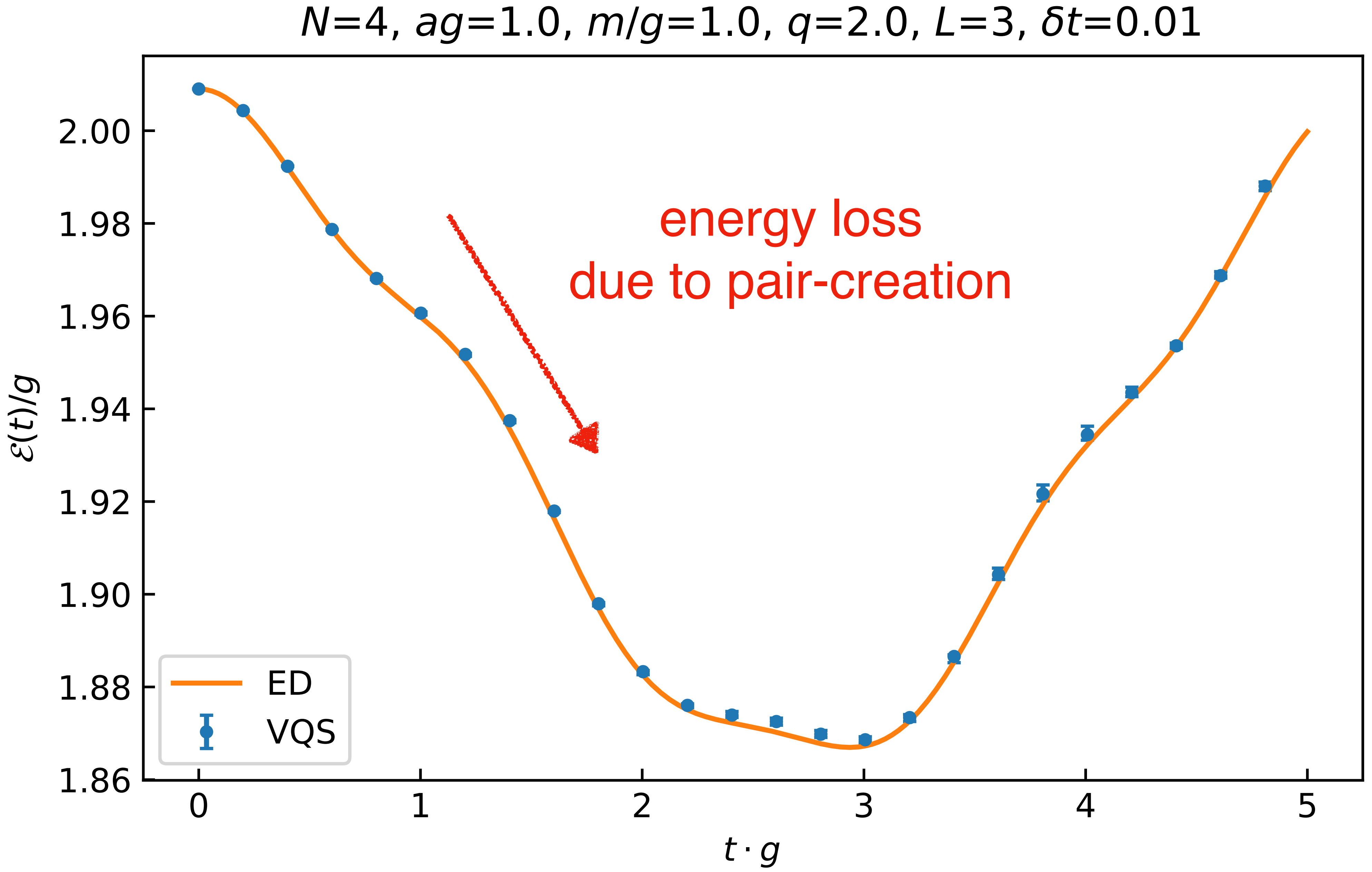
Real-time evolution via VQS



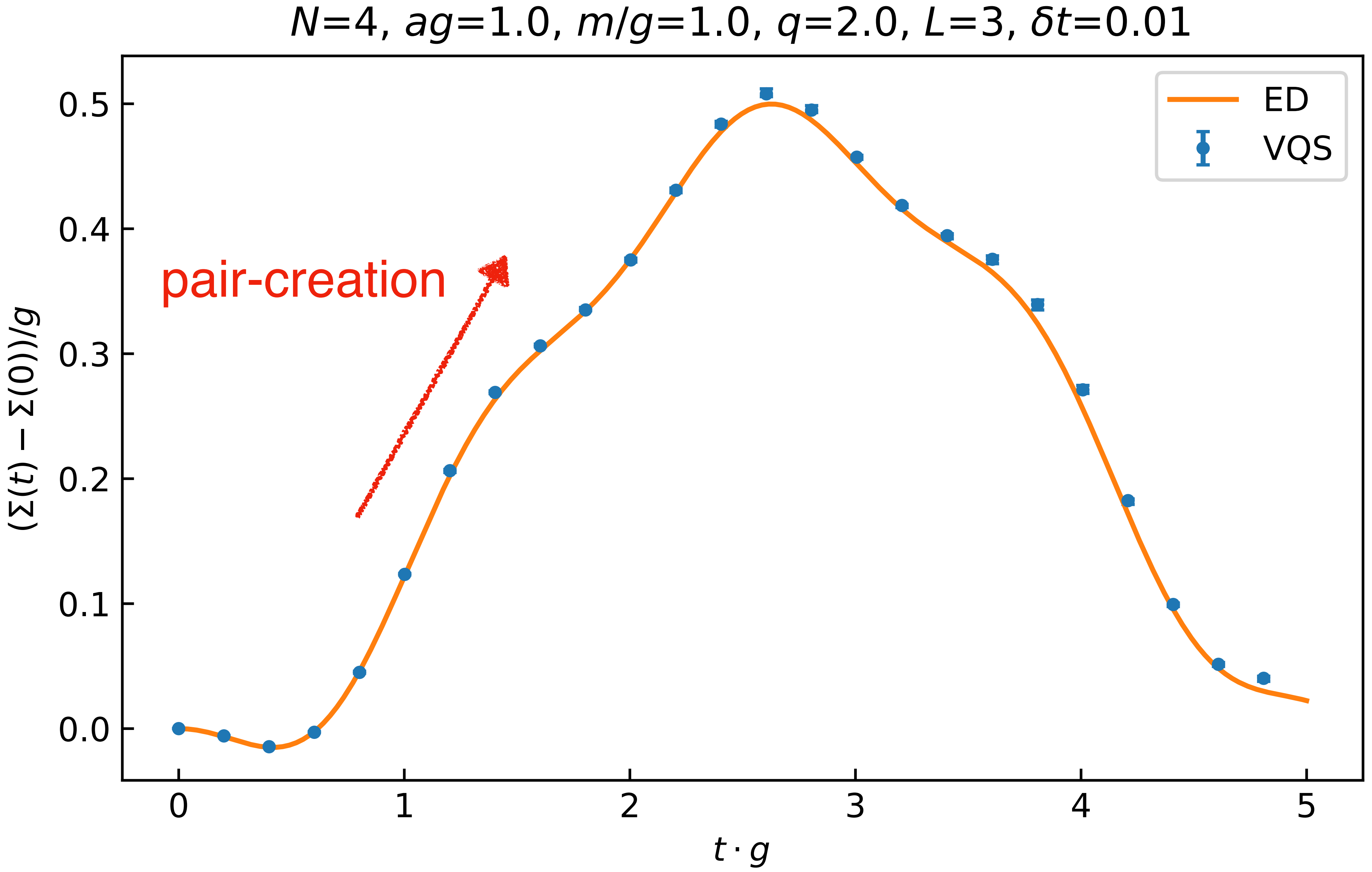
- two observables:
  - total electric field $\mathscr{E}$
  - chiral condensation $\langle \bar{\psi}\psi \rangle$ (~particle number density)
- observing energy loss and pair-creation!

- 20 samples with different initialization
- dots/bars represent medians and 25-75 percentiles

electric field

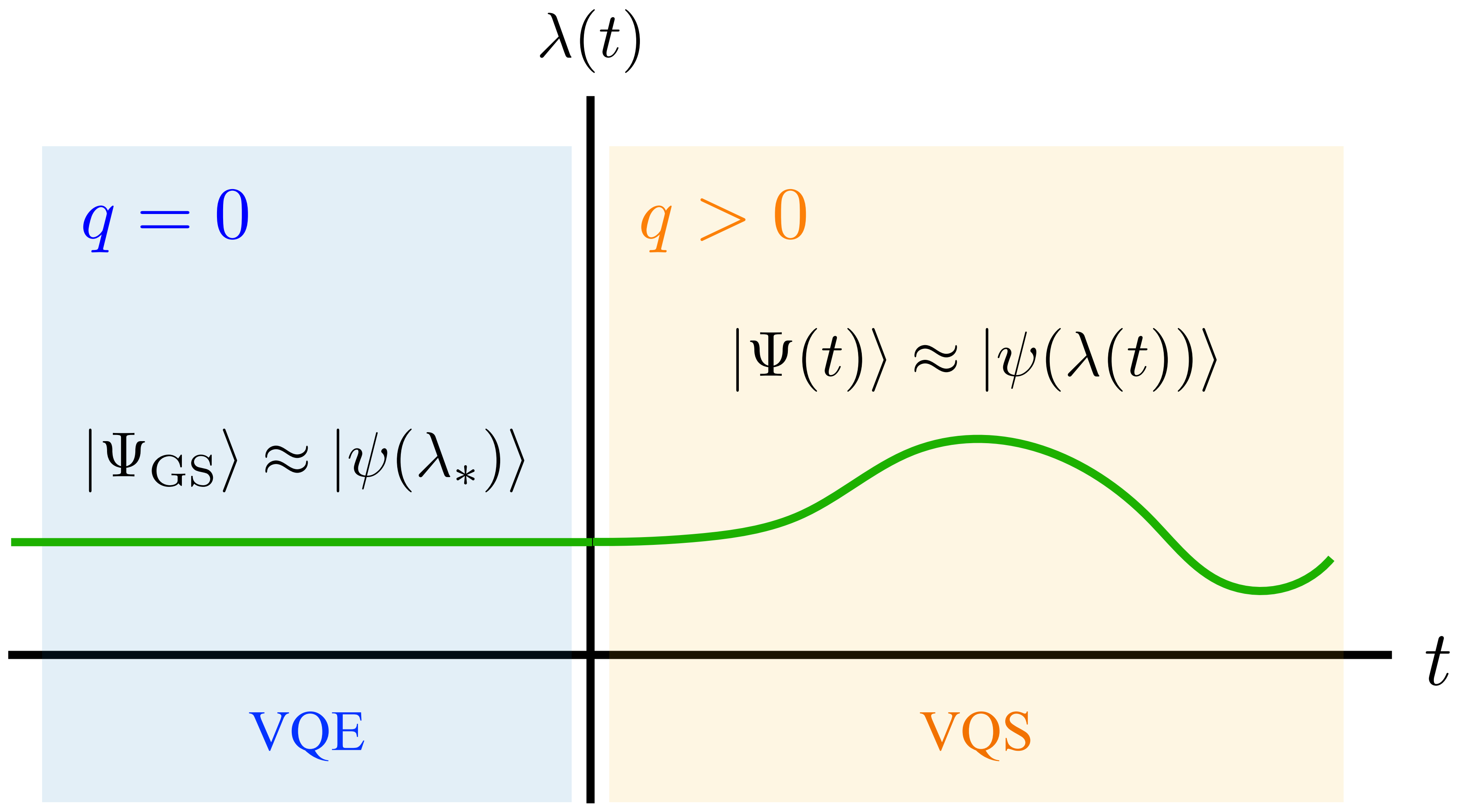


chiral condensation



Summary and outlooks

- quench dynamics in the Schwinger model via VQAs
  - ground state w/o external field q via VQE
  - time evolution via Hamiltonian w/ external field q via VQS
  - we can reduce circuit depth
  - VQA results agree well with ED
- **future directions**:
  - understanding scaling of resources
  - effects of errors (shot noise, quantum noise)
  - reducing sampling cost [in progress]
  - extension to higher dimensional and/or non-Abelian theories



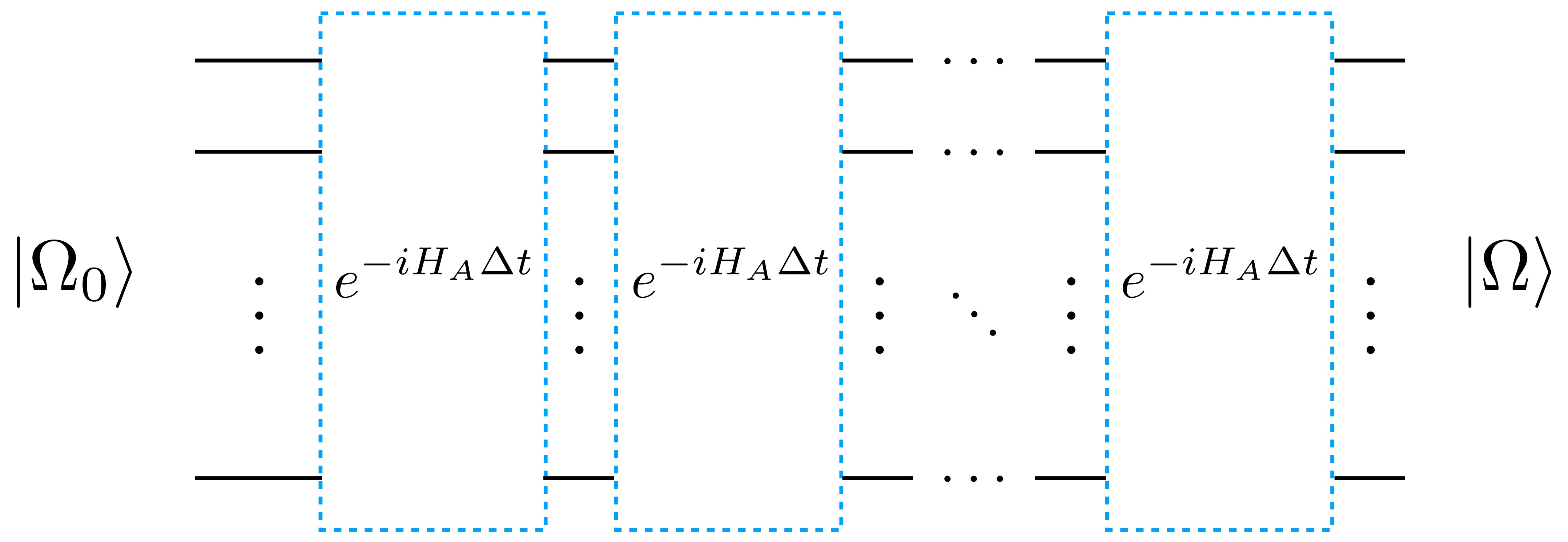
Backups (including preliminary results)

Suzuki-Trotter decomposition

- adiabatic state preparation

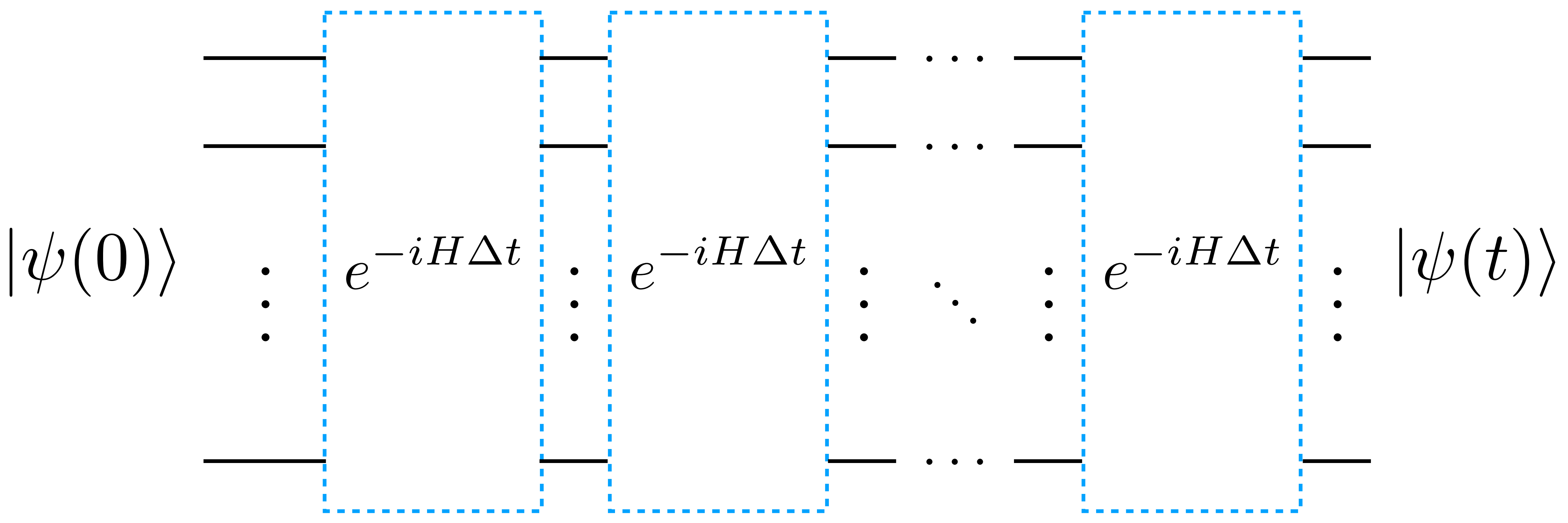
$$|\Omega\rangle = \lim_{T\to\infty} \mathrm{T}\exp\left(-\mathrm{i}\int_0^T \mathrm{d}t H_A(t)\right)|\Omega_0\rangle$$

$$\simeq \prod_s e^{-\mathrm{i}H_A(s\Delta t)\Delta t}|\Omega_0\rangle$$



- real-time evolution

$$|\psi(t)\rangle = e^{-\mathrm{i}Ht}|\psi(0)\rangle = \left(e^{-\mathrm{i}H\Delta t}\right)^s|\psi(0)\rangle$$



- drawback: #depth grows with #steps

McLachlan's variational principle

$$\delta \left\| \left(\frac{d}{dt} + iH \right) |\psi(\lambda)\rangle \right\| = 0$$

$$\Rightarrow \sum_j M_{ij} \dot{\lambda}_j = V_i$$

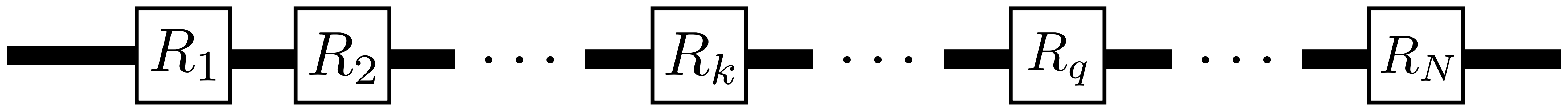
$$M_{ij} = \mathrm{Re} \frac{\partial \langle \psi(\lambda)|}{\partial \lambda_i} \frac{\partial |\psi(\lambda)\rangle}{\partial \lambda_j}$$

$$V_i = \mathrm{Im} \frac{\partial \langle \psi(\lambda)|}{\partial \lambda_i} H |\psi(\lambda)\rangle$$

Quantum circuit for VQS

[Li, Benjamin, Phys. Rev. X 7, 021050 (2017)]
[Yuan et al., Quantum 3, 191 (2019)]

$$U(\lambda) = R_N(\lambda_N) \cdots R_1(\lambda_N)$$



$$|\psi(\lambda)\rangle = U(\lambda)|\psi_0\rangle$$

- evaluation of matrix elements $M_{kq} = \mathrm{Re}\frac{\partial\langle\psi(\lambda)|}{\partial\lambda_k}\frac{\partial|\psi(\lambda)\rangle}{\partial\lambda_q}$
- derivative of each component w.r.t. parameters $\frac{\partial}{\partial\lambda_k}R_k(\lambda) = U_k R_k(\lambda)$
- quantum circuit:

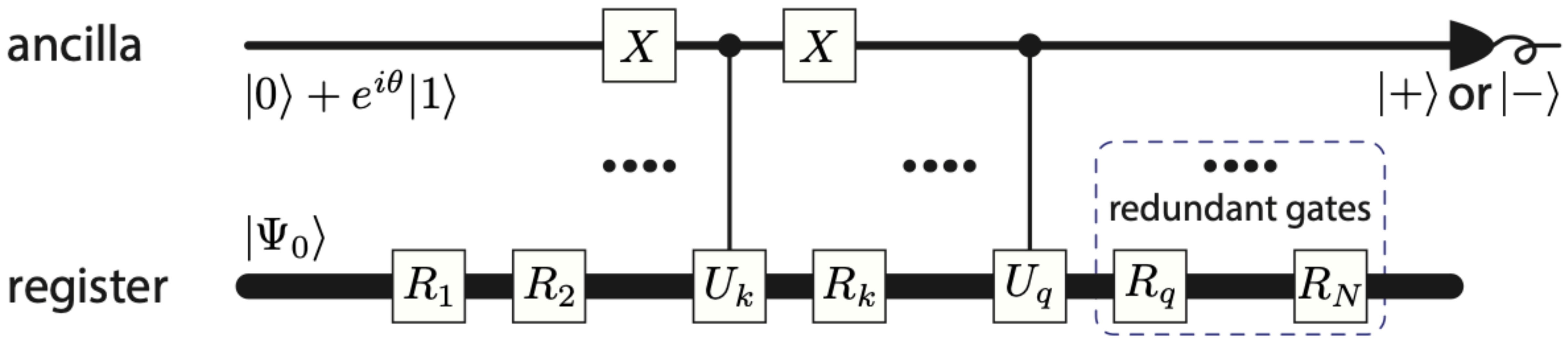
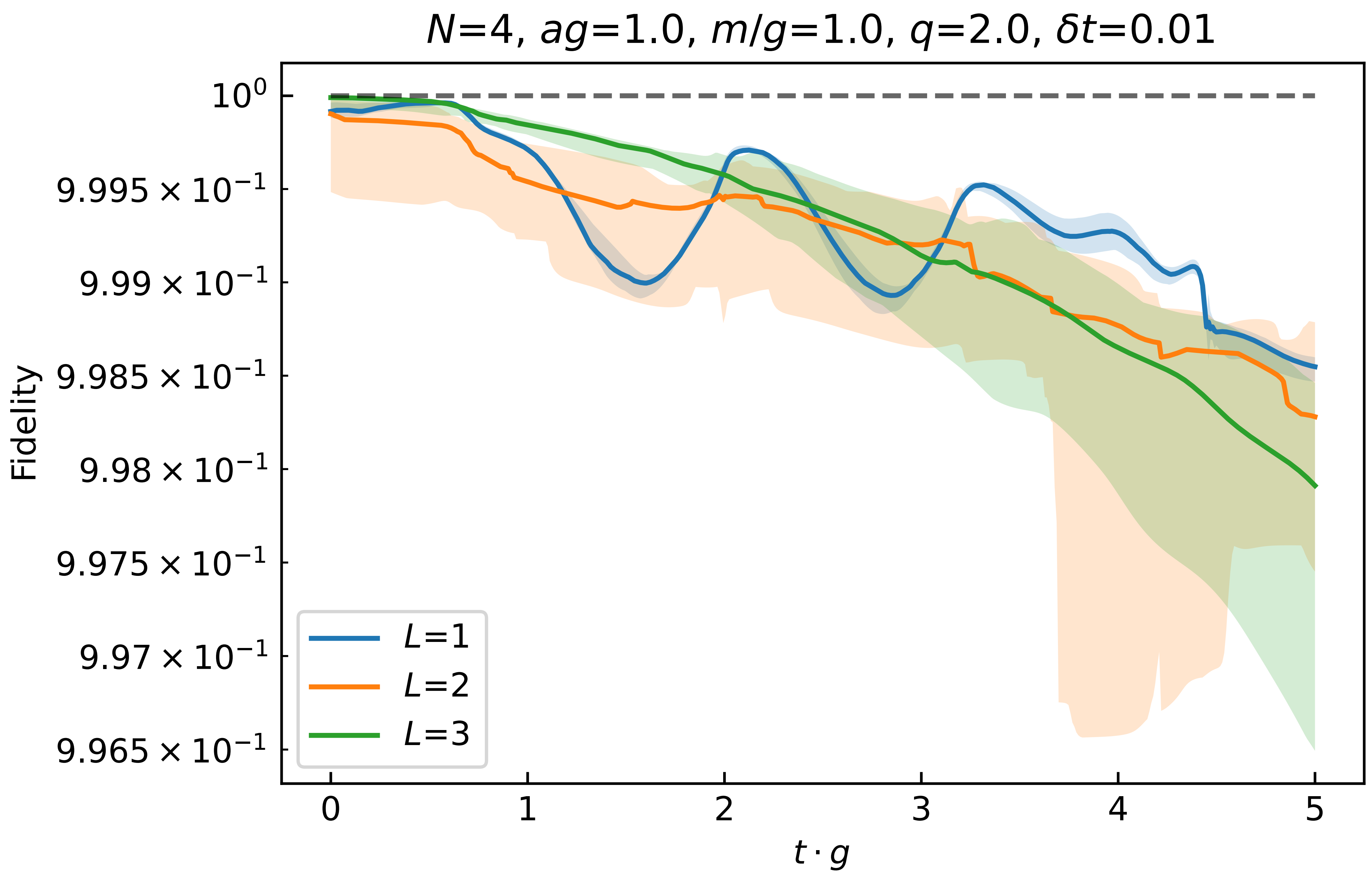
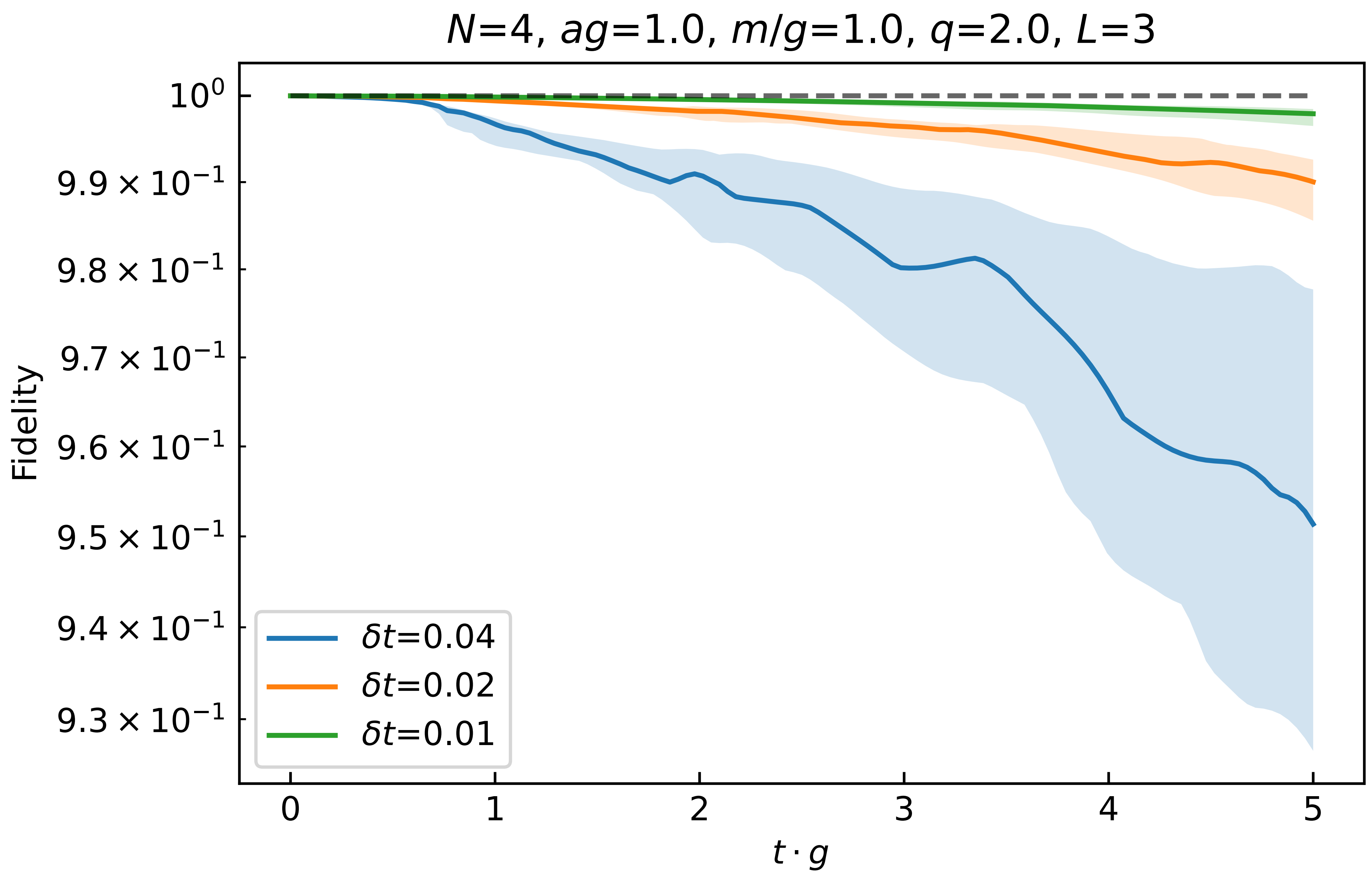


Figure from Yuan et al.

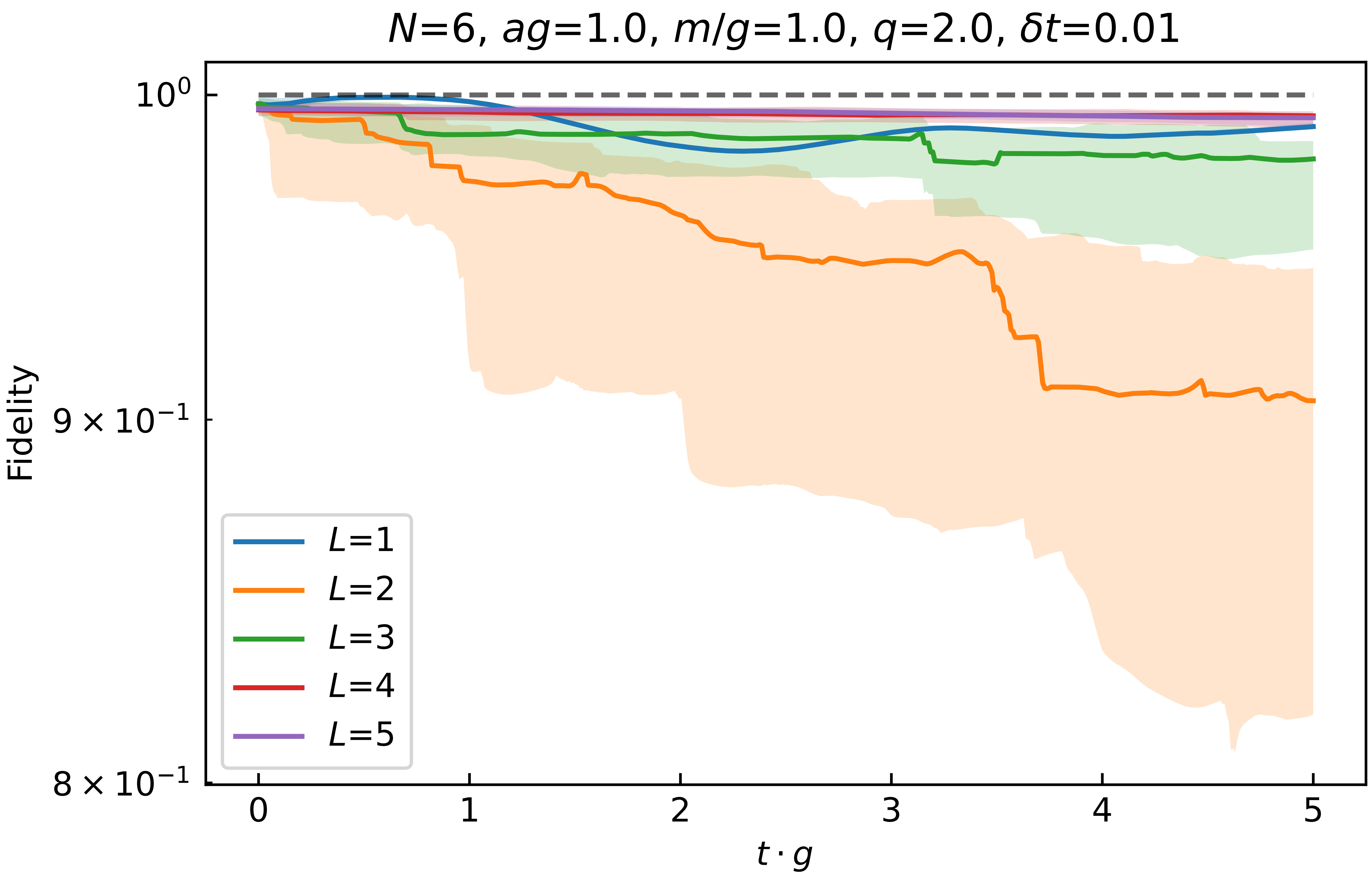
Fidelity and algorithmic errors

- (avaraged) fidelity improves as increasing L and/or decreasing $\delta t = T_{\max}/N_{\text{step}}$
- effects from δt is significant

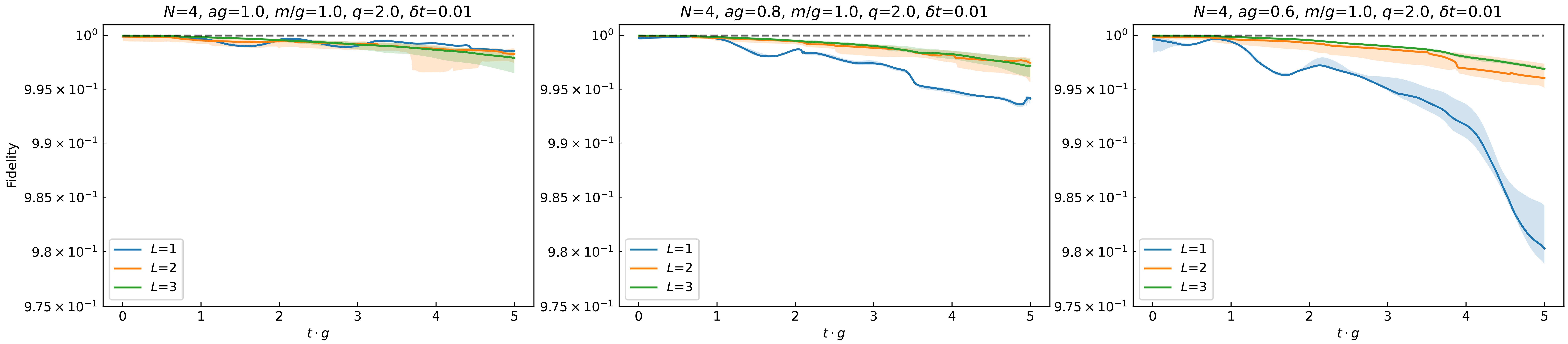




L-dependence ($N = 6$)

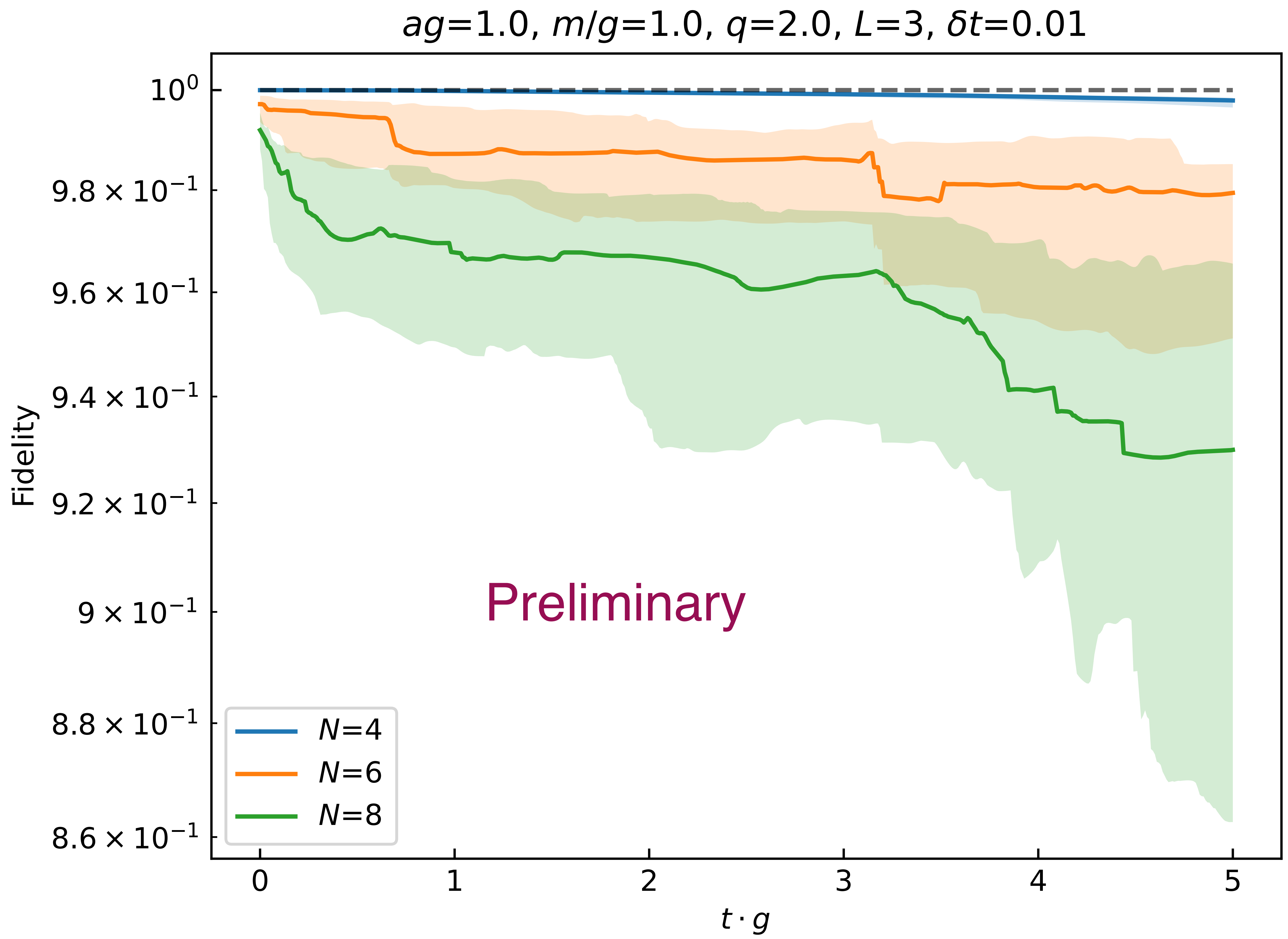


Lattice spacing dependence

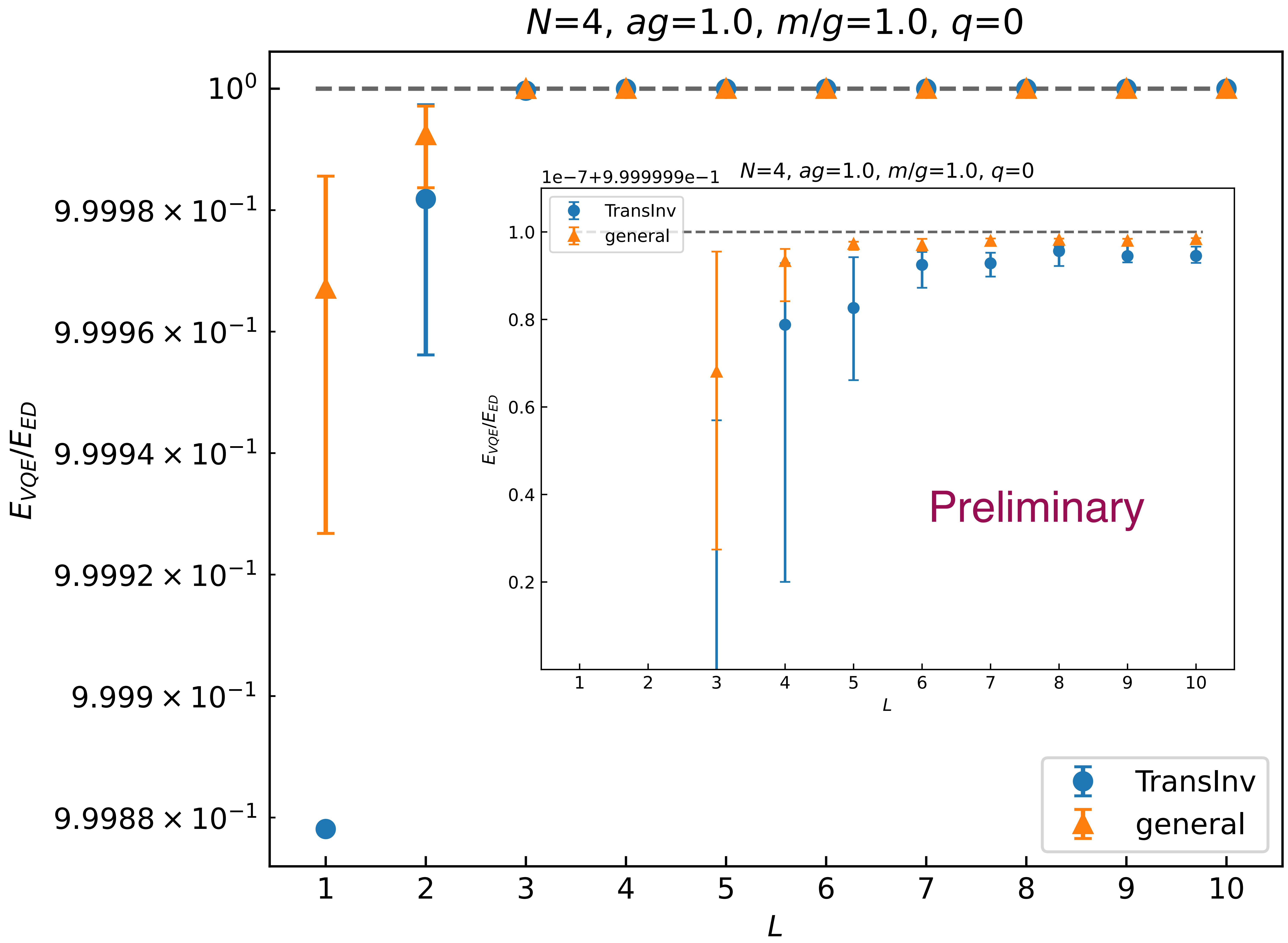


Comments on scalability

- results get worse with increasing N…
  - VQE accuracy (#iteration, optimizer)
  - VQS accuracy (choice of δt, L)
- at least $N \leq 8$, $F > 0.9$ with $L = 3$

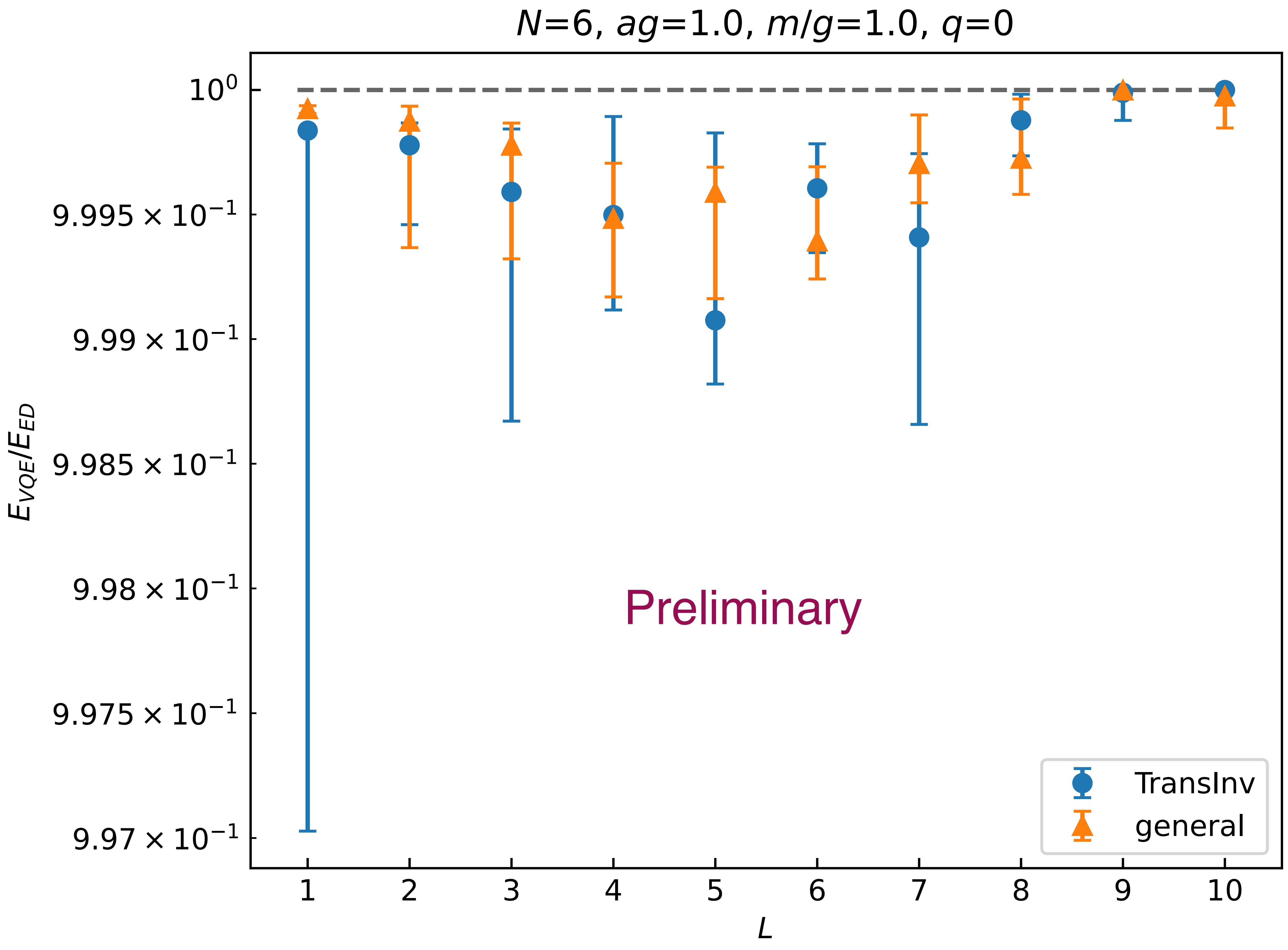


Translational invariant parameters ($N = 4$)



#iter=5,000

Translational invariant parameters ($N = 6$)



#iter=5,000