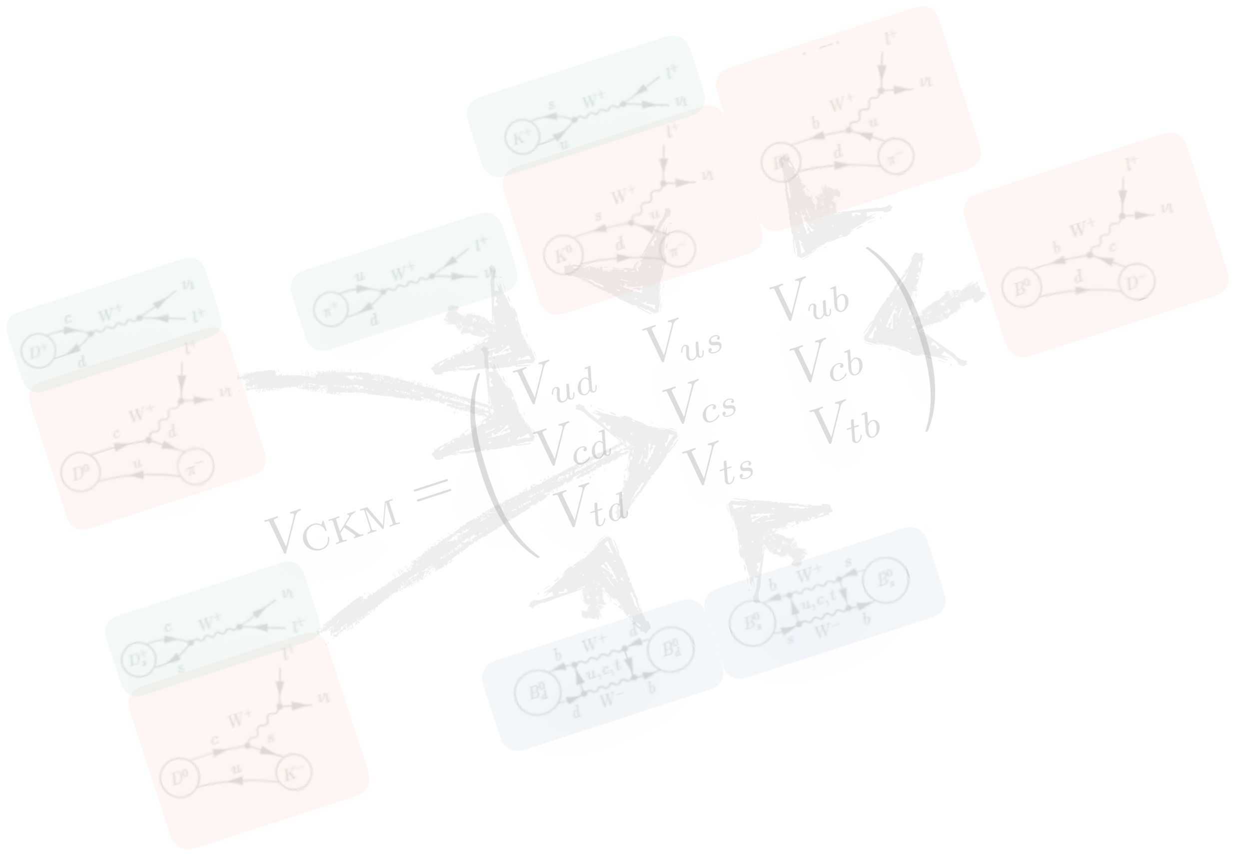
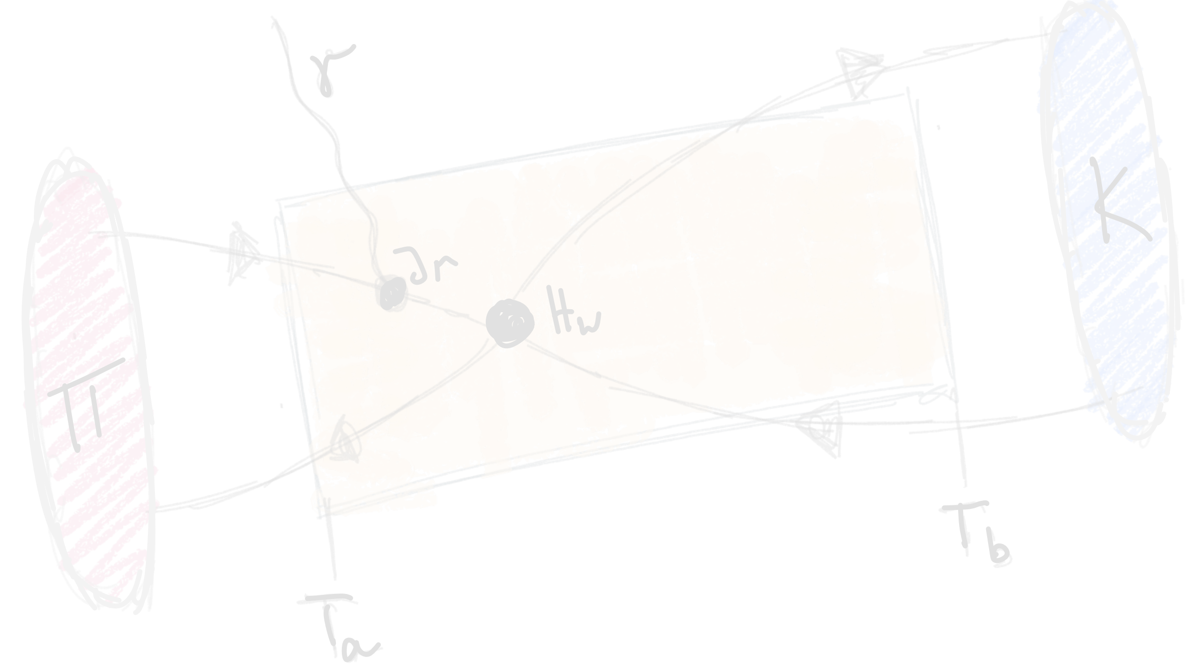
Bayesian inference for form-factor fits regulated by unitarity and analyticity



2023 LATTICE

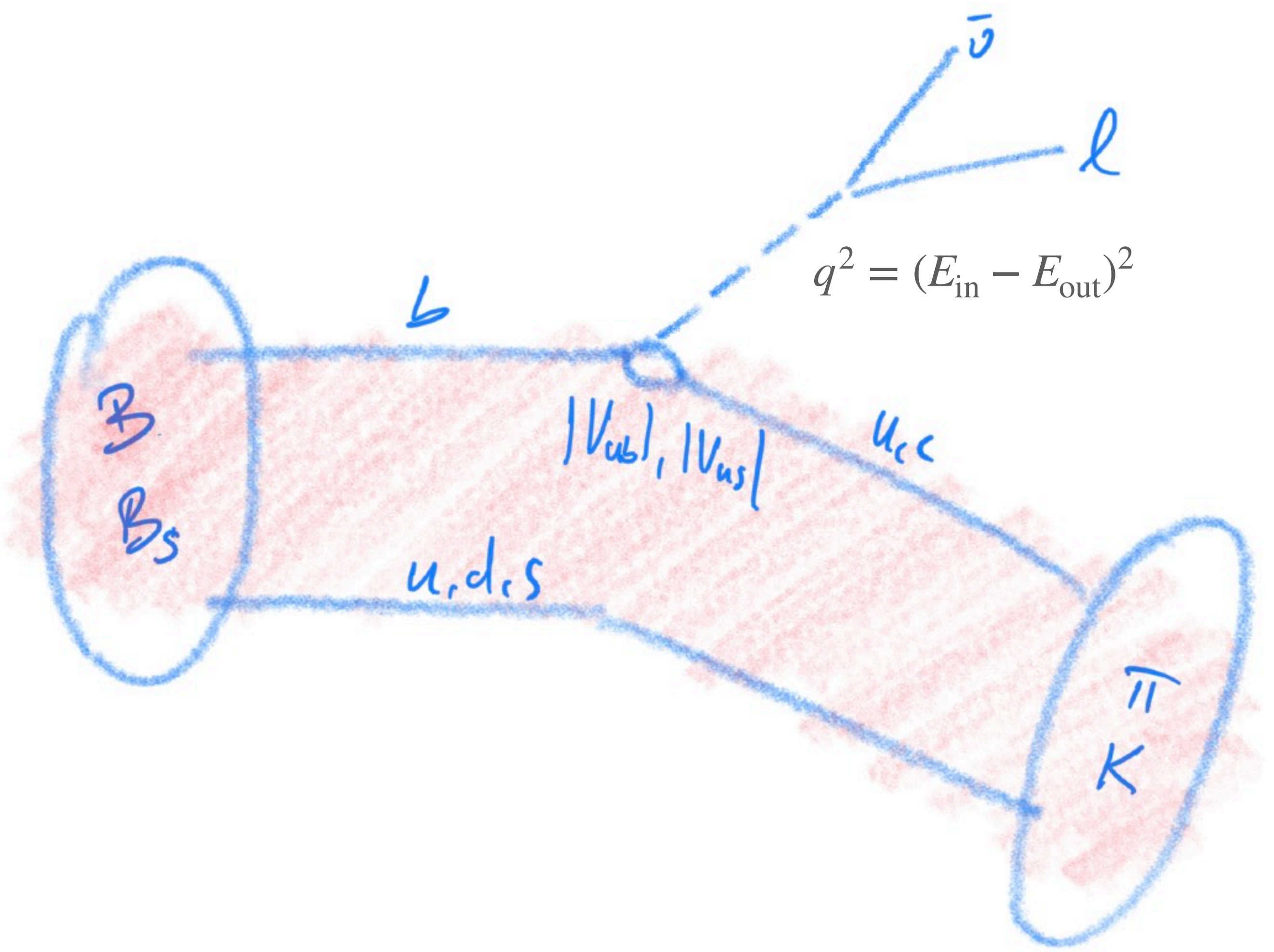


Andreas Jüttner
with
Jonathan Flynn and Tobi Tsang

CERN University of Southampton

Flynn, AJ, Tsang, arXiv:2303.11285

Intro: exclusive semileptonic meson decay



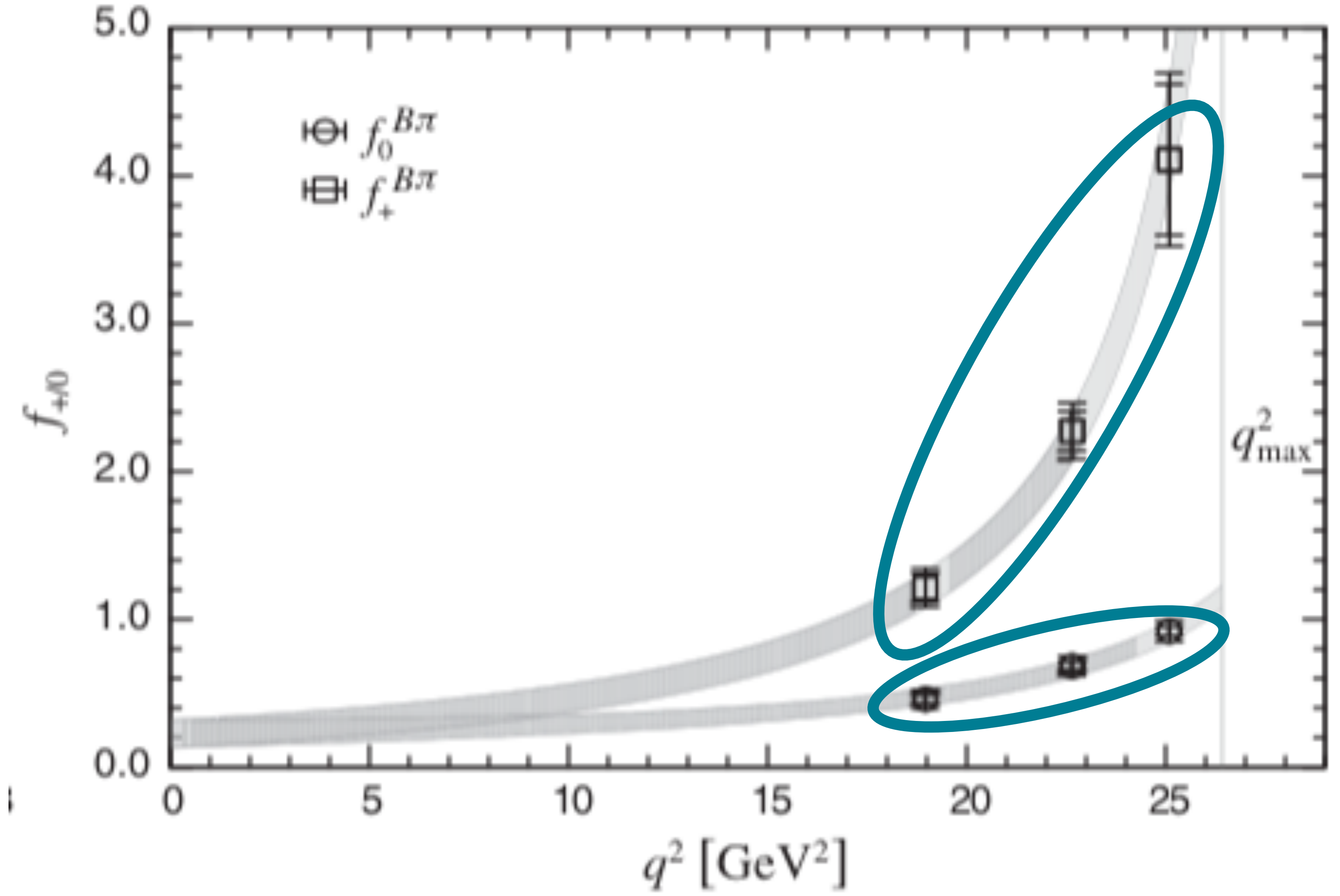
For $B_{(s)}$ decay the kinematics towards physical point are such that

- finite cutoff
- finite volume
- worsening signal-to-noise

→ **limited kinematic reach**

↓

need extrapolation



RBC/UKQCD PRD 91, 074510 (2015)

BGL fitting strategies

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n \qquad |\mathbf{a}_X|^2 \leq 1$$

Boyd, Grinstein, Lebed, PRL 74 (1995)

Determine all $a_{X,n}$ from finite set of experimental/theory data

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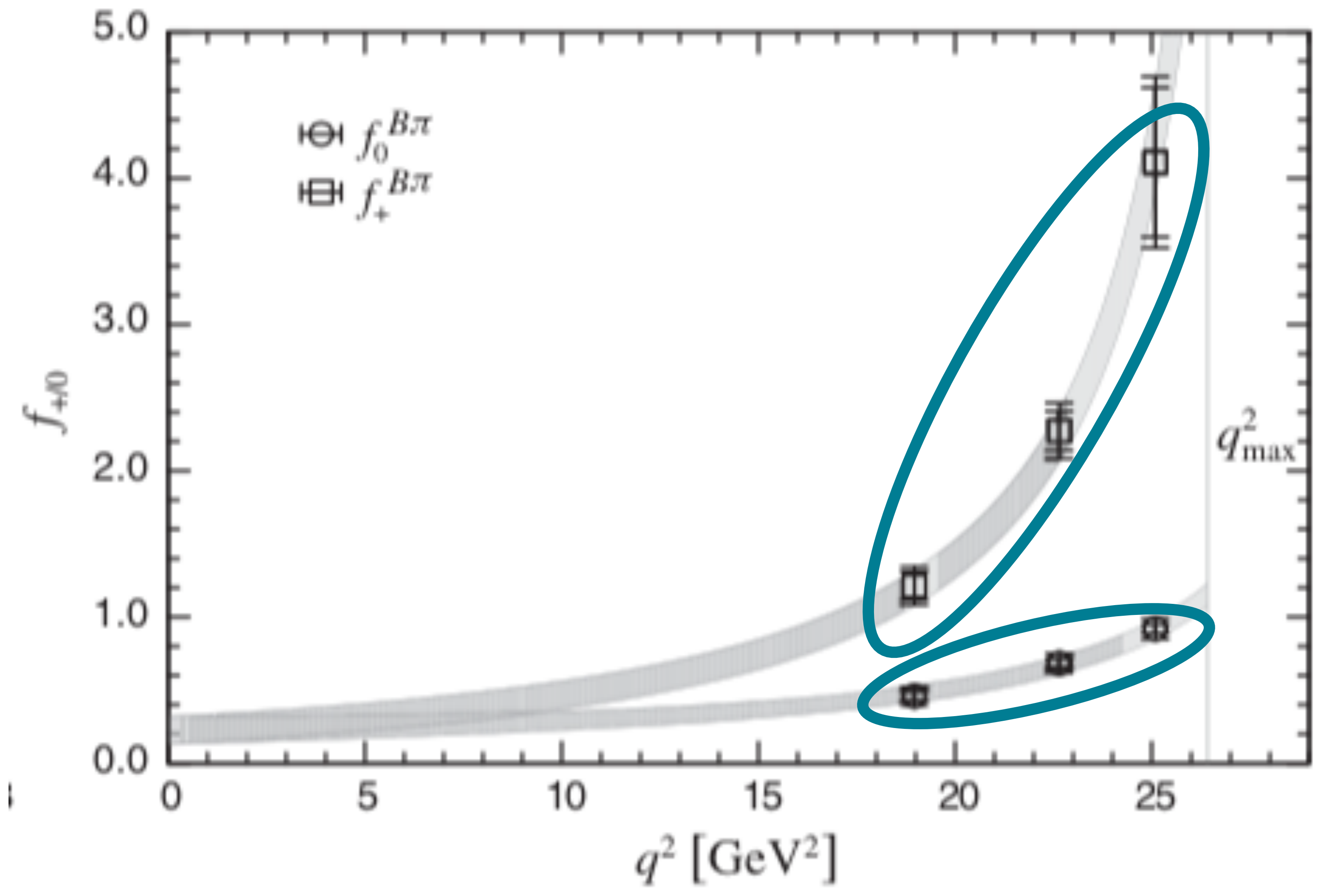
Determine all $a_{X,n}$ from finite set of experimental/theory data

Frequentist fit:

- $N_{\text{dof}} = N_{\text{data}} - N_{\text{params}} \geq 1 \rightarrow$ in practice truncation of z expansion@low order
- induced systematic difficult to estimate
- meaning of Frequentist with unitarity constraint?



RBC/UKQCD PRD 91, 074510 (2015)

BGL fitting strategies

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n \qquad |\mathbf{a}_X|^2 \leq 1$$

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Determine all $a_{X,n}$ from finite set of experimental/theory data

Frequentist fit:

- $N_{\text{dof}} = N_{\text{data}} - N_{\text{params}} \geq 1$ → in practice truncation of z expansion@low order
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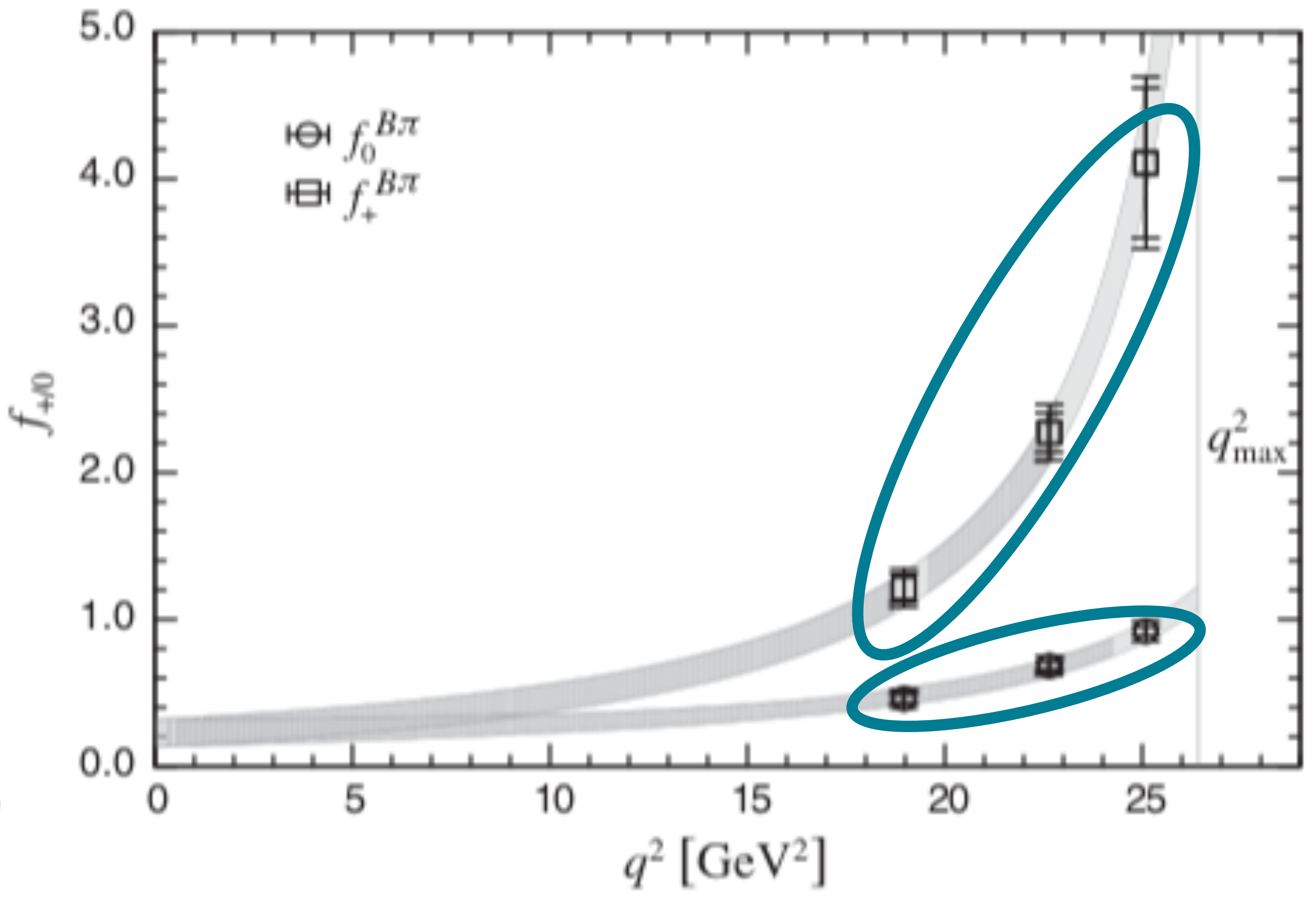
- meaning of Frequentist with unitarity constraint?

Bayesian fit:

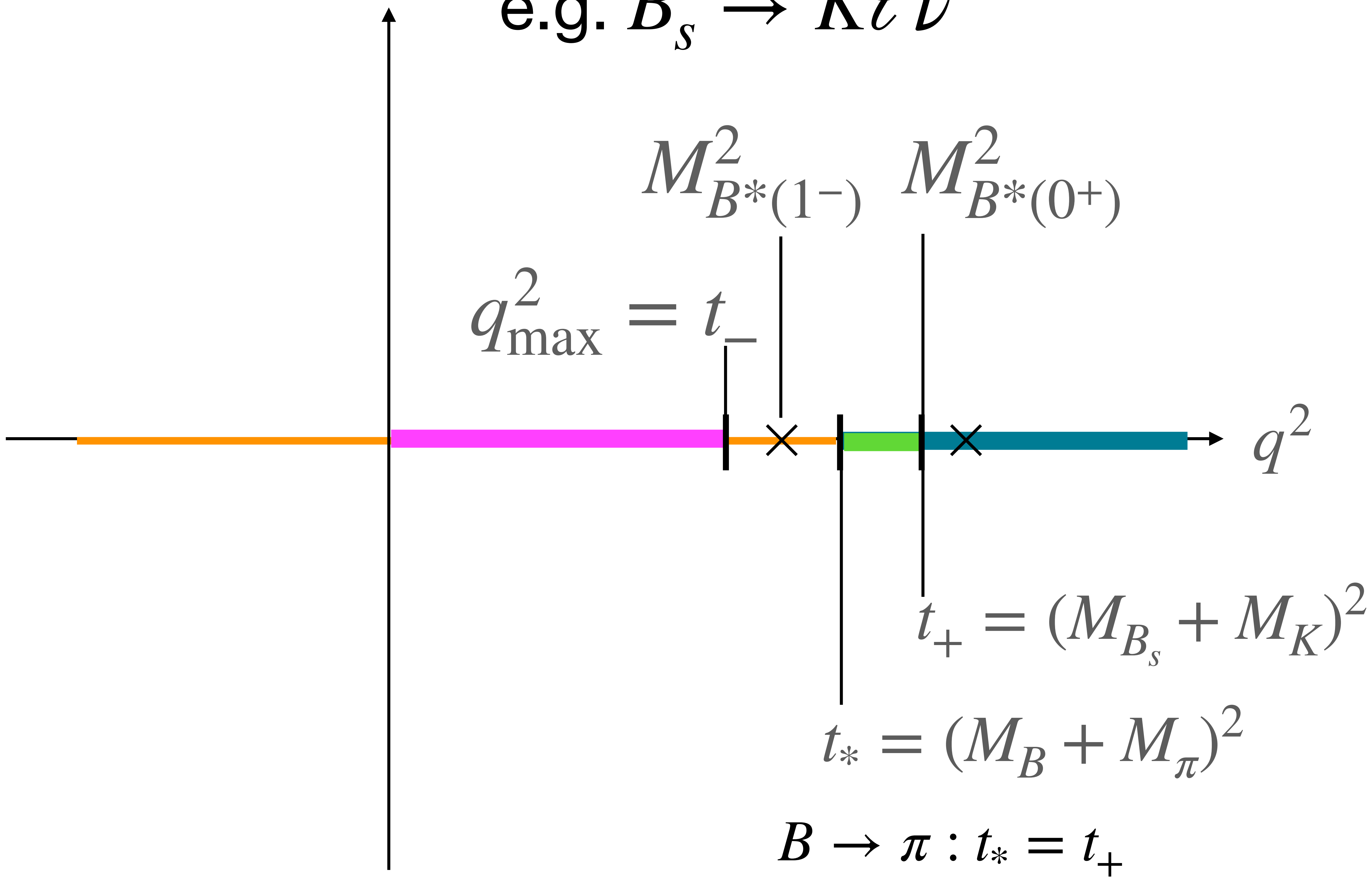
- **Here:** fit full z expansion (i.e. arbitrary high (→ no) truncation)
- need regulator to control higher-order coefficients
- well-defined meaning of unitarity constraint



RBC/UKQCD PRD 91, 074510 (2015)

An aside: correct BGL unitarity constraint for for $B_s \to K\ell\nu$

e.g. $B_s \to K\ell\nu$



$B \to \pi : t_* = t_+$

$B_s \to K : t_* < t_+$

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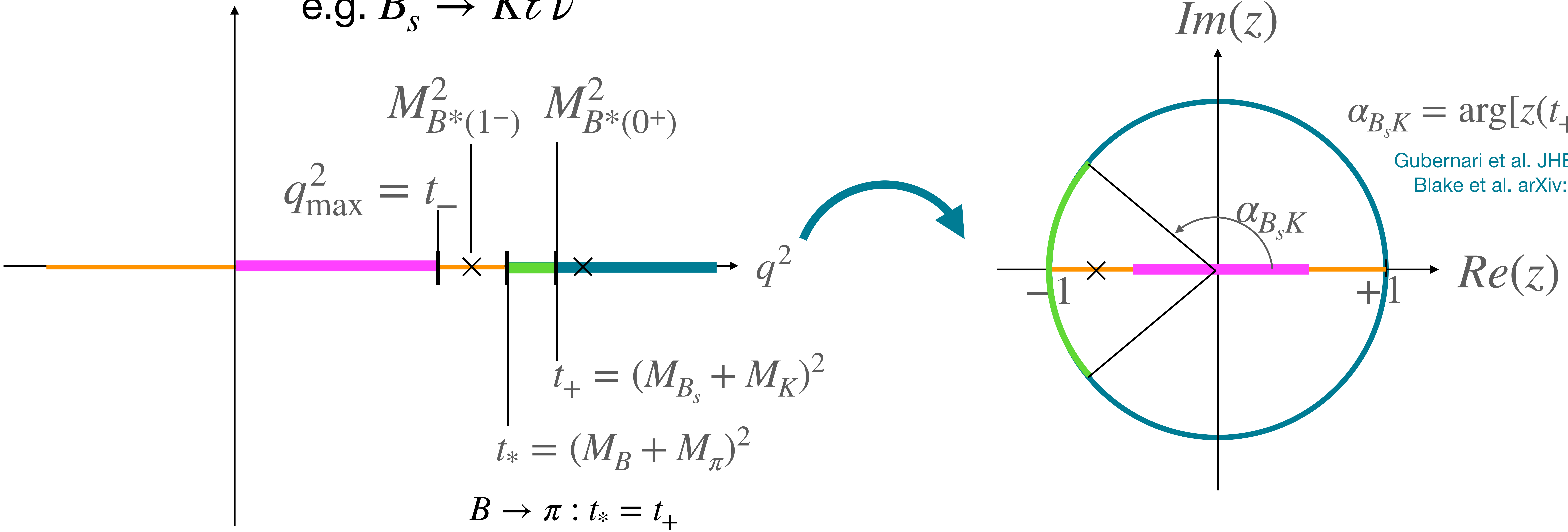

$\alpha_{B_sK} = \arg[z(t_+)]$

Gubernari et al. JHEP 2021,2022,
Blake et al. arXiv:2205.06041

$B \to \pi : t_* = t_+$
$B_s \to K : t_* < t_+$

Okubo, PRD 3, 2807 (1971), PRD 4, 725 (1971).
Okubo, Shih, PRD 4, 2020 (1971).
Boyd, Grinstein, Lebed, PLB 353, 306 (1995).
NPB461, 493 (1996). PRD 56, 6895 (1997).

An aside: correct BGL unitarity constraint for for $B_s \to K\ell\nu$

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n \qquad X = +,0$$

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unitarity constraint

e.g. $$\frac{1}{\pi\chi_J^T(q^2)} \int_{t_+}^{\infty} dt \frac{W(t)\,|f_+(t)|^2}{(t-q^2)^3} \le 1 \longrightarrow \frac{1}{2\pi i}\oint_C \frac{dz}{z}\theta_{B_sK}\,|B_X(q^2)\phi_X(q^2,t_0)f_X(q^2)|^2 \le 1$$

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$$\downarrow$$

$$a_{X,i}\langle z^i | z^j \rangle a_{X,j} \le 1$$

$$\langle z^i|z^j\rangle_\alpha = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} d\phi\,(z^i)^* z^j|_{z=e^{i\phi}} = \begin{cases} \dfrac{\sin(\alpha(i-j))}{\pi(i-j)} & i \neq j \\ \dfrac{\alpha}{\pi} & i = j \end{cases}$$

Flynn, AJ, Tsang, arXiv:2303.11285

modified unitarity constraint for channels where relevant threshold t_* not the lowest

Gubernari et al. JHEP 2021,2022,
Blake et al. arXiv:2205.06041

BGL — conventions

$$f_X(q_i^2) = \frac{1}{B_X(q_i^2)\phi_X(q_i^2, t_0)} \sum_{n=0}^{K_X-1} a_{X,n} z(q_i^2)^n = Z_{XX,in} a_{X,n}$$

Input (e.g. lattice ff): $\mathbf{f}^T = (\mathbf{f}_+^T, \mathbf{f}_0^T) = (f_+(q_0^2), f_+(q_1^2), \ldots, f_+(q_{N_+-1}^2), f_0(q_0^2), f_0(q_1^2), \ldots, f_0(q_{N_0-1}^2))$

Output (BGL params): $\mathbf{a}^T = (\mathbf{a}_+^T, \mathbf{a}_0^T) = (a_{+,0}, a_{+,1}, a_{+,2}, \ldots, a_{+,K_+-1}, a_{0,1}, \ldots, a_{0,K_0-1})$

BGL — frequentist fit with kinematical constraint

Flynn, AJ, Tsang, arXiv:2303.11285

Frequentist fit $\chi^2(\mathbf{a}, \mathbf{f}) = [\mathbf{f} - Z\mathbf{a}]^T C_{\mathbf{f}}^{-1} [\mathbf{f} - Z\mathbf{a}]$ $f_X(q_i^2) = Z_{XX,in} a_{X,n}$

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For combined fit over $f_+(q^2)$ and $f_0(q^2)$
with constraint $f_+(0) = f_0(0)$:

$$Z = \begin{pmatrix} Z_{++} & Z_{+0} \\ Z_{0+} & Z_{00} \end{pmatrix}$$

$$a_{0,0} = B_0(0)\phi_0(0,t_0)f_+(0) - \sum_{k=1}^{K_0-1} a_{0,k} z^k(0)$$

expressions in arXiv:2303.11285

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expressions in arXiv:2303.11285

Solution:

$$\mathbf{a} = \left(Z^T C_{\mathbf{f}}^{-1} Z\right)^{-1} Z C_{\mathbf{f}}^{-1} \mathbf{f}$$

$$C_{\mathbf{a}} = \left(Z^T C_{\mathbf{f}}^{-1} Z\right)^{-1}$$

New: Bayesian form-factor fit

Flynn, AJ, Tsang, arXiv:2303.11285

Compute BGL parameters as expectation values $\langle g(\mathbf{a})\rangle = \mathcal{N}\int d\mathbf{a}\, g(\mathbf{a})\, \pi(\mathbf{a}\,|\,\mathbf{f}, C_{\mathbf{f}})\, \pi_{\mathbf{a}}$

where *probability for parameters given model and data*

$$\pi(\mathbf{a}\,|\,\mathbf{f}, C_{\mathbf{f}}) \propto \exp\left(-\frac{1}{2}\chi^2(\mathbf{a},\mathbf{f})\right) \quad \text{where} \quad \chi^2(\mathbf{a},\mathbf{f}) = (\mathbf{f} - Z\mathbf{a})^T C_{\mathbf{f}}^{-1}(\mathbf{f} - Z\mathbf{a})$$

where *prior knowledge just QFT:*

$$\pi_{\mathbf{a}} \propto \theta\left(1 - |\mathbf{a}_+|^2_{\alpha_{B_sK}}\right)\theta\left(1 - |\mathbf{a}_0|^2_{\alpha_{B_sK}}\right)$$

In practice MC integration: draw samples for **a** from multivariate normal distribution and drop samples not compatible with unitarity

New: Bayesian form-factor fit

Flynn, AJ, Tsang, arXiv:2303.11285

In practice high-dimensional → low probability of drawing random number compatible with constraint

$$\pi_{\mathbf{a}}(\mathbf{a}\,|\,\mathbf{f}_p, C_{\mathbf{f}_p})\pi_{\mathbf{a}}(\mathbf{a}\,|\,\mathbf{a}_p, M) \propto \theta(\mathbf{a})\exp\left(-\frac{1}{2}(\mathbf{f}_p - Z\mathbf{a})^T C_{\mathbf{f}_p}^{-1}(\mathbf{f}_p - Z\mathbf{a}) - \frac{1}{2}\mathbf{a}^T M/\sigma^2\,\mathbf{a}\right)$$

- Add ‘technical’ prior
- Choose M such that $\mathbf{a}^T M\mathbf{a} \leq 1$ in presence of kinematical constraint
- Correct towards ‘flat unitarity-only prior’ with accept-reject step with probability

$$p \leq \frac{\exp(-1/\sigma^2)}{\exp(-\mathbf{a}^T M\mathbf{a}/2\sigma^2)}$$

FINAL RESULT INDEPENDENT OF *TECHNICAL* PRIOR

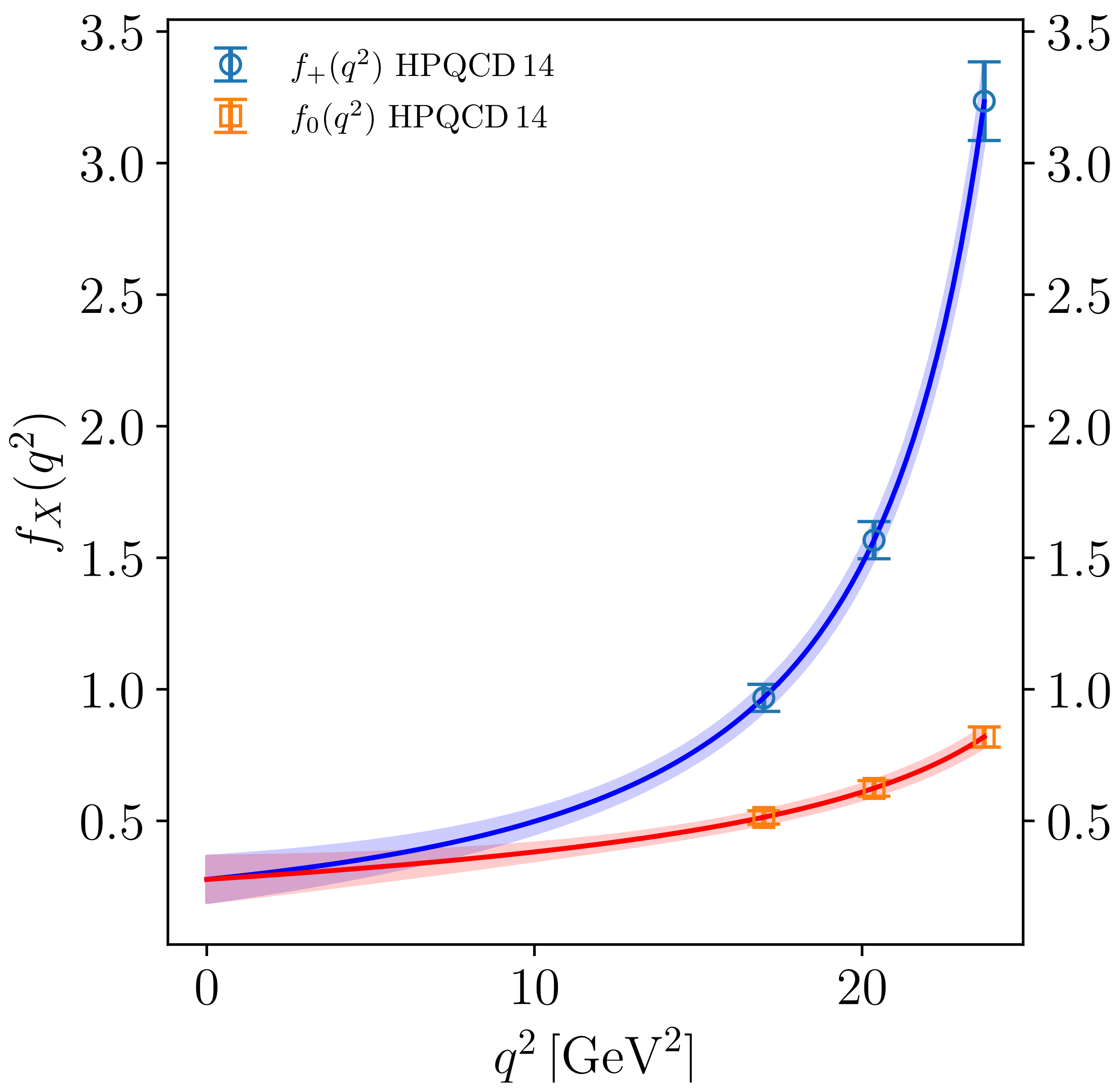
Example 1: $B_s \to K\ell\nu$

HPQCD 14
PRD 90 (2014) 054506

HPQCD 14 – $\mathbf{a}_+$

Frequentist

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.0270(13)	-0.0792(50)	-	0.03	2.93	3
2	3	0.0273(13)	-0.0760(63)	-	0.02	4.06	2
3	2	0.0257(14)	-0.0805(50)	0.068(31)	0.15	1.89	2
3	3	0.0262(14)	-0.0727(64)	0.096(34)	0.97	0.00	1



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2	2	0.0883(44)	-0.250(17)	-	0.03	2.93	3
2	3	0.0880(44)	-0.242(19)	0.053(65)	0.02	4.06	2
3	2	0.0906(45)	-0.240(17)	-	0.15	1.89	2
3	3	0.0908(46)	-0.215(22)	0.138(71)	0.97	0.00	1

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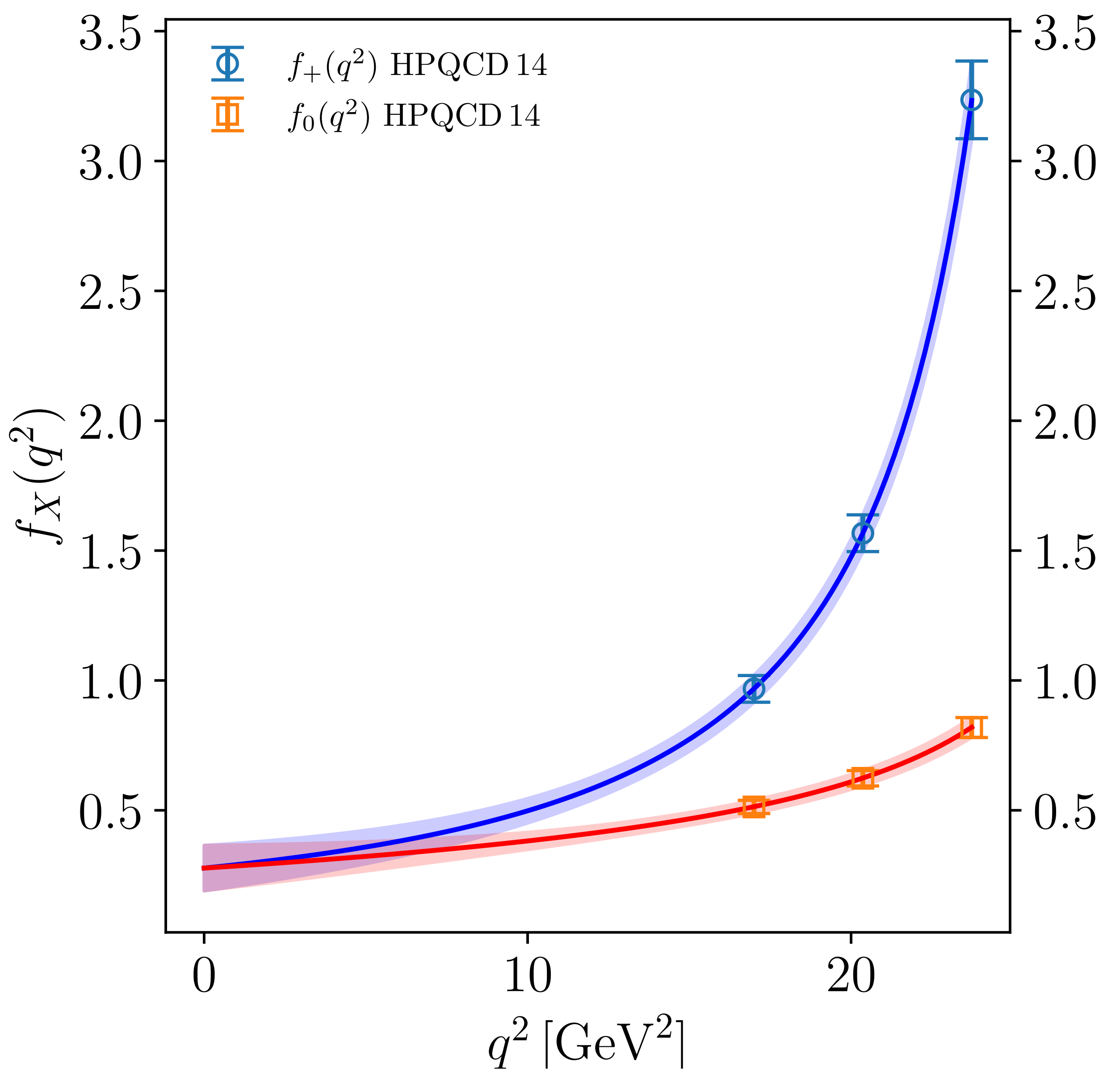

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Frequentist

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Flynn, AJ, Tsang, arXiv:2303.11285

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	$a_{+,5}$	$a_{+,6}$	$a_{+,7}$	$a_{+,8}$	$a_{+,9}$
2	2	0.0270(12)	-0.0792(49)	-	-	-	-	-	-	-	-
2	3	0.0273(13)	-0.0761(63)	-	-	-	-	-	-	-	-
3	2	0.0257(14)	-0.0805(49)	0.069(30)	-	-	-	-	-	-	-
3	3	0.0261(14)	-0.0728(64)	0.096(34)	-	-	-	-	-	-	-
3	4	0.0261(14)	-0.0728(76)	0.096(39)	-	-	-	-	-	-	-
4	3	0.0261(14)	-0.0729(68)	0.096(35)	0.008(90)	-	-	-	-	-	-
4	4	0.0261(14)	-0.0730(77)	0.091(62)	-0.02(20)	-	-	-	-	-	-
5	5	0.0262(15)	-0.0735(79)	0.084(67)	-0.03(19)	0.03(68)	-	-	-	-	-
6	6	0.0261(14)	-0.0735(79)	0.086(69)	-0.03(19)	-0.00(64)	0.01(65)	-	-	-	-
7	7	0.0262(14)	-0.0732(84)	0.088(69)	-0.02(18)	0.01(65)	0.02(73)	-0.03(70)	-	-	-
8	8	0.0261(14)	-0.0732(80)	0.089(72)	-0.02(18)	-0.00(66)	0.03(86)	-0.04(90)	0.03(73)	-	-
9	9	0.0261(14)	-0.0729(84)	0.095(75)	-0.02(19)	-0.04(68)	0.1(1.0)	-0.1(1.2)	0.1(1.1)	-0.06(79)	-
10	10	0.0261(14)	-0.0726(89)	0.101(79)	-0.01(20)	-0.09(73)	0.2(1.3)	-0.3(1.7)	0.2(1.8)	-0.2(1.4)	0.08(87)

Bayesian

HPQCD 14 – $\mathbf{a}_0$

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$	$a_{0,6}$	$a_{0,7}$	$a_{0,8}$	$a_{0,9}$
2	2	0.0883(44)	-0.250(17)	-	-	-	-	-	-	-	-
2	3	0.0880(44)	-0.243(19)	0.052(65)	-	-	-	-	-	-	-
3	2	0.0907(46)	-0.240(17)	-	-	-	-	-	-	-	-
3	3	0.0906(44)	-0.215(22)	0.137(73)	-	-	-	-	-	-	-
3	4	0.0907(47)	-0.215(22)	0.14(11)	-0.01(31)	-	-	-	-	-	-
4	3	0.0907(45)	-0.214(22)	0.139(72)	-	-	-	-	-	-	-
4	4	0.0907(46)	-0.215(25)	0.12(19)	-0.08(60)	-	-	-	-	-	-
5	5	0.0909(46)	-0.218(25)	0.10(19)	-0.12(55)	0.04(63)	-	-	-	-	-
6	6	0.0907(45)	-0.217(25)	0.10(19)	-0.11(53)	0.06(66)	-0.02(66)	-	-	-	-
7	7	0.0907(46)	-0.217(26)	0.11(20)	-0.08(51)	0.03(73)	0.03(81)	-0.04(70)	-	-	-
8	8	0.0908(46)	-0.217(25)	0.11(20)	-0.08(50)	-0.01(84)	0.1(1.0)	-0.09(96)	0.08(74)	-	-
9	9	0.0907(46)	-0.215(25)	0.13(22)	-0.05(50)	-0.06(95)	0.2(1.4)	-0.2(1.5)	0.1(1.2)	-0.05(82)	-
10	10	0.0907(46)	-0.214(27)	0.15(24)	-0.03(49)	-0.2(1.1)	0.4(1.8)	-0.5(2.2)	0.4(2.1)	-0.3(1.6)	0.13(90)

Bayesian

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2	3	0.0273(13)	-0.0760(63)	-	0.02	4.06	2
3	2	0.0257(14)	-0.0805(50)	0.068(31)	0.15	1.89	2
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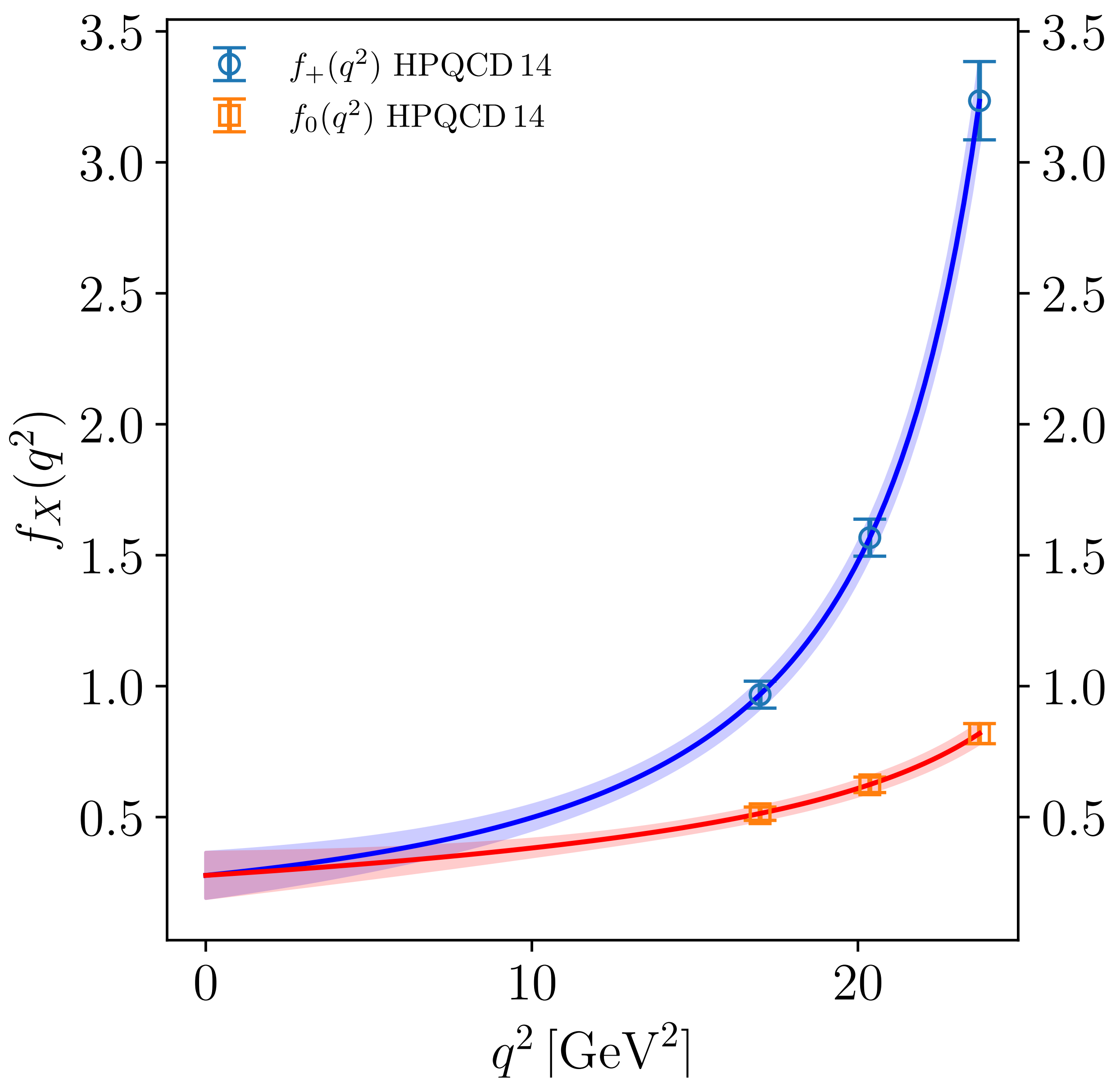

HPQCD 14 – $\mathbf{a}_0$

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Flynn, AJ, Tsang, arXiv:2303.11285

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2	2	0.0270(12)	-0.0792(49)	-	-	-	-	-	-	-	-
2	3	0.0273(13)	-0.0761(63)	-	-	-	-	-	-	-	-
3	2	0.0257(14)	-0.0805(49)	0.069(30)	-	-	-	-	-	-	-
3	3	0.0261(14)	-0.0728(64)	0.096(34)	-	-	-	-	-	-	-
3	4	0.0261(14)	-0.0728(76)	0.096(39)	-	-	-	-	-	-	-
4	3	0.0261(14)	-0.0729(68)	0.096(35)	0.008(90)	-	-	-	-	-	-
4	4	0.0261(14)	-0.0730(77)	0.091(62)	-0.02(20)	-	-	-	-	-	-
5	5	0.0262(15)	-0.0735(79)	0.084(67)	-0.03(19)	0.03(68)	-	-	-	-	-
6	6	0.0261(14)	-0.0735(79)	0.086(69)	-0.03(19)	-0.00(64)	0.01(65)	-	-	-	-
7	7	0.0262(14)	-0.0732(84)	0.088(69)	-0.02(18)	0.01(65)	0.02(73)	-0.03(70)	-	-	-
8	8	0.0261(14)	-0.0732(80)	0.089(72)	-0.02(18)	-0.00(66)	0.03(86)	-0.04(90)	0.03(73)	-	-
9	9	0.0261(14)	-0.0729(84)	0.095(75)	-0.02(19)	-0.04(68)	0.1(1.0)	-0.1(1.2)	0.1(1.1)	-0.06(79)	-
10	10	0.0261(14)	-0.0726(89)	0.101(79)	-0.01(20)	-0.09(73)	0.2(1.3)	-0.3(1.7)	0.2(1.8)	-0.2(1.4)	0.08(87)

Can increase to any truncation — results stable!

HPQCD 14 – $\mathbf{a}_0$

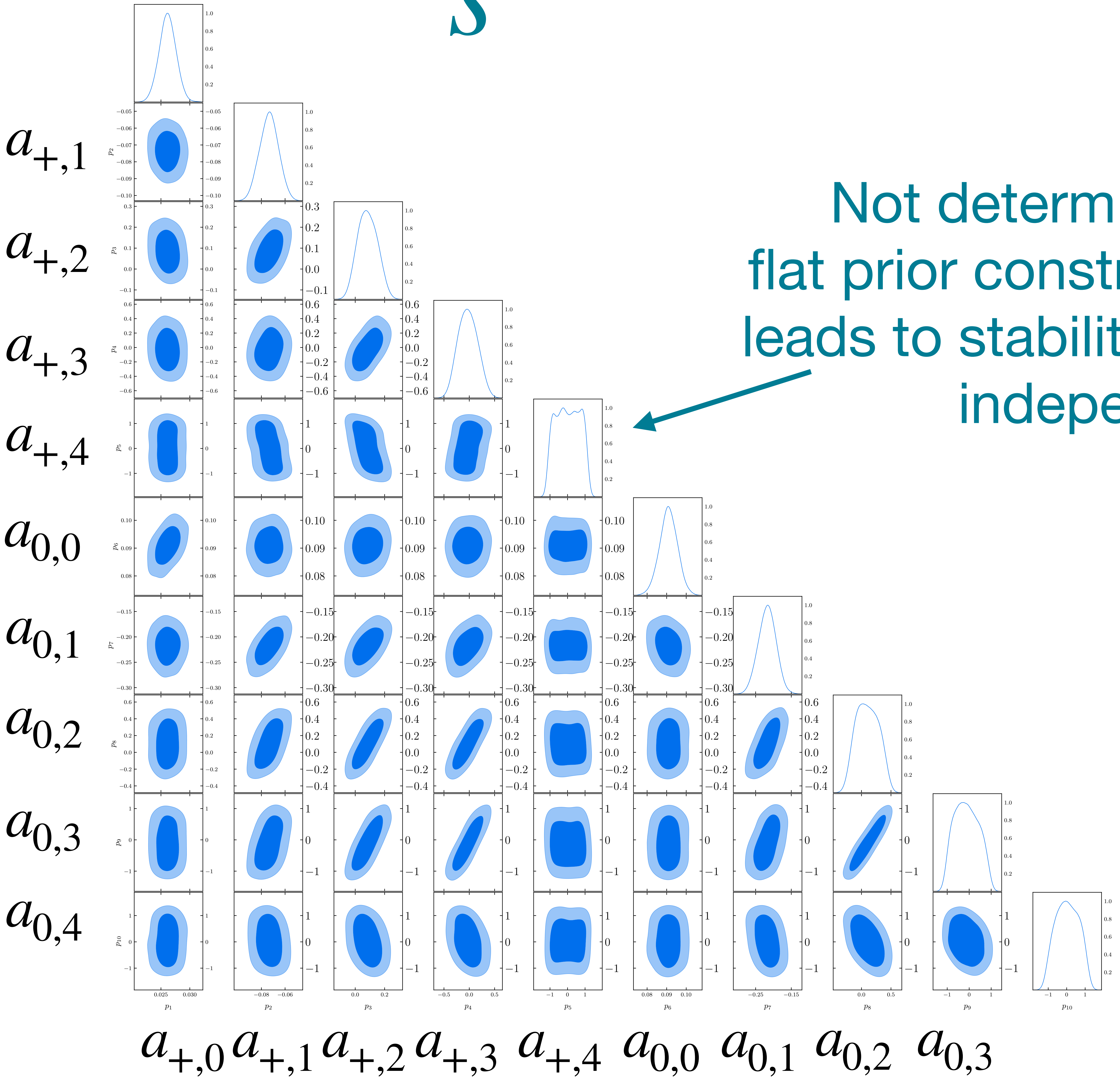
Bayesian

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	$a_{0,5}$	$a_{0,6}$	$a_{0,7}$	$a_{0,8}$	$a_{0,9}$
2	2	0.0883(44)	-0.250(17)	-	-	-	-	-	-	-	-
2	3	0.0880(44)	-0.243(19)	0.052(65)	-	-	-	-	-	-	-
3	2	0.0907(46)	-0.240(17)	-	-	-	-	-	-	-	-
3	3	0.0906(44)	-0.215(22)	0.137(73)	-	-	-	-	-	-	-
3	4	0.0907(47)	-0.215(22)	0.14(11)	-0.01(31)	-	-	-	-	-	-
4	3	0.0907(45)	-0.214(22)	0.139(72)	-	-	-	-	-	-	-
4	4	0.0907(46)	-0.215(25)	0.12(19)	-0.08(60)	-	-	-	-	-	-
5	5	0.0909(46)	-0.218(25)	0.10(19)	-0.12(55)	0.04(63)	-	-	-	-	-
6	6	0.0907(45)	-0.217(25)	0.10(19)	-0.11(53)	0.06(66)	-0.02(66)	-	-	-	-
7	7	0.0907(46)	-0.217(26)	0.11(20)	-0.08(51)	0.03(73)	0.03(81)	-0.04(70)	-	-	-
8	8	0.0908(46)	-0.217(25)	0.11(20)	-0.08(50)	-0.01(84)	0.1(1.0)	-0.09(96)	0.08(74)	-	-
9	9	0.0907(46)	-0.215(25)	0.13(22)	-0.05(50)	-0.06(95)	0.2(1.4)	-0.2(1.5)	0.1(1.2)	-0.05(82)	-
10	10	0.0907(46)	-0.214(27)	0.15(24)	-0.03(49)	-0.2(1.1)	0.4(1.8)	-0.5(2.2)	0.4(2.1)	-0.3(1.6)	0.13(90)

Example 1: $B_s \to K\ell\nu$

Distribution and correlation of BGL coefficients

truncation, e.g. $(K_+, K_0) = (5,5)$



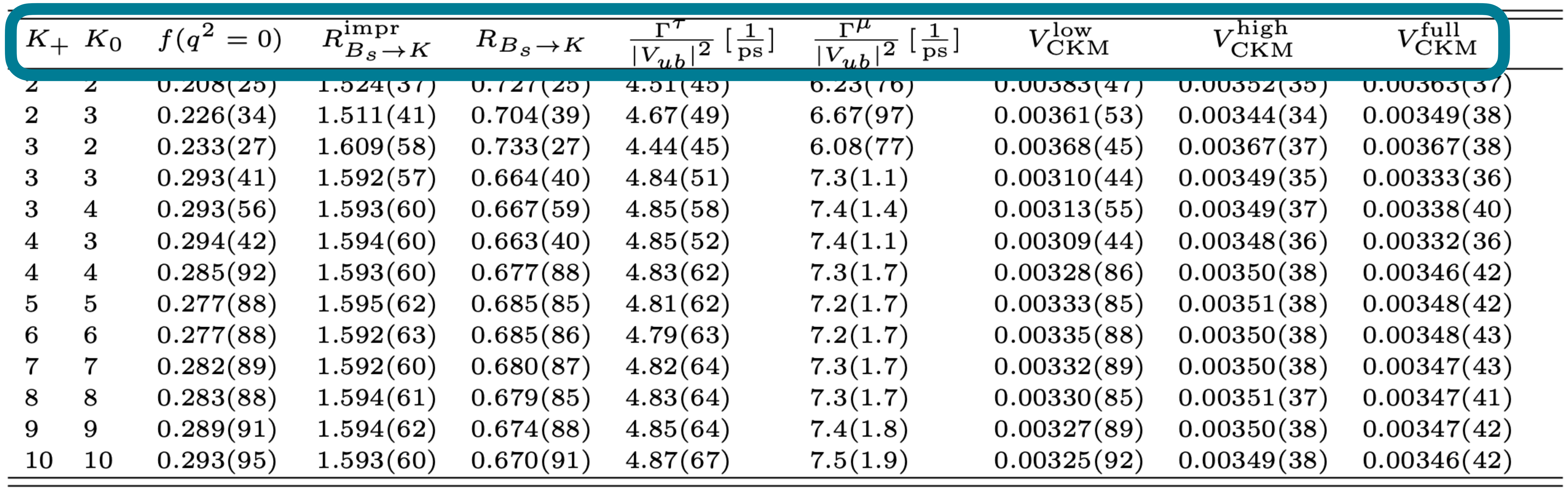
Not determined by data,
flat prior constrains coefficient,
leads to stability and truncation
independence

Example 1: $B_s \to K\ell\nu$

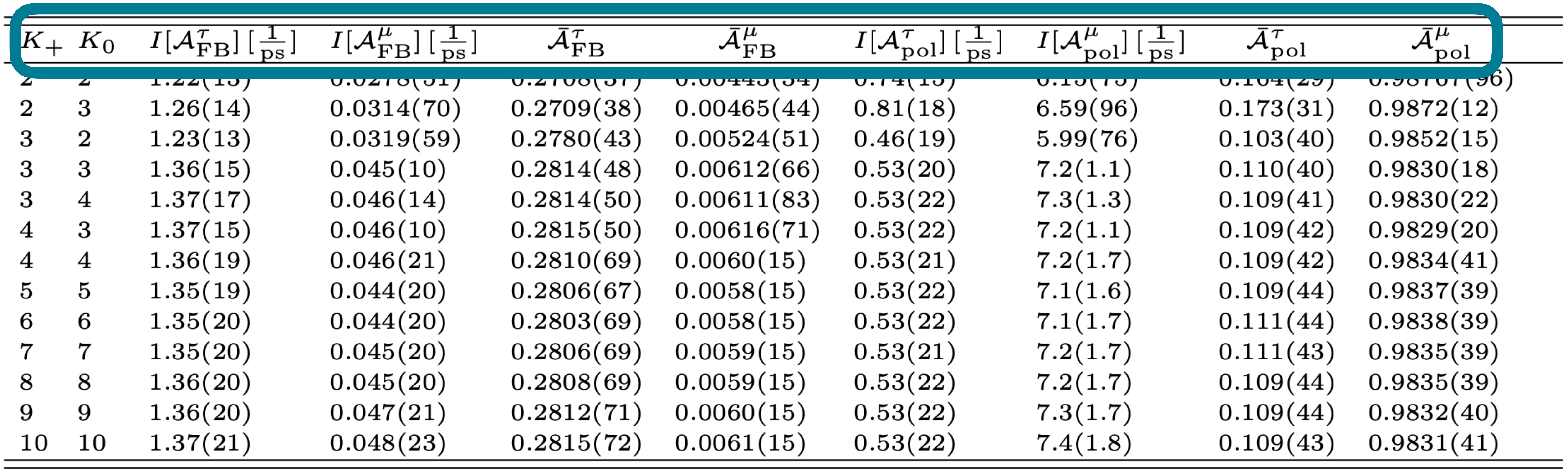
increasing truncation

increasing truncation

results for
phenomenology
independent
of truncation



K_+	K_0	$f(q^2=0)$	$R^{\text{impr}}_{B_s\to K}$	$R_{B_s\to K}$	$\frac{\Gamma^\tau}{\lvert V_{ub}\rvert^2}\,[\frac{1}{\text{ps}}]$	$\frac{\Gamma^\mu}{\lvert V_{ub}\rvert^2}\,[\frac{1}{\text{ps}}]$	$V^{\text{low}}_{\text{CKM}}$	$V^{\text{high}}_{\text{CKM}}$	$V^{\text{full}}_{\text{CKM}}$
2	2	0.208(25)	1.524(37)	0.727(25)	4.51(45)	6.23(76)	0.00383(47)	0.00352(35)	0.00363(37)
2	3	0.226(34)	1.511(41)	0.704(39)	4.67(49)	6.67(97)	0.00361(53)	0.00344(34)	0.00349(38)
3	2	0.233(27)	1.609(58)	0.733(27)	4.44(45)	6.08(77)	0.00368(45)	0.00367(37)	0.00367(38)
3	3	0.293(41)	1.592(57)	0.664(40)	4.84(51)	7.3(1.1)	0.00310(44)	0.00349(35)	0.00333(36)
3	4	0.293(56)	1.593(60)	0.667(59)	4.85(58)	7.4(1.4)	0.00313(55)	0.00349(37)	0.00338(40)
4	3	0.294(42)	1.594(60)	0.663(40)	4.85(52)	7.4(1.1)	0.00309(44)	0.00348(36)	0.00332(36)
4	4	0.285(92)	1.593(60)	0.677(88)	4.83(62)	7.3(1.7)	0.00328(86)	0.00350(38)	0.00346(42)
5	5	0.277(88)	1.595(62)	0.685(85)	4.81(62)	7.2(1.7)	0.00333(85)	0.00351(38)	0.00348(42)
6	6	0.277(88)	1.592(63)	0.685(86)	4.79(63)	7.2(1.7)	0.00335(88)	0.00350(38)	0.00348(43)
7	7	0.282(89)	1.592(60)	0.680(87)	4.82(64)	7.3(1.7)	0.00332(89)	0.00350(38)	0.00347(43)
8	8	0.283(88)	1.594(61)	0.679(85)	4.83(64)	7.3(1.7)	0.00330(85)	0.00351(37)	0.00347(41)
9	9	0.289(91)	1.594(62)	0.674(88)	4.85(64)	7.4(1.8)	0.00327(89)	0.00350(38)	0.00347(42)
10	10	0.293(95)	1.593(60)	0.670(91)	4.87(67)	7.5(1.9)	0.00325(92)	0.00349(38)	0.00346(42)



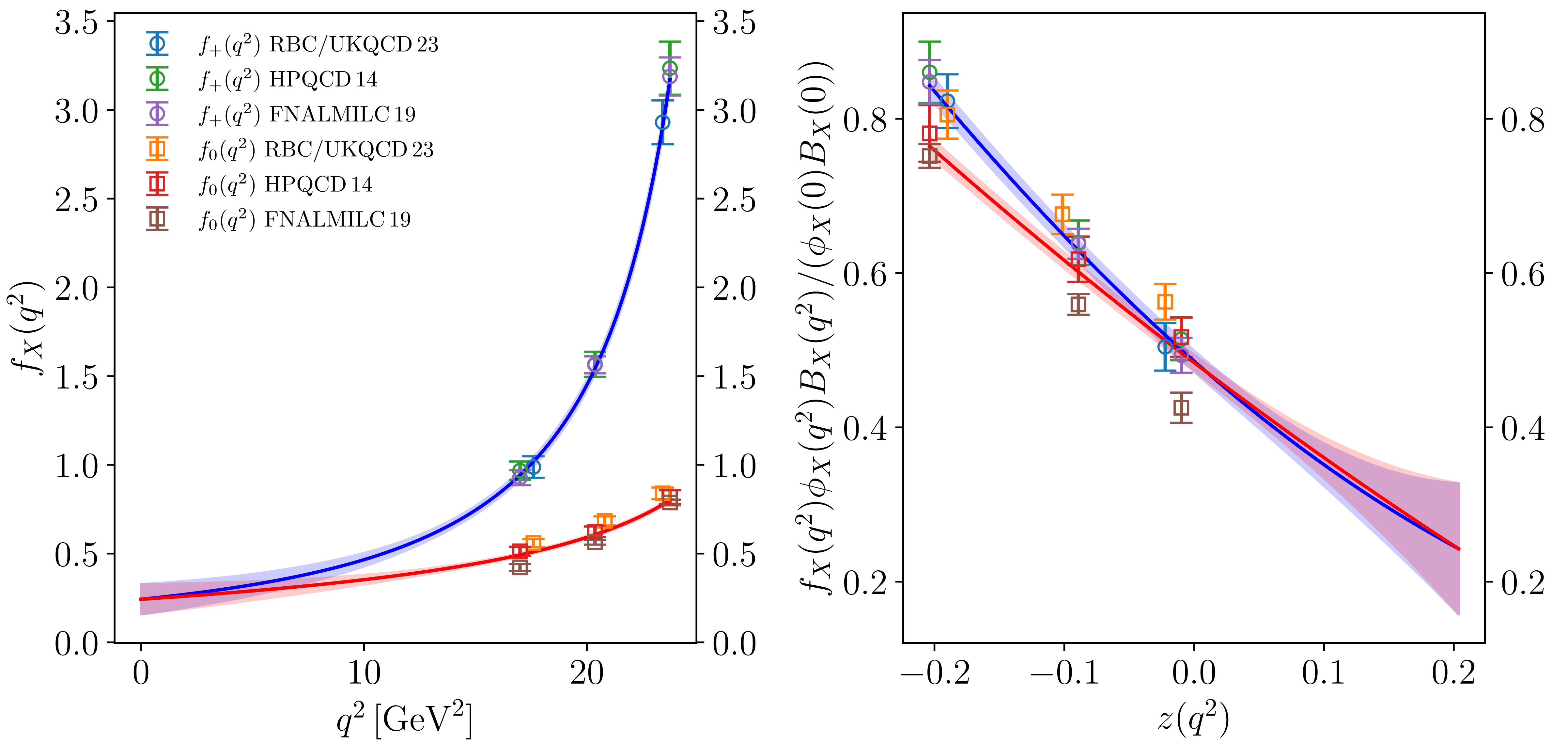
K_+	K_0	$I[\mathcal{A}^\tau_{\text{FB}}]\,[\frac{1}{\text{ps}}]$	$I[\mathcal{A}^\mu_{\text{FB}}]\,[\frac{1}{\text{ps}}]$	$\bar{\mathcal{A}}^\tau_{\text{FB}}$	$\bar{\mathcal{A}}^\mu_{\text{FB}}$	$I[\mathcal{A}^\tau_{\text{pol}}]\,[\frac{1}{\text{ps}}]$	$I[\mathcal{A}^\mu_{\text{pol}}]\,[\frac{1}{\text{ps}}]$	$\bar{\mathcal{A}}^\tau_{\text{pol}}$	$\bar{\mathcal{A}}^\mu_{\text{pol}}$
2	2	1.22(13)	0.0278(51)	0.2708(37)	0.00443(34)	0.74(15)	6.15(75)	0.164(29)	0.98767(96)
2	3	1.26(14)	0.0314(70)	0.2709(38)	0.00465(44)	0.81(18)	6.59(96)	0.173(31)	0.9872(12)
3	2	1.23(13)	0.0319(59)	0.2780(43)	0.00524(51)	0.46(19)	5.99(76)	0.103(40)	0.9852(15)
3	3	1.36(15)	0.045(10)	0.2814(48)	0.00612(66)	0.53(20)	7.2(1.1)	0.110(40)	0.9830(18)
3	4	1.37(17)	0.046(14)	0.2814(50)	0.00611(83)	0.53(22)	7.3(1.3)	0.109(41)	0.9830(22)
4	3	1.37(15)	0.046(10)	0.2815(50)	0.00616(71)	0.53(22)	7.2(1.1)	0.109(42)	0.9829(20)
4	4	1.36(19)	0.046(21)	0.2810(69)	0.0060(15)	0.53(21)	7.2(1.7)	0.109(42)	0.9834(41)
5	5	1.35(19)	0.044(20)	0.2806(67)	0.0058(15)	0.53(22)	7.1(1.6)	0.109(44)	0.9837(39)
6	6	1.35(20)	0.044(20)	0.2803(69)	0.0058(15)	0.53(22)	7.1(1.7)	0.111(44)	0.9838(39)
7	7	1.35(20)	0.045(20)	0.2806(69)	0.0059(15)	0.53(21)	7.2(1.7)	0.111(43)	0.9835(39)
8	8	1.36(20)	0.045(20)	0.2808(69)	0.0059(15)	0.53(22)	7.2(1.7)	0.109(44)	0.9835(39)
9	9	1.36(20)	0.047(21)	0.2812(71)	0.0060(15)	0.53(22)	7.3(1.7)	0.109(44)	0.9832(40)
10	10	1.37(21)	0.048(23)	0.2815(72)	0.0061(15)	0.53(22)	7.4(1.8)	0.109(43)	0.9831(41)

Example 1: $B_s \to K\ell\nu$

Easy to combine independent or correlated data sets:

K_+	K_0	$a_{+,0}$	$a_{+,1}$	$a_{+,2}$	$a_{+,3}$	$a_{+,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.02641(58)	-0.0824(26)	-	-	-	0.00	5.15	14
2	3	0.02668(68)	-0.0811(31)	-	-	-	0.00	5.50	13
3	2	0.02477(68)	-0.0829(26)	0.054(12)	-	-	0.00	3.95	13
3	3	0.02534(73)	-0.0792(31)	0.062(12)	-	-	0.00	3.89	12
3	4	0.02534(73)	-0.0781(34)	0.067(14)	-	-	0.00	4.19	11
4	3	0.02535(73)	-0.0776(38)	0.074(20)	0.023(30)	-	0.00	4.19	11
4	4	0.02592(97)	-0.033(50)	0.69(69)	2.1(2.3)	-	0.00	4.53	10
5	5	0.0266(10)	0.052(65)	2.21(97)	11.1(5.6)	17.2(15.1)	0.00	5.04	8

K_+	K_0	$a_{0,0}$	$a_{0,1}$	$a_{0,2}$	$a_{0,3}$	$a_{0,4}$	p	$\chi^2/N_{\rm dof}$	$N_{\rm dof}$
2	2	0.0854(17)	-0.2565(75)	-	-	-	0.00	5.15	14
2	3	0.0856(18)	-0.2527(91)	0.021(27)	-	-	0.00	5.50	13
3	2	0.0858(18)	-0.2501(77)	-	-	-	0.00	3.95	13
3	3	0.0864(18)	-0.2379(95)	0.061(28)	-	-	0.00	3.89	12
3	4	0.0869(19)	-0.231(13)	0.067(29)	-0.08(10)	-	0.00	4.19	11
4	3	0.0869(19)	-0.229(15)	0.091(48)	-	-	0.00	4.19	11
4	4	0.0887(27)	-0.08(17)	2.2(2.4)	7.0(7.9)	-	0.00	4.53	10
5	5	0.0887(28)	0.07(20)	6.1(3.3)	41.5(19.0)	93.3(44.0)	0.00	5.04	8

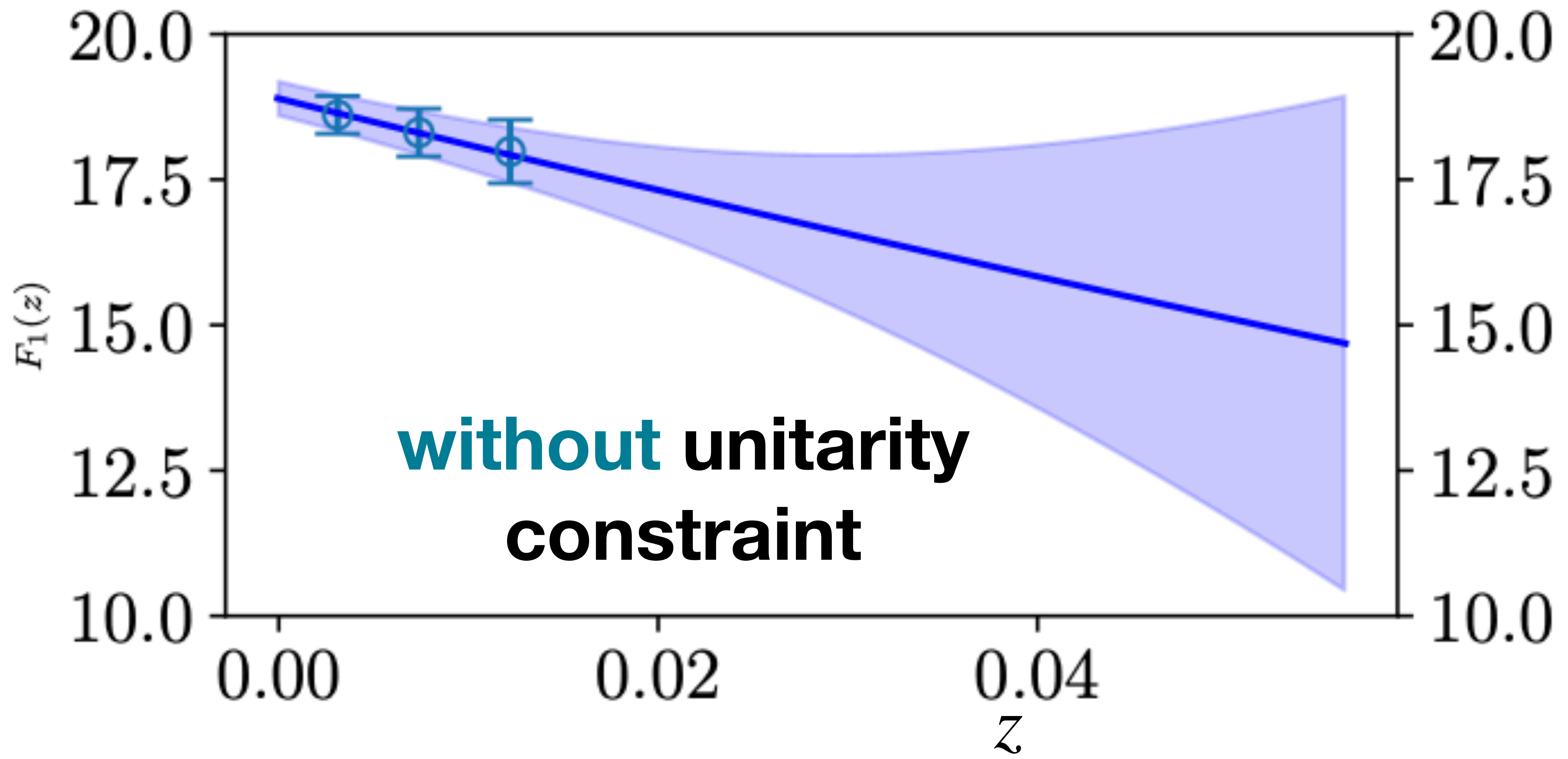


Bayesian and frequentist provide complementary information — consider both simultaneously!

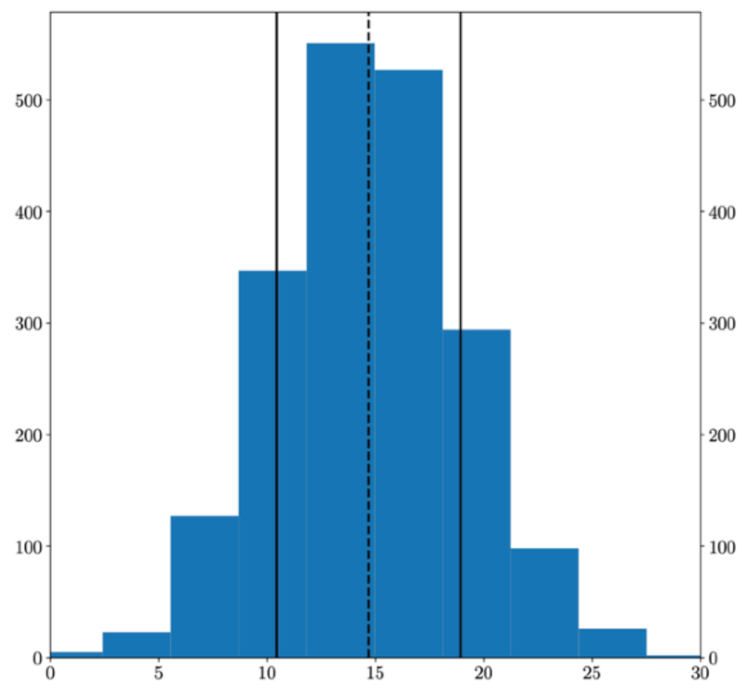
Observation: World lattice data for $B_s \to K\ell\nu$ is in somewhat bad shape…

Example 2: $B \to D^* l\nu$

JLQCD 23 data 2306.05657



without unitarity constraint

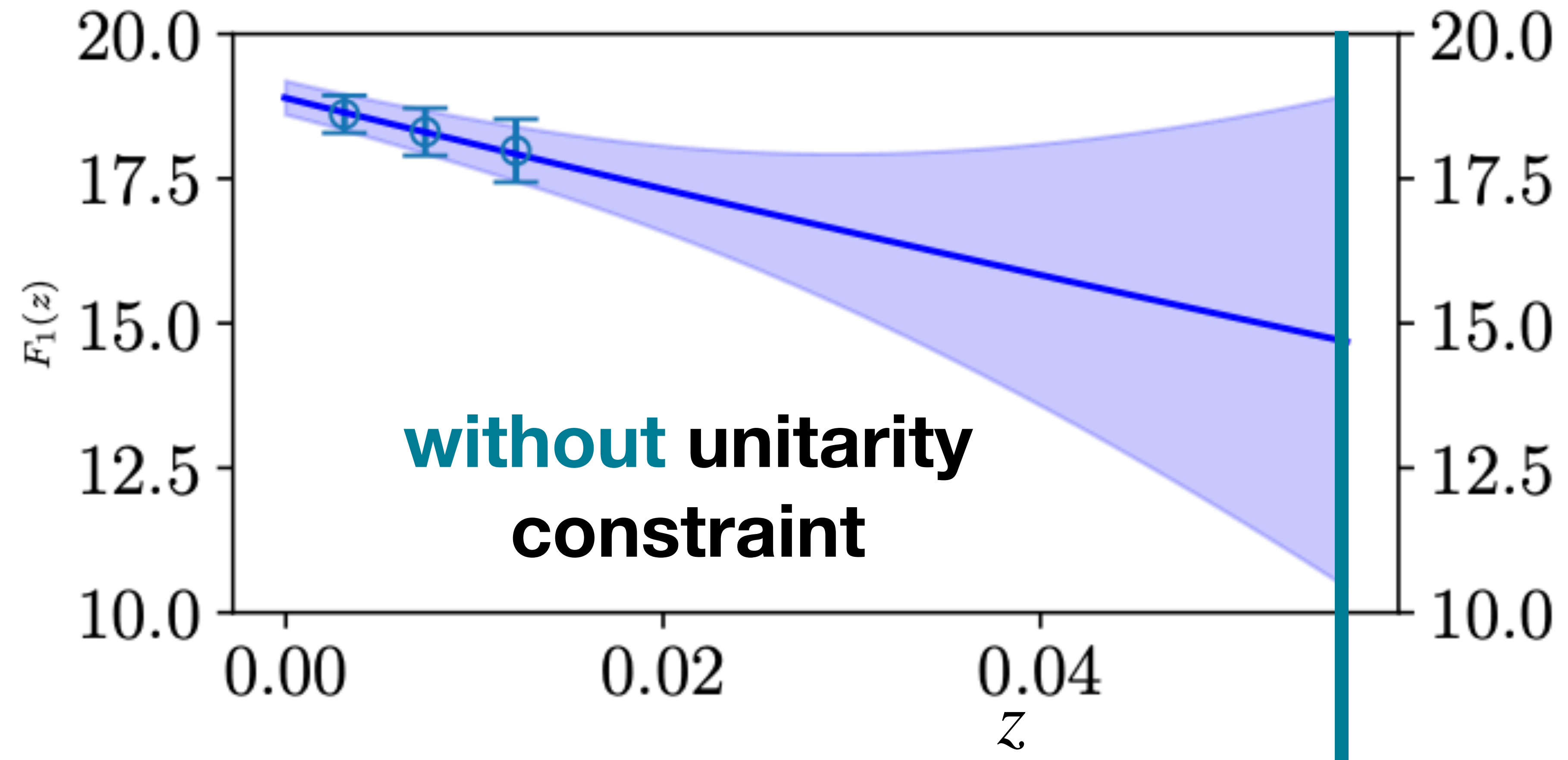


$F_1(q^2 = 0)$

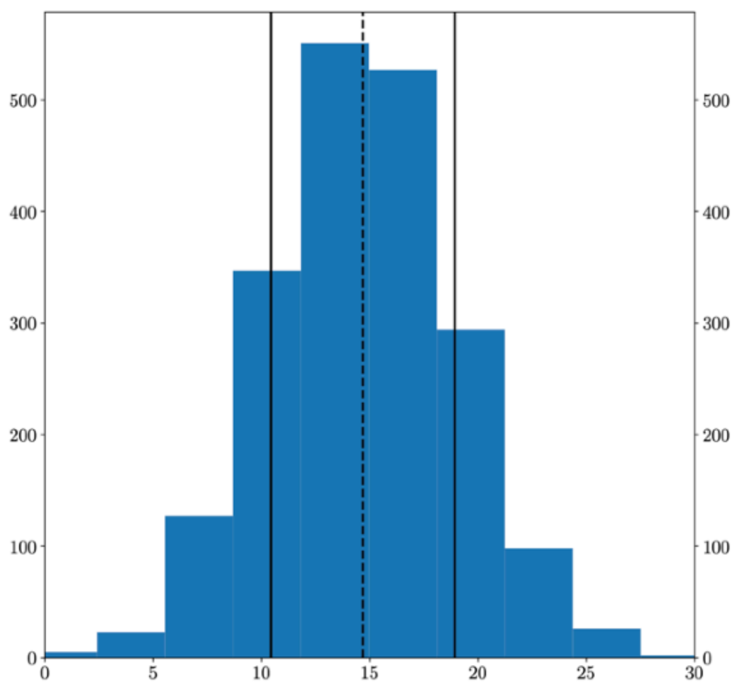
Combined fit over 3 form factors including two unitarity constraints

Example 2: $B \to D^*l\nu$

JLQCD 23 data 2306.05657

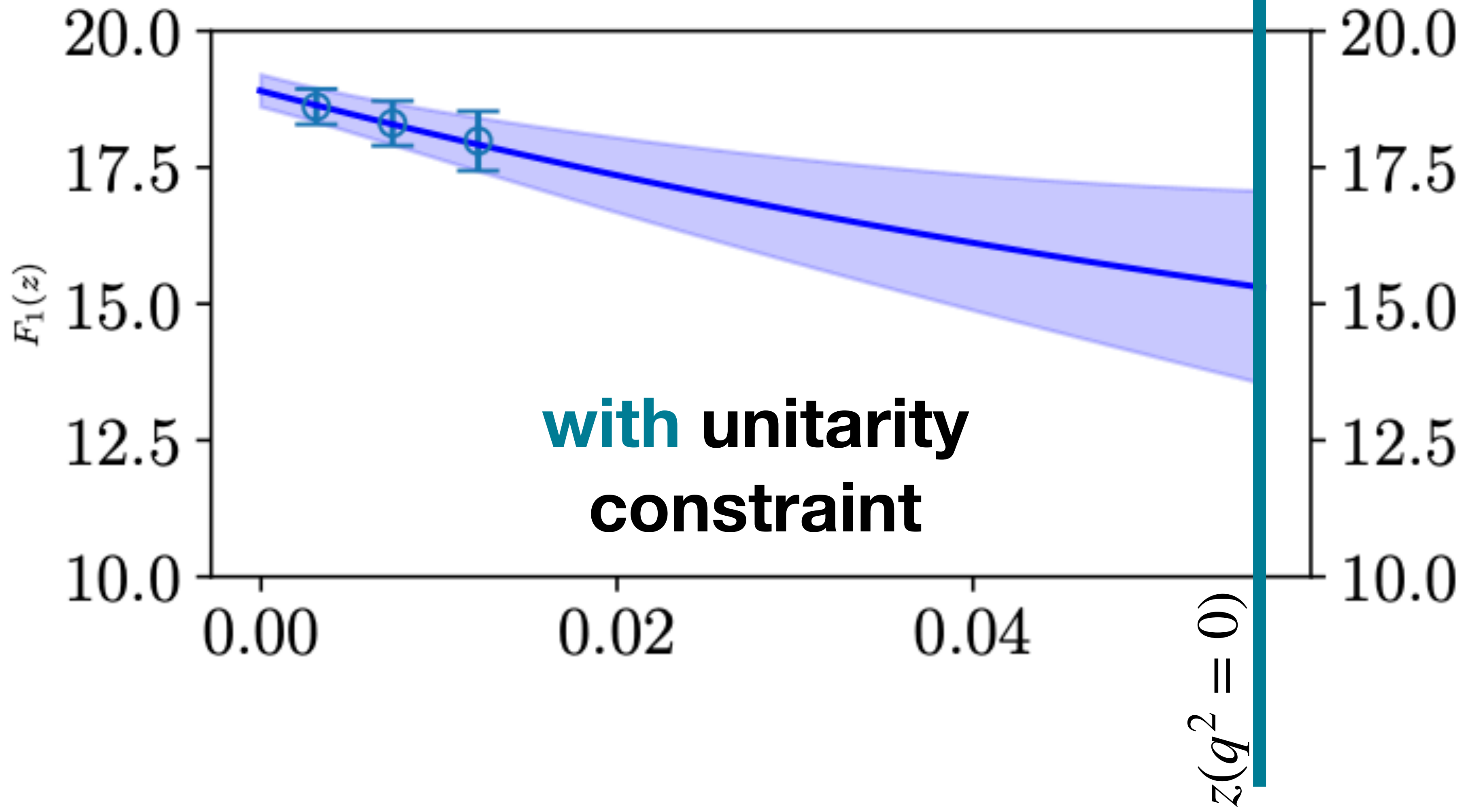


without **unitarity constraint**

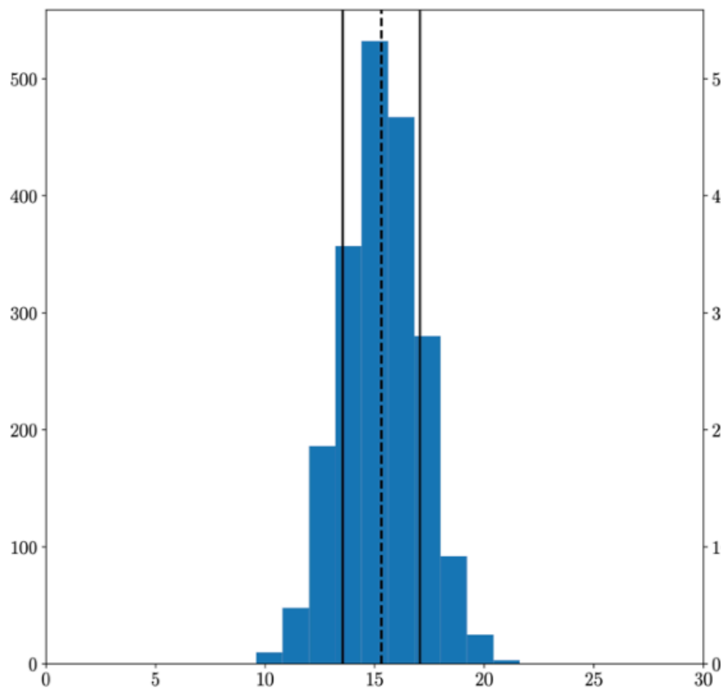


$F_1(q^2 = 0)$

Combined fit over 3 form factors including two unitarity constraints



with **unitarity constraint**

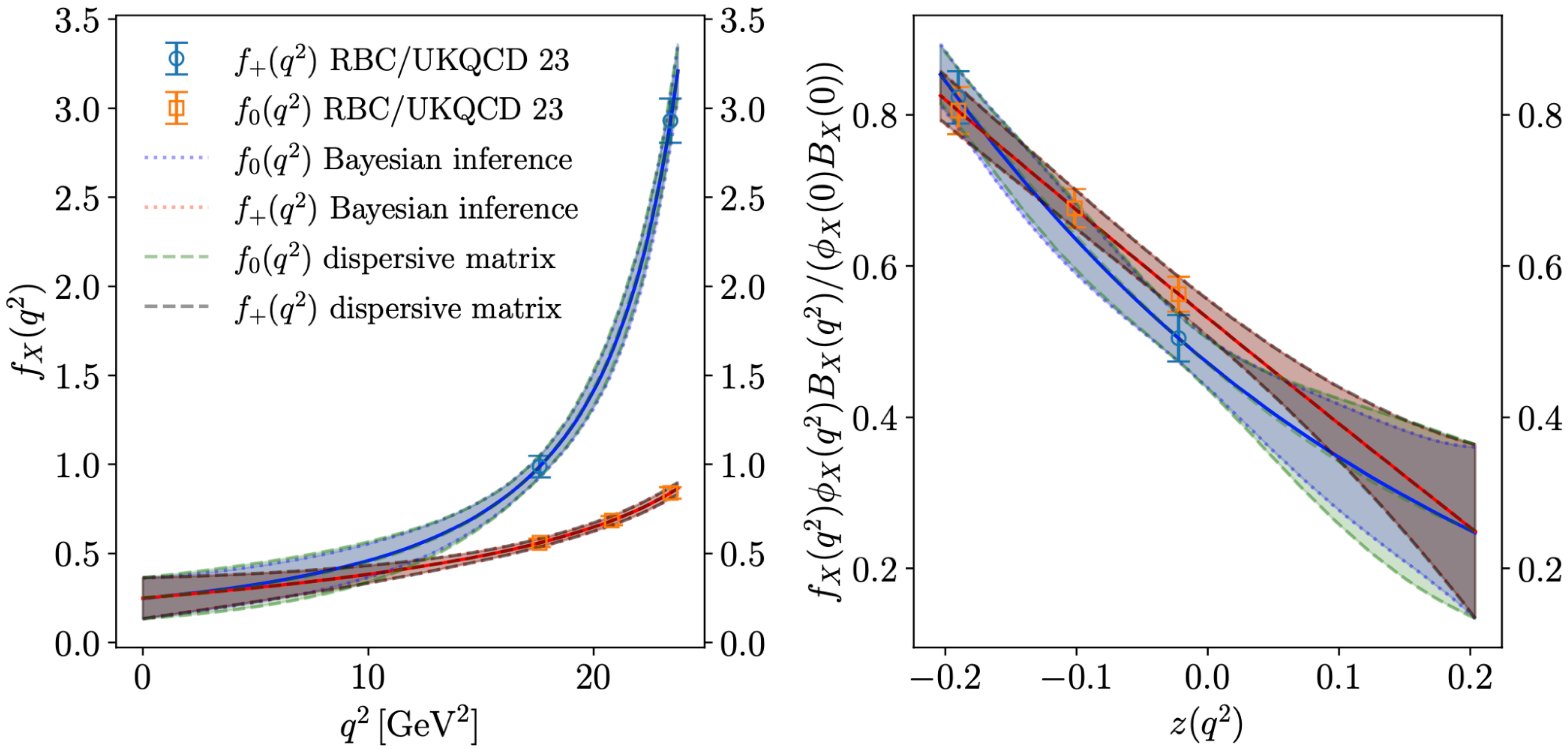


$F_1(q^2 = 0)$

Unitarity constrains higher-order noise — substantial reduction in stat. error

Relation to dispersive-matrix method?

Di Carlo, Martinelli, Naviglio et al. PRD 104 (2021) 054502



Bayesian inference and dispersive-matrix method produce essentially the same results.

Practical advantages of Bayesian inference:

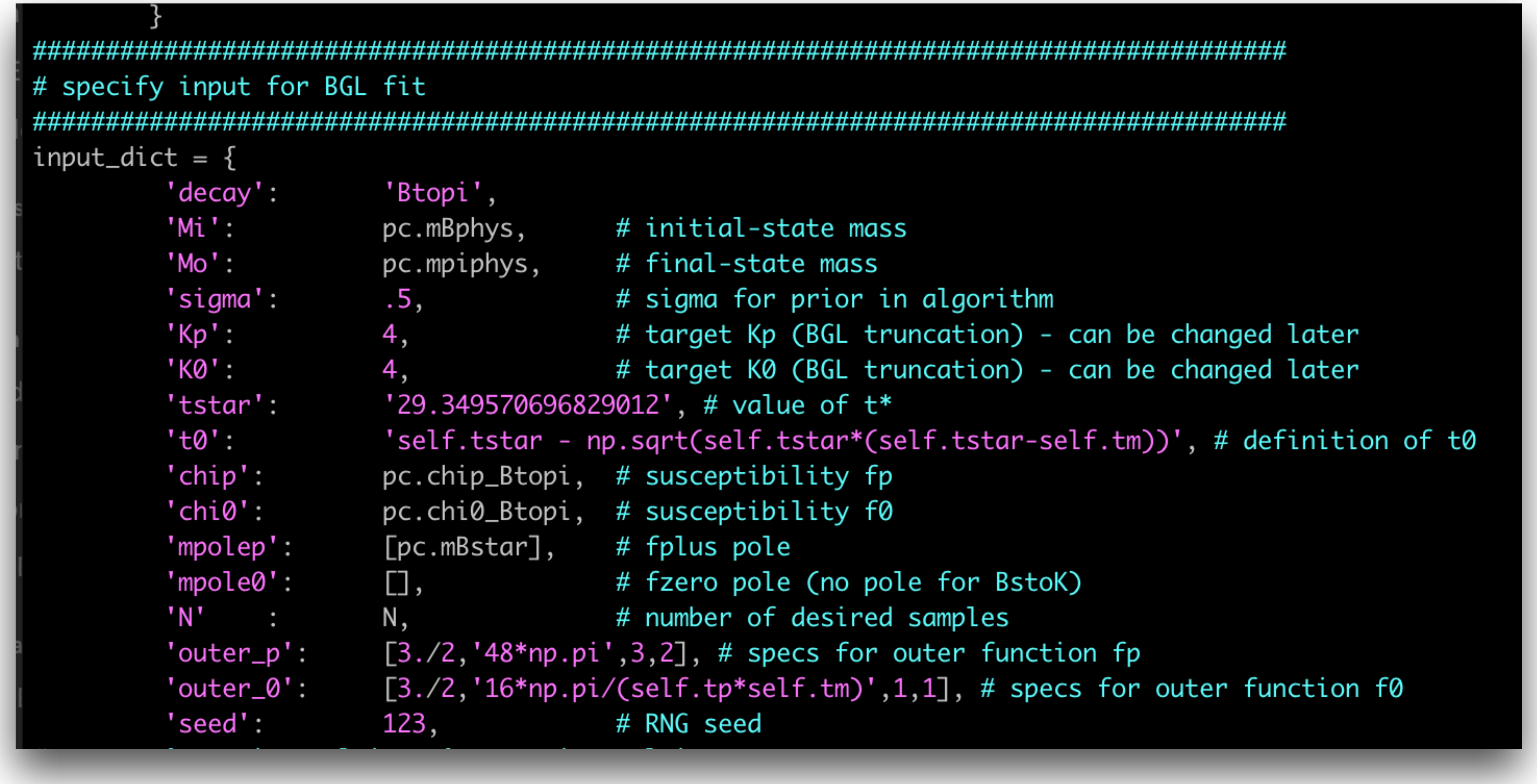
- kinematical constraints exactly and cleanly implemented
- simultaneous fit over various (correlated) data sets possible
- clean statistical underpinning
- produces set of coefficients for further use

Code: `BFF`

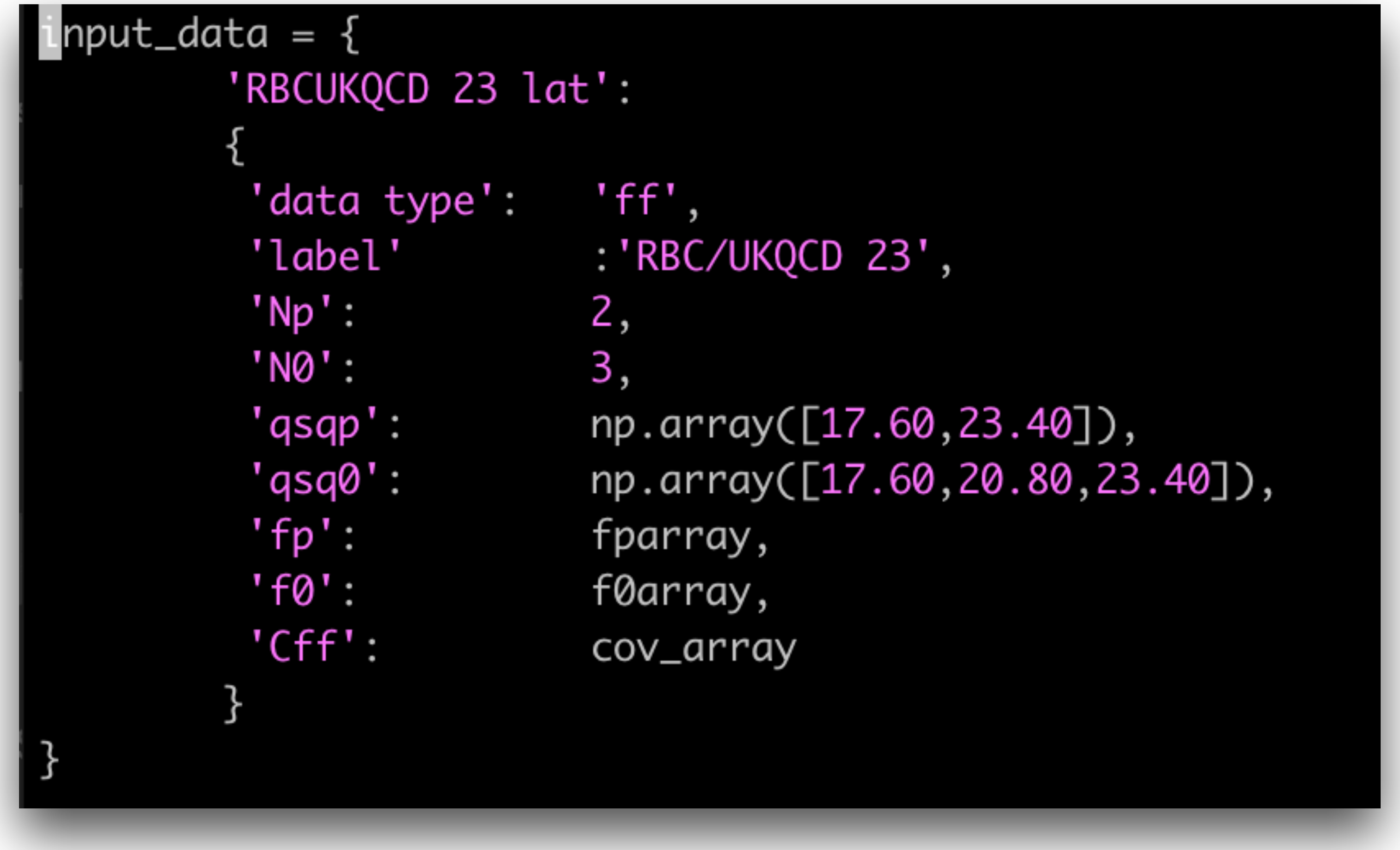
Flynn, AJ, Tsang, arXiv:2303.11285

Python3 available via github/Zenodo

https://github.com/andreasjuettner/BFF **https://zenodo.org/record/7799543#.ZEezTy8Ro80**



```
        }
######################################################################################
# specify input for BGL fit
######################################################################################
input_dict = {
         'decay':        'Btopi',
         'Mi':           pc.mBphys,      # initial-state mass
         'Mo':           pc.mpiphys,     # final-state mass
         'sigma':        .5,             # sigma for prior in algorithm
         'Kp':           4,              # target Kp (BGL truncation) - can be changed later
         'K0':           4,              # target K0 (BGL truncation) - can be changed later
         'tstar':        '29.349570696829012', # value of t*
         't0':           'self.tstar - np.sqrt(self.tstar*(self.tstar-self.tm))', # definition of t0
         'chip':         pc.chip_Btopi,  # susceptibility fp
         'chi0':         pc.chi0_Btopi,  # susceptibility f0
         'mpolep':       [pc.mBstar],    # fplus pole
         'mpole0':       [],             # fzero pole (no pole for BstoK)
         'N'     :       N,              # number of desired samples
         'outer_p':      [3./2,'48*np.pi',3,2], # specs for outer function fp
         'outer_0':      [3./2,'16*np.pi/(self.tp*self.tm)',1,1], # specs for outer function f0
         'seed':         123,            # RNG seed
```



```
input_data = {
        'RBCUKQCD 23 lat':
        {
         'data type':    'ff',
         'label'         :'RBC/UKQCD 23',
         'Np':           2,
         'N0':           3,
         'qsqp':         np.array([17.60,23.40]),
         'qsq0':         np.array([17.60,20.80,23.40]),
         'fp':           fparray,
         'f0':           f0array,
         'Cff':          cov_array
        }
}
```

Summary

Novel framework for truncation- and model-independent form-factor fitting

Implemented and demonstrated for $P \to P$ transitions [2303.11285]

First applied for $B_s \to K\ell\nu$ [*PRD* 107 (2023) 11, 114512 2303.11280]
Talk by Rayn Hill Monday 31/07 13:50

Now also implemented for $P \to V$ (e.g. $B \to D^*\ell\nu$)

Apply method to a wider range of meson decays and beyond,
e.g. nucleon decay form factors

TWO Postdoctoral opportunities at CERN:

- **3-year CERN Fellowship — application deadline 03. September!!! [link]**
- **2-year Marie Skłodowska-Curie European Postdoctoral Fellowship — internal deadline passed but talk to me if interested!! [link]**