New gauge-independent transition dividing the confinement phase in the lattice gauge-adjoint scalar model.

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Based on arXiv:2307.15953 [hep-lat]
We investigate the gauge-scalar model to clarify the mechanism of confinement in the Yang-Mills theory in the presence of matter fields.

We also investigate non-perturbative characterization of the Brout-Englert-Higgs (BEH) mechanism providing the gauge field with the mass, in the gauge-independent way (without gauge fixing).

We reexamine the lattice $SU(2)$ gauge-scalar model with a radially-fixed scalar field (no Higgs mode) which transforms according to the adjoint representation of the gauge group $SU(2)$ without any gauge fixing.

Note that it was impossible to realize the conventional BEH mechanism on the lattice unless the gauge fixing condition is imposed, since gauge non-invariant operators have vanishing vacuum expectation value on the lattice without gauge fixing due to the Elitzur theorem.

This difficulty can be avoided by using the gauge-independent description of the BEH mechanism proposed recently by one of the authors, which needs neither the spontaneous breaking of gauge symmetry,

Therefore, we can study the Higgs phase in the gauge-invariant way on the lattice without gauge fixing based on the lattice construction of gauge-independent description of the BEH mechanism.
In case of fundamental scalar field (talk by Ikeda)

- Confinement and Higgs regions are subregions of analytically continued single phase.  
  E. Fradkin and S.H. Shenker, PRD 19, 3682 (1979)  
- We found a new transition line (red) which separates confinement and Higgs regions completely.

In case of adjoint scalar fields. (this talk)

- Confinement and Higgs regions are completely separated into the two different phases by continuous transition line.  
  R.C. Brower et. al. PRD 25, 3319 (1982)  
- We found a new transition line (red) that divides completely the confinement phase into two parts.

Note that these results are obtained by investigating the correlation functions between the gauge-invariant composite operators constructed from the scalar field and the color-direction field obtained through the gauge-covariant decomposition.
The $SU(2)$ gauge-scalar model with a radially-fixed scalar field in the adjoint representation of the gauge group:

$$S_{GS} := S_g[U] + S_\phi[U, \phi],$$

$$S_g[U] := \sum_x \sum_{\mu < \nu} \frac{\beta}{2} \text{tr} \left( 1 - U_{x,\mu} U_{x+\mu,v} U_{x+v,\mu}^\dagger U_{x,v}^\dagger \right) + c.c.,$$

$$S_\phi[U, \phi] := \sum_{x,\mu} \frac{\gamma}{2} \text{tr} \left( (D_\mu^e[U] \phi_x)^\dagger (D_\mu^e[U] \phi_x) \right),$$

where $U_{x,\mu} = \exp(-i\epsilon A_{x,\mu}) \in SU(2)$ represents a gauge variable on a link $< x, \mu >$, $\phi_x = \phi_x^A \sigma^A \in su(2) (A = 1, 2, 3)$ represents a scalar field on a site $x$ in the adjoint representation subject to the radially-fixed condition: $\phi_x \cdot \phi_x = \phi_x^A \phi_x^A = 1$, and $D_\mu^e[U] \phi_x$ represents the covariant derivative in the adjoint representation defined as

$$D_\mu^e[U] \phi_x = U_{x,\mu} \phi_{x+\epsilon \hat{\mu}} - \phi_x U_{x,\mu}.$$

In the naive continuum limit this action reproduces $\epsilon \to 0$ the continuum gauge-scalar theory with a radially-fixed scalar field $|\phi(x)| = \nu$ and a gauge coupling constant $g$ where $\beta = 4/g^2$ and $\gamma = \nu^2/2$. 
The numerical simulation can be performed by updating link variables and scalar fields alternately.

For link variable $U_{x,\mu}$ we can apply the standard HMC algorithm.

For scalar field we reparametrized the variable $\phi_x \in su(2)$ according to the adjoint-orbit representation:

$$\phi_x := Y_x \sigma^3 Y_x^\dagger, \ Y_x \in SU(2),$$

which satisfies the normalization condition $\phi_x \cdot \phi_x = 1$ automatically.

The Haar measure is replaced by $D[\phi] \prod_x \delta(\phi_x \cdot \phi_x - 1)$ to $D[Y]$, and we can apply the standard HMC algorithm for $Y_x$, to update configurations of the scalar fields $\phi_x$. 
We introduce the site variable \( n_x := n^A_x \sigma_A \in SU(2) / U(1) \) which is called the color-direction (vector) field, in addition to the original link variable \( U_{x,\mu} \in SU(2) \). The link variable \( U_{x,\mu} \) and the site variable \( n_x \) transforms under the gauge transformation \( \Omega_x \in SU(2) \) as

\[
U_{x,\mu} \rightarrow \Omega_x U_{x,\mu} \Omega^\dagger_x = U'_{x,\mu}, \quad n_x \rightarrow \Omega_x n_x \Omega^\dagger_x = n'_x.
\]

In the decomposition, a link variable \( U_{x,\mu} \) is decomposed into two parts:

\[
U_{x,\mu} := X_{x,\mu} V_{x,\mu}.
\]

\[
V_{x,\mu} \rightarrow \Omega_x V_{x,\mu} \Omega^\dagger_x = V'_{x,\mu} \quad \text{and} \quad X_{x,\mu} \rightarrow \Omega_x X_{x,\mu} \Omega^\dagger_x = X'_{x,\mu}.
\]

Such decomposition is obtained by solving the defining equations:

\[
D_\mu[V]n_x := V_{x,\mu} n_{x+\mu} - n_x V_{x,\mu} = 0, \quad \text{tr}(n_x X_{x,\mu}) = 0.
\]

This defining equation has been solved exactly and the resulting link variable \( V_{x,\mu} \) and site variable \( X_{x,\mu} \) are of the form

\[
V_{x,\mu} := \tilde{V}_{x,\mu} / \sqrt{\text{tr}[\tilde{V}^\dagger_{x,\mu} \tilde{V}_{x,\mu}]/2}, \quad \tilde{V}_{x,\mu} := U_{x,\mu} + n_x U_{x,\mu} n_{x+\mu},
\]

\[
X_{x,\mu} := U_{x,\mu} V_{x,\mu}^{-1}.
\]

This decomposition is obtained uniquely for given set of link variable \( U_{x,\mu} \) once the site variable \( n_x \) is given.
Gauge-covariant decomposition (cont’)

The configurations of the color-direction field \( \{n_x\} \) are obtained by minimizing the functional:

\[
F_{\text{red}}[\{n_x\}|\{U_{x,\mu}\}] := \sum_{x,\mu} \text{tr} \left\{ (D_{x,\mu}[U]n_x)\dagger (D_{x,\mu}[U]n_x) \right\},
\]

which we call the \textit{reduction condition.}

Note that this functional has the same form as the action of the scalar field:

\[
S_{\phi} = \frac{\gamma}{2} F_{\text{red}}[\{\phi_x\}|\{U_{x,\mu}\}].
\]
Lattice result and gauge-independent analyses

![Simulation points in $\beta$-$\gamma$ plane](image)

**Figure:** Simulation points in $\beta$-$\gamma$ plane

- **Simulation**
  - $16^4$ lattice with PBC in the gauge-independent way (without gauge fixing).
  - After thermalization 5000 sweeps and store 800 configurations every 25 sweeps.

- **The search for the phase boundary**
  - by measuring the expectation value $\langle O \rangle$ of a chosen operator $O$ by changing $\gamma$ (or $\beta$) along the $\beta =$const. (or $\gamma =$const.) lines.

- **Identify the boundary,**
  - Used the bent, step, and gap observed in the graph of the plots for $\langle O \rangle$. 

Numerical Result I

**plaquette-action density**

\[
P = \frac{1}{6N_{\text{site}}} \sum_x \sum_{\mu < \nu} \frac{1}{2} \text{tr}(U_{x,\mu\nu}), \quad U_{x,\mu\nu} = U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu} U_{x,\nu}^{\dagger},
\]

**Figure:** Average of the plaquette-action density \( \langle P \rangle \): (Left) \( \langle P \rangle \) versus \( \gamma \) on various \( \beta = \text{const.} \) lines, (Right) \( \langle P \rangle \) versus \( \beta \) on various \( \gamma = \text{const.} \) lines.
scalar-action density

\[ M = \frac{1}{4N_{\text{site}}} \sum_{x} \sum_{\mu} \frac{1}{2} \text{tr} \left( (D_{\mu}[U_{x,\mu}] \phi_{x})^{\dagger} (D_{\mu}[U_{x,\mu}] \phi_{x}) \right), \]

**Figure:** Average of the scalar-action density \( \langle M \rangle \): (Left) \( \langle M \rangle \) versus \( \gamma \) on various \( \beta = \text{const.} \) lines, (Right) \( \langle M \rangle \) versus \( \beta \) on various \( \gamma = \text{const.} \) lines.
Figure: The phase boundary determined by the action densities: (Left) $\langle P \rangle$, (Right) $\langle M \rangle$.

We find that the gauge-independent numerical simulations reproduce the critical line obtained by Brower et al.
We investigate the correlations between the scalar field and the color-direction field through the gauge covariant decomposition:

\[ Q = \frac{1}{N_{\text{site}}} \sum_x \frac{1}{2} \text{tr}(n_x \phi_x), \]

We need to solve the reduction condition to obtain the color-direction field \( n_x \), which however has two kinds of ambiguity. One comes from so-called the Gribov copies that are the local minimal solutions of the reduction condition. Another comes from the choice of a global sign factor, which originates from the fact that whenever a configuration \( \{n_x\} \) is a solution, the flipped one \( \{-n_x\} \) is also a solution, since the reduction functional is quadratic in the color fields. To avoid these issues, we propose to use \( \langle |Q| \rangle \) and \( \langle Q^2 \rangle \). The phase boundary is searched for based on two ways:

(i) the location at which \( \langle |Q| \rangle \) changes from \( \langle |Q| \rangle \simeq 0 \) to \( \langle |Q| \rangle > 0 \). This is also the case for \( \langle Q^2 \rangle \).

(ii) the location at which \( \langle |Q| \rangle \) changes abruptly, as was done for \( \langle P \rangle \) and \( \langle M \rangle \). This is also the case for \( \langle Q^2 \rangle \).
The scalar-color correlation is given by:

\[ Q = \frac{1}{N_{\text{site}}} \sum_x \frac{1}{2} \text{tr}(n_x \phi_x), \]

Figure: Average of the scalar-color composite field \( \langle |Q| \rangle \): (Left) \( \langle |Q| \rangle \) versus \( \gamma \) on various \( \beta = \text{const.} \) lines, (Right) \( \langle |Q| \rangle \) versus \( \beta \) on various \( \gamma = \text{const.} \) lines.
Phase boundary (critical line) from scalar-color correlation

**Figure:** Critical lines determined (Left) from $\langle |Q| \rangle$, (Right) from $\langle Q^2 \rangle$. 
Understanding the new phase structure

(I) confinement phase
$\gamma < \gamma_c(\beta)$: the effect of the scalar field would be relatively small and confinement would occur in the way similar to the pure $SU(2)$ gauge theory.

Disordered phase: in the sense that the color direction field $n_x$ takes various directions with no specific direction in color space, i.e., very small or vanishing value of the average $\langle Q \rangle = 0$.

Confinement is expected to occur due to vacuum condensations of non-Abelian magnetic monopoles.

(II) Higgs phase
The gauge fields become massive due to different physical origins. That is, the off-diagonal gauge fields for the modes $SU(2)/U(1)$ become massive due to the BEH mechanism, which is a consequence of the (partial) spontaneous symmetry breaking $SU(2) \to U(1)$ according to the conventional understanding of the BEH mechanism.

The diagonal gauge field for the mode $U(1)$ always remains massless everywhere.
(III) confinement phase
the gauge fields become massive due to self-interactions among the gauge fields, as in the phase (I).

difference between (II) and (III)

dordered phase: In these phases, the color-direction field $n_x$ correlates strongly with the given scalar field $\phi_x$ which tends to align to a specific direction. Consequently, the $SU(2)$ gauge-scalar model reduces to the pure compact $U(1)$ gauge model.

The pure compact $U(1)$ gauge model in four space-time dimensions has two phases: confinement phase with massive $U(1)$ gauge field in the strong gauge coupling region $\beta < \beta^*$ and the Coulomb phase with massless $U(1)$ gauge field in the weak gauge coupling region $\beta > \beta^*$, which has been proved rigorously. (A.H. Guth 1980, Fröhlich and T. Spencer 1982)
Conclusion and discussion

- We have investigated the lattice $SU(2)$ gauge-scalar model with the scalar field in the adjoint representation of the gauge group in a gauge-independent way.
- We have re-examined this phase structure in the gauge-independent way based on the numerical simulations performed without any gauge fixing, which should be compared with the preceding studies by Brower et.al..
- This is motivated to confirm the recently proposed gauge-independent Brout-Englert-Higgs mechanics for giving the mass of the gauge field without relying on any spontaneous symmetry breaking.
- For this purpose we have investigated correlation functions between gauge-invariant operators obtained by combining the original adjoint scalar field and the new field called the color-direction field which is constructed from the gauge field based on the gauge-covariant decomposition of the gauge field due to Cho-Duan-Ge-Shabanov and Faddeev-Niemi.
- We have reproduced gauge-independently the transition line separating confinement and Higgs phase obtained in the preseeding study, and show surprisingly the existence of a new transition line that divides completely the confinement phase into two parts.
- The result obtained in this paper should be compared with the lattice $SU(2)$ gauge-scalar model with the scalar field in the fundamental representation of the gauge group in a gauge-independent way.