Towards Unpolarized GPDs from Pseudo-Distributions

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Parton Structure

One for each flavor and spin combination

Wigner Distribution/
Generalized Transverse Momentum Distribution (GTMD)
$F(x, b_T, k_T)$

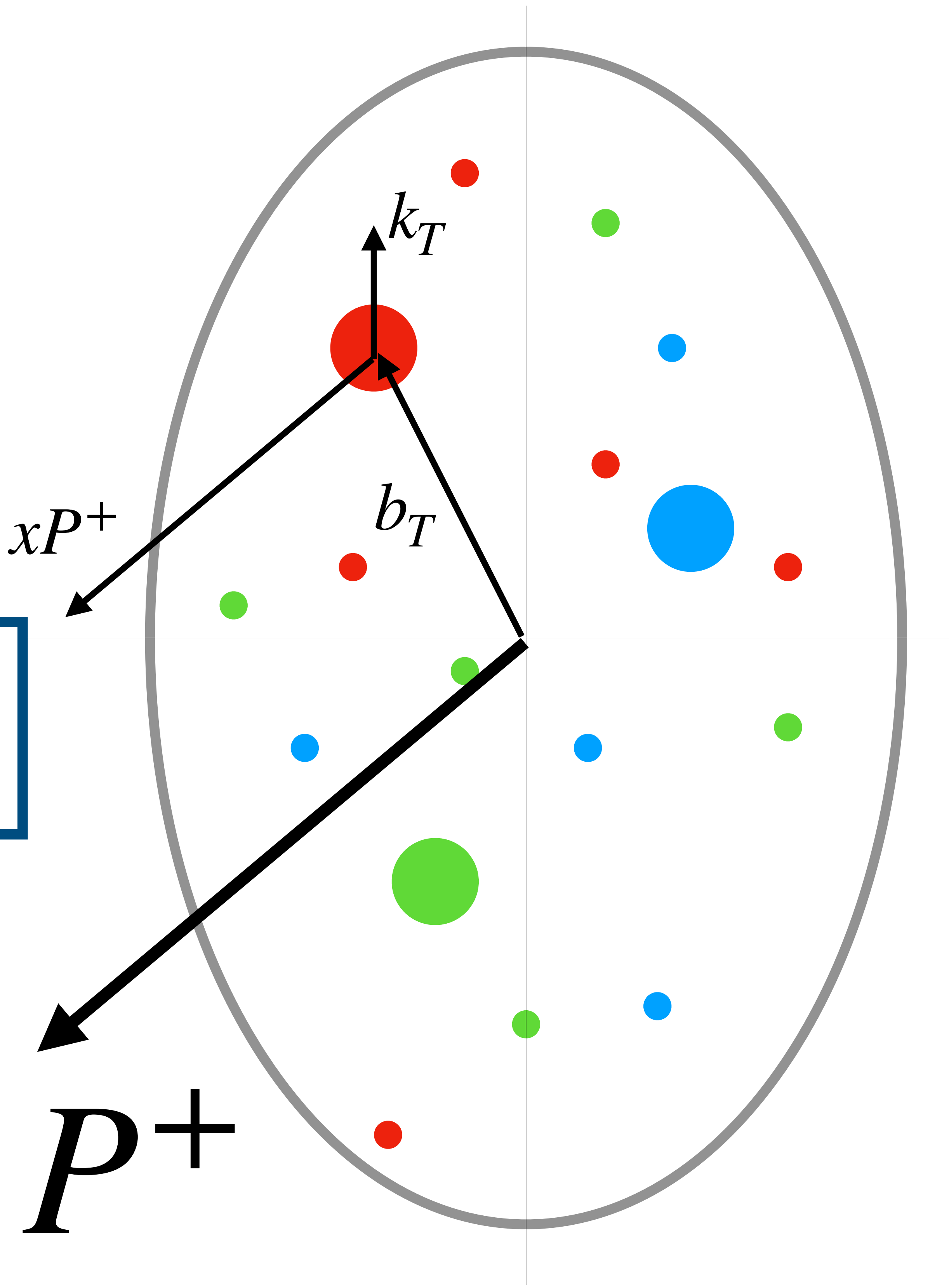
$\int d^2 b_t$ ↙ ↘ $\int d^2 k_t$

Transverse Momentum Distribution (TMD)
$f(x, k_T)$

Generalized Parton Distribution (GPD)
$f(x, b_T)$

$\int d^2 k_t$ ↘ ↙ $\int d^2 b_t$

Parton Distribution Function (PDF)
$f(x)$



Generalized Parton Distributions

- Generalized Ioffe time distributions

$$I^{\mu}(p', p, z = z^{-}, \mu^2) = \langle p' | \bar{q}\Big(-\frac{z^{-}}{2}\Big)\gamma^{\mu} W\Big(-\frac{z^{-}}{2}, \frac{z^{-}}{2}\Big) q\Big(\frac{z^{-}}{2}\Big) | p \rangle_{\mu^2}$$

Ioffe Time

$$\nu = \frac{p + p'}{2} \cdot z = P \cdot z$$

Momentum Transfer

squared

$$t = (p' - p)^2 = q^2$$

Skewness

$$\xi = \frac{q \cdot z}{P \cdot z}$$

- Generalized Parton Distributions

$$\sigma^{\mu a} = \sigma^{\mu\nu} a_{\nu}$$

$$\bullet \int \frac{d\nu}{2\pi} e^{i\nu x} I^{\mu}(\nu, t, \xi, \mu^2) = H(x, t, \xi)\, \bar{u}' \gamma^{\mu} u + E(x, t, \xi)\, \bar{u}' \frac{i\sigma^{\mu q}}{2m} u + \ldots$$

X. Ji PRL 78 (1997) 610-613

Lorentz decomposition important!

Double Distributions

- Generalized Ioffe time distributions

$$I^\mu(p', p, z = z^-, \mu^2) = \langle p' | \bar{q}\Big(-\frac{z^-}{2}\Big)\gamma^\mu W\Big(-\frac{z^-}{2}, \frac{z^-}{2}\Big) q\Big(\frac{z^-}{2}\Big) | p \rangle_{\mu^2}$$

Ioffe Time

$$\nu = \frac{p + p'}{2} \cdot z = P \cdot z$$

Momentum Transfer
squared

$$t = (p' - p)^2 = q^2$$

Skewness

$$\xi = \frac{q \cdot z}{P \cdot z}$$

- Double Distributions

$$\sigma^{ba} = b_\mu \sigma^{\mu\nu} a_\nu$$

$$z_\mu I^\mu(\nu, t, \xi, \mu^2) = \int d\beta d\alpha e^{i\nu(\beta+\xi\alpha)} \Big[f_q(\alpha, \beta, t)\, \bar{u}' z_\mu \gamma^\mu u + k_q(\alpha, \beta, t)\, \bar{u}' \frac{i\sigma^{zq}}{2m} u$$

$$+ \delta(\beta) D(\alpha, t)\, \xi \nu\, \bar{u}' u \Big]$$

A.V. Radyushkin Phys Lett B380 (1996) 417-425

M.V. Polyakov and C. Weiss PRD 60, 114017 (1999)

Two Faced Distributions

Radon Transform

GPDs: ⟷ **DDs:**

x, ξ β, α

$$f(x,\xi) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha\, \delta(x-\beta-\xi\alpha)\tilde{f}(\alpha,\beta)$$

- Interpretation: average/change in parton momentum fraction
- Mellin moments give Form Factors and Angular Momentum decomposition
- Complex interrelation of variables
  - ERBL/DGLAP regions and polynomiality

Statue of Janus Bifrons (Wikipedia)

- Interpretation: Hybridize PDFs/DAs
- β acts like PDF x
- α acts like DA x
- GPD evolution and polynomiality arise naturally from parameterized DD

Lorentz off the Lightcone (PDFs)

- Unpolarized Quark Ioffe time pseudo-distributions (pseudo-ITD)

$$M^\mu(p,z) = \langle p \,|\, \bar{q}\Big(-\frac{z}{2}\Big)\gamma^\mu W\Big(-\frac{z}{2},\frac{z}{2}\Big)q\Big(\frac{z}{2}\Big)\,|\, p\rangle$$

$$= 2p^\mu \mathscr{M}(\nu,z^2) + 2z^\mu \mathscr{N}(\nu,z^2)$$

A. Radyushkin *PRD* 96 (2017) 3, 034025

Exist for lightcone at $O(\alpha_s^0)$

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- Polarized Quark pseudo-ITD

C. Egerer et al (HadStruc) PRD 105 (2022) 3, 034507

R. Edwards et al (HadStruc) JHEP 03 (2023) 086

$$M^\mu(p,z) = \langle p\,|\,\bar{q}\big(-\frac{z}{2}\big)\gamma^\mu\gamma^5 W\big(-\frac{z}{2},\frac{z}{2}\big)q\big(\frac{z}{2}\big)\,|\,p\rangle \qquad S^\mu = \bar{u}\gamma^\mu\gamma^5 u$$

$$= 2mS^\mu\,\mathscr{M}(\nu,z^2) - 2imp^\mu z\cdot S\,\mathscr{N}(\nu,z^2) + 2m^3 z^\mu z\cdot S\,\mathscr{R}(\nu,z^2)$$

Exist for lightcone at $O(\alpha_s^0)$

$$I^+(\nu,\mu^2) = 2mP^+\left[\mathscr{M}(\nu,\mu^2) + i\nu\mathscr{N}(\nu,\mu^2)\right]$$

Lorentz off the Lightcone (PDFs)

- Unpolarized Quark Ioffe time pseudo-distributions (pseudo-ITD)

$$M^\mu(p,z) = \langle p\,|\,\bar{q}\Big(-\frac{z}{2}\Big)\gamma^\mu W\Big(-\frac{z}{2},\frac{z}{2}\Big)q\Big(\frac{z}{2}\Big)\,|\,p\rangle$$

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$$= 2mS^\mu\,\mathscr{M}(\nu, z^2) - 2imp^\mu z\cdot S\,\mathscr{N}(\nu, z^2) + 2m^3 z^\mu z\cdot S\,\mathscr{R}(\nu, z^2)$$

- Helicity Gluon pseudo-ITD

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
C. Egerer et al (HadStruc) PRD 106 (2022) 9, 094511

$$M^{\mu\nu;\alpha\beta}(p,z) = \langle p\,|\,F^{\mu\nu}\Big(-\frac{z}{2}\Big)W\Big(-\frac{z}{2},\frac{z}{2}\Big)\tilde{F}^{\alpha\beta}\Big(\frac{z}{2}\Big)\,|\,p\rangle$$

$$= \sum_{i=1}^{12} K_i^{\mu\nu;\alpha\beta}(p,z)\mathscr{M}_i(\nu, z^2)$$

Only 2 exist for lightcone at $O(\alpha_s^0)$

Lorentz off the Lightcone (GPDs)

- Generalized pseudo-ITD

Rest frame has same renormalization constant

$$M^{\mu}(p',p,z) = \langle p' | \bar{q}\Big(-\frac{z}{2}\Big)\gamma^{\mu}W\Big(-\frac{z}{2},\frac{z}{2}\Big)q\Big(\frac{z}{2}\Big) | p\rangle$$

$$\mathfrak{M}^{\mu}(p',p,z) = \sum_{i=1}^{8} K^{\mu}(p',p,z)A_i(\nu,\xi,t)$$

$$\mathfrak{M}^{\mu} = \frac{M^{\mu}(p',p,z)}{M^{4}(p'=0,p=0,z)}$$

S. Bhattacharya et al PRD 106 (2022) 11, 114512

Gordon Identity means decomposition is not unique

- Leading twist amplitude for Double Distributions

$$A_1 \to f_q : \qquad K_1 = \bar{u}'\gamma^{\mu}u$$

Isolation of Amplitudes: SVD

$$\mathfrak{M}^\mu(p', p, z) = \sum_{i=1}^{8} K^\mu(p', p, z) A_i(\nu, \xi, t)$$

Calculate for fixed p', p, z and vary initial/final spin and μ to build matrix equation

$$\mathfrak{M} = KA$$

Pseudo-inverse solution $\tilde{A}$ gives minimum of $\chi^2 = (K\tilde{A} - \mathfrak{M})^2$

$$\tilde{A} = K^+\mathfrak{M}$$

Already used for Lattice Form Factor calculations

Inverse Problems: In all Parton Physics

- **Structure Functions**

$$F_2(x, Q^2) = \sum_i \int_x^1 d\xi\, C(\xi, \frac{\mu^2}{Q^2})\, q(\frac{x}{\xi}, \mu^2)$$

- **LaMET (on the lattice)**

$$M(p_z, z) = \int_{-\infty}^{\infty} dy\, e^{iyp_z z}\, \tilde{q}(y, p_z^2)$$

- **pseudo-Distributions / Good Lattice Cross Sections**

$$\mathfrak{M}(\nu, z^2) = \int_{-1}^{1} dx\, C(x\nu, \mu^2 z^2)\, q(x, \mu^2)$$

- **Local Matrix elements / HOPE / OPE-without-OPE**

$$a_n(\mu^2) = \int_{-1}^{1} dx\, x^{n-1}\, q(x, \mu^2)$$

- **Hadronic Tensor**

$$\tilde{W}_{\mu\nu}(\tau) = \int d\nu\, e^{-\nu\tau}\, W_{\mu\nu}(\nu)$$

Inverse Problems: Most probable solution

JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos JHEP 04 (2019) 057

- Exists because information is lost. Must add new “prior information”
- Find the most probable distribution $\langle q(x)\rangle = \int D[q]q(x)P[q\,|\,\mathcal{M}, I]$
- Posterior Prob by Bayes’s Theorem $P[q\,|\,\mathcal{M}, I] \propto \exp[-\frac{\chi^2}{2}]P[q\,|\,I]$
- Choice of prior introduce biases
  - Bayesian Model averaging to amortize the biases
  - Allow for rigorous comparison of cuts in z^2

W. Jay and E. Neil PRD 103, 114502 (2021)

Inverse Problems: Most probable solution

JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos JHEP 04 (2019) 057

- Exists because information is lost. Must add new "prior information"
- Find the most probable distribution $\langle q(x) \rangle = \int D[q] q(x) P[q \,|\, \mathcal{M}, I]$
- Parametrization

$$Q(x; \alpha, \beta) = x^{\alpha}(1-x)^{\beta}$$

$$P[q \,|\, I] = \int d[c]\, \delta(q(x) - Q(x; c)) P[c \,|\, I] \qquad Q(x; w, b) = NN(x; w, b)$$

- Bayesian Reconstruction / Maximum Entropy Method

$$P[q \,|\, I] = \int dq_i\, \delta(q(x) - Q(x; q_i)) P[q_i \,|\, I] \qquad Q(x; q_i) = q_i \quad \text{if } x = x_i$$

else interpolate

- Discrete Fourier transform

$$P[q \,|\, I] = \int d\tilde{M}_i\, \delta(q(x) - Q(x; \tilde{M}_i)) \qquad Q(x; \tilde{M}_i) = \sum_i e^{i\nu_i x} \tilde{M}_i$$

- BONUS: Hybrid solutions $q(x) = q_{z<z_l}(x) + q_{z>z_l}(x)$

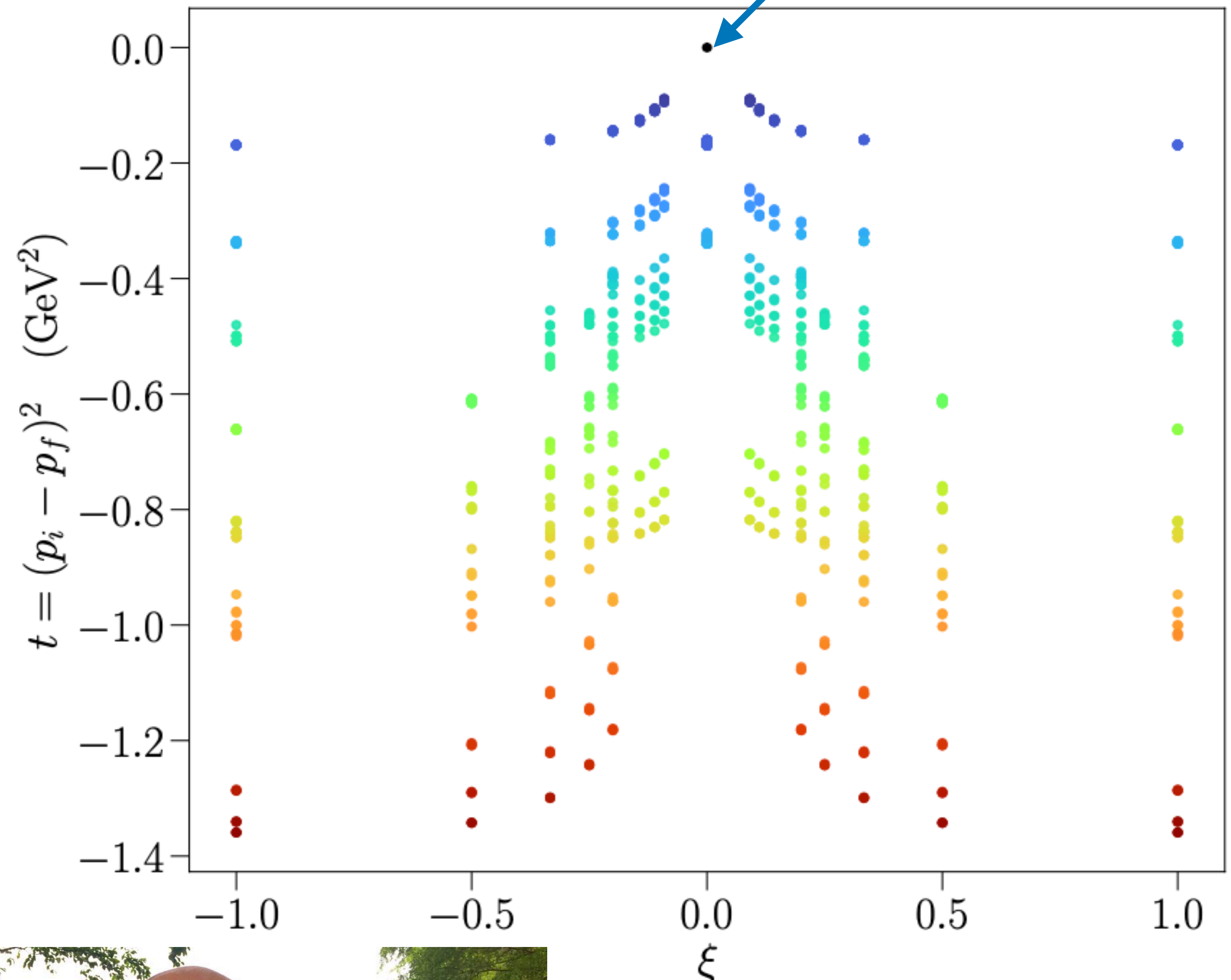
X. Gao et al PRL 128 (2022) 14, 142003

C. Egerer et al (HadStruc) JHEP 11 (2021) 148

Lattice Data

- Used Distillation to improve signal and allow easy substitution of different p
- Summation method for controlling excited states
- Calculated 270 (p', p) combinations
- Separations $z \in [-8,8]\hat{3}$
- Different γ^μ and spins to perform SVD
  - $\mu = 1, 2, 4$ and 2 spin combinations
- O(25k) matrix elements

PDF Calculation with 72 matrix elements



$a = 0.094(1)\,\mathrm{fm}$
$m_\pi = 358(3)\mathrm{MeV}$
$L^3 \times T = 32^3 \times 64$

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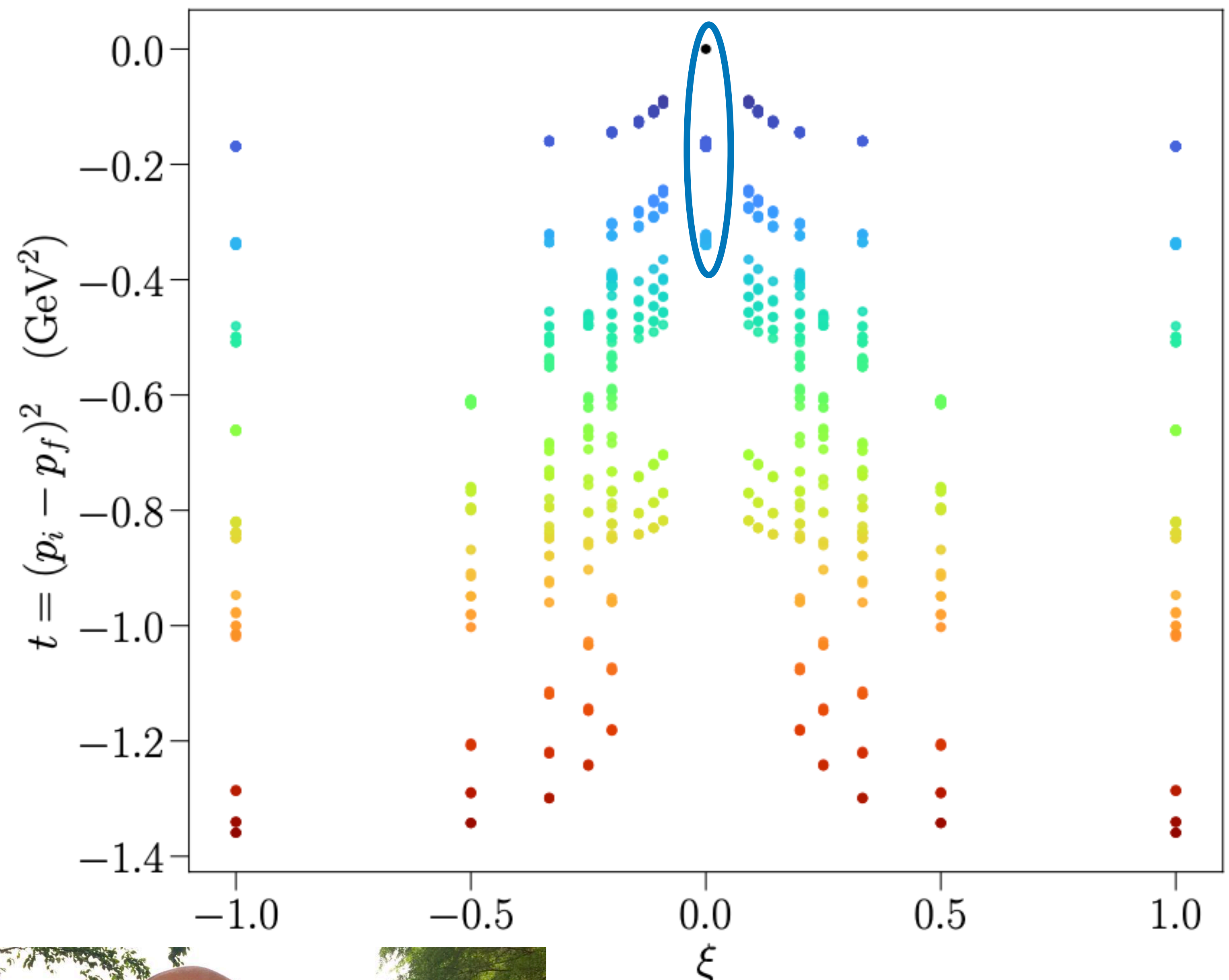
  - $\mu = 1, 2, 4$ and 2 spin combinations
- O(30k) matrix elements



$$a = 0.094(1)\ \text{fm}$$
$$m_\pi = 358(3)\text{MeV}$$
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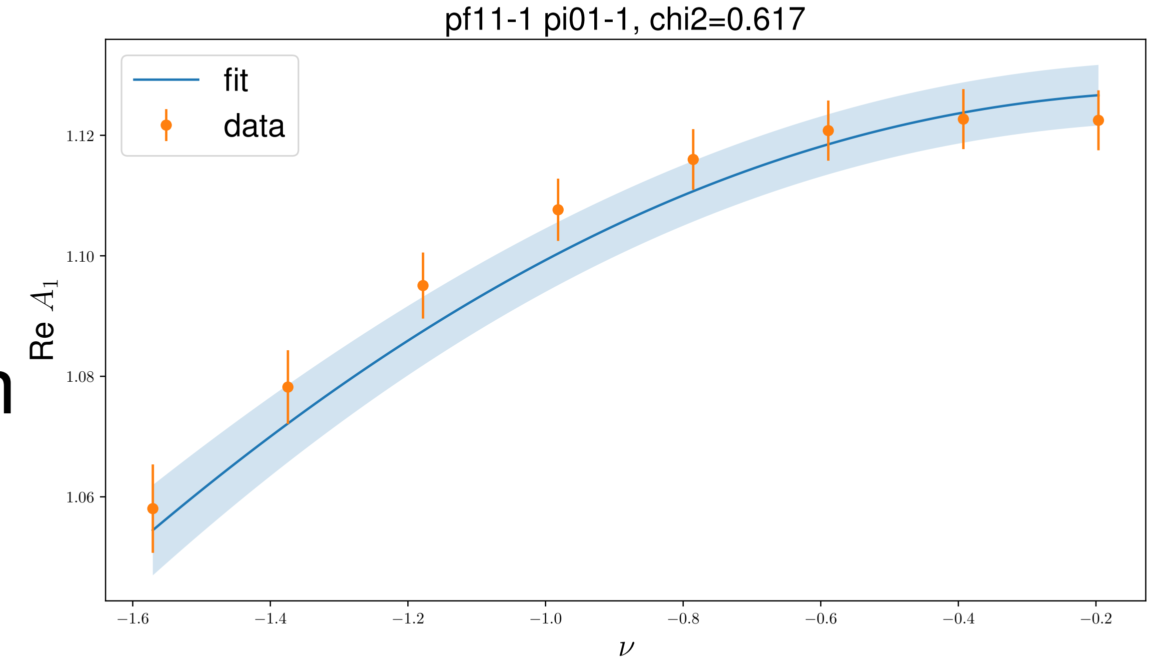
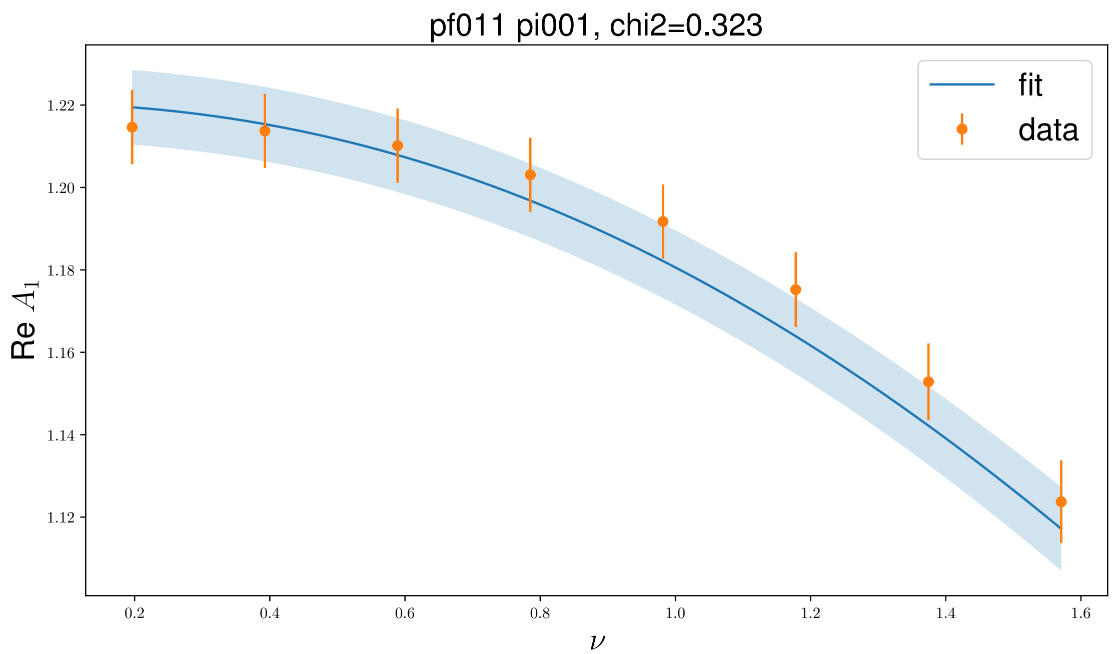
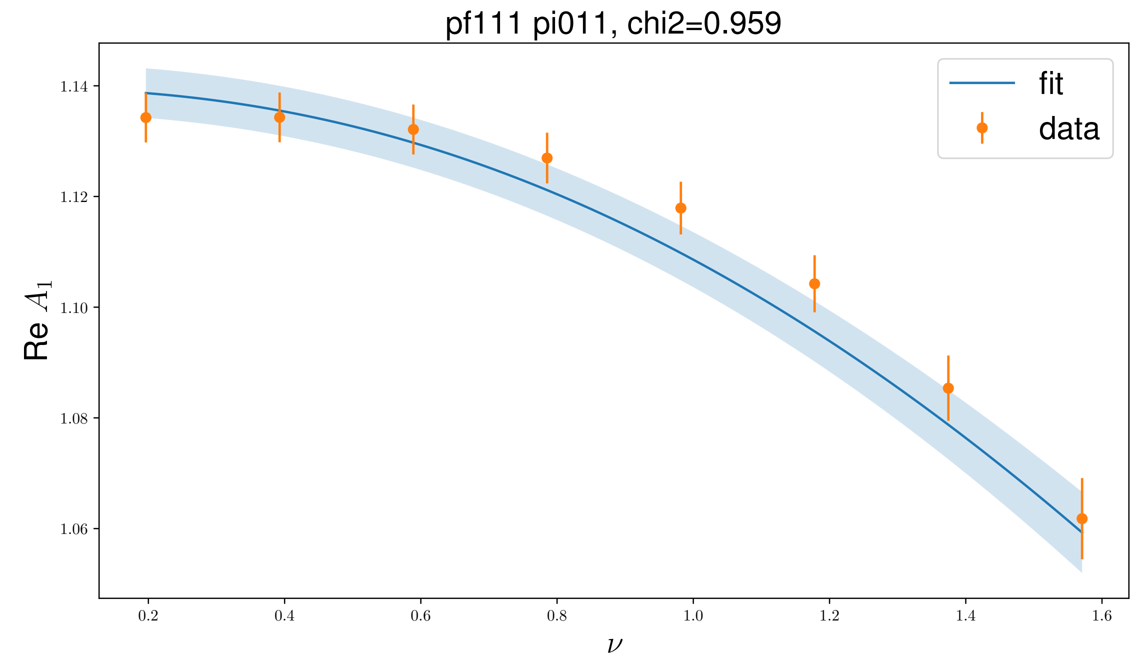
f_q at $\xi = 0$

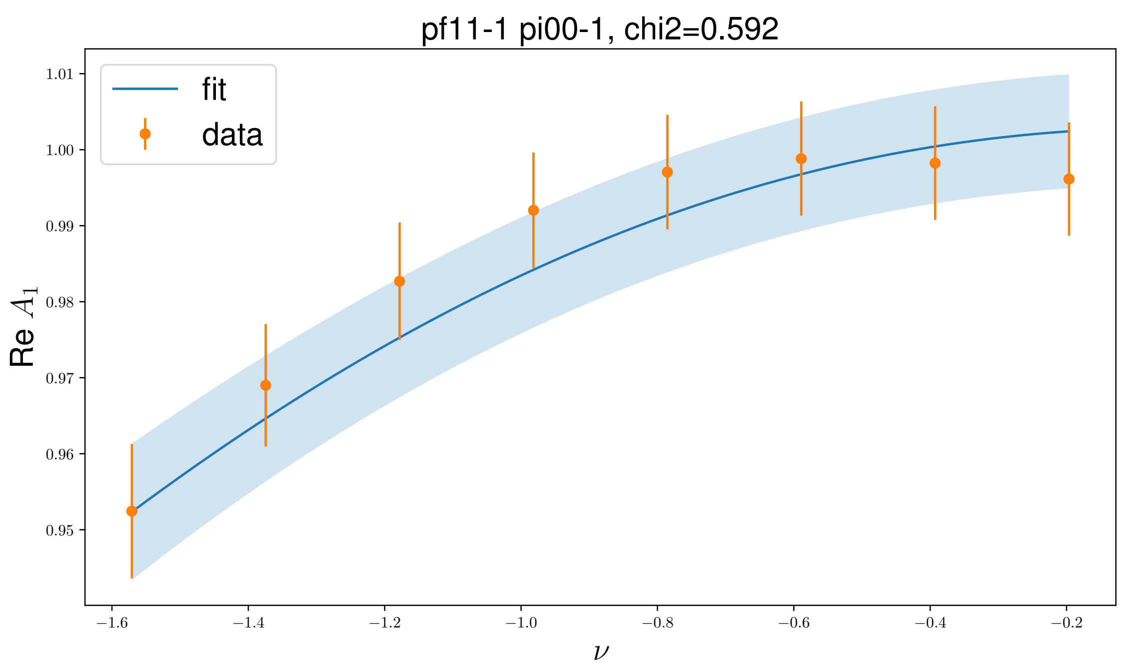
- Parametrize the DD with simplest form

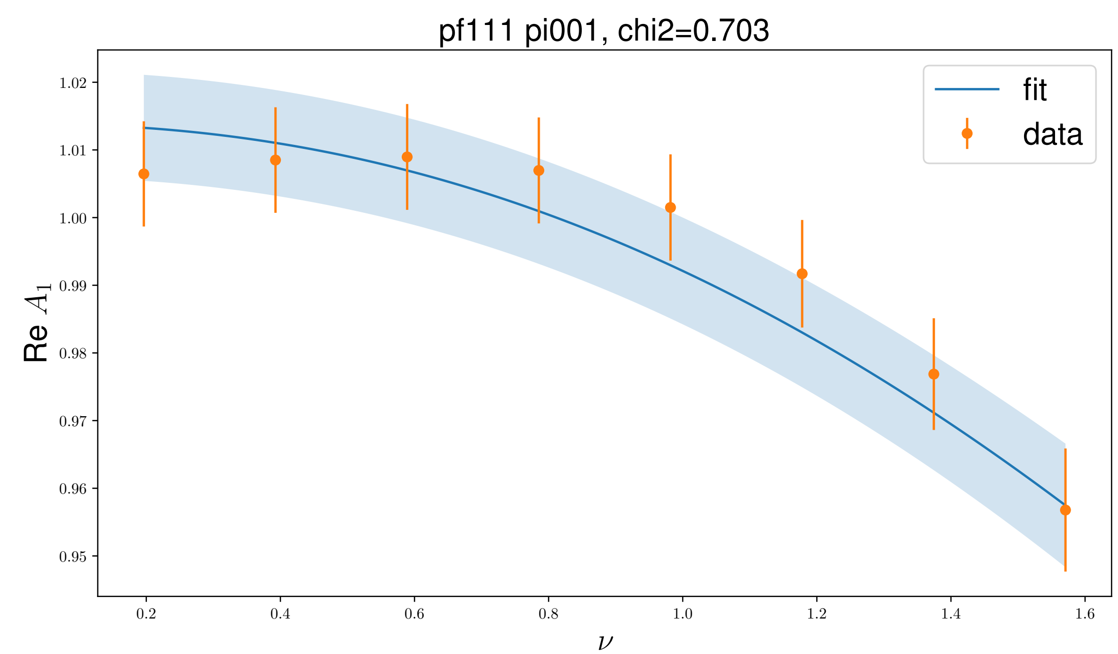
$$A_1(\nu, t, \xi, z^2) = N \int d\beta e^{i\nu\beta} \beta^a (1-\beta)^b$$











f_q at $\xi = 0$

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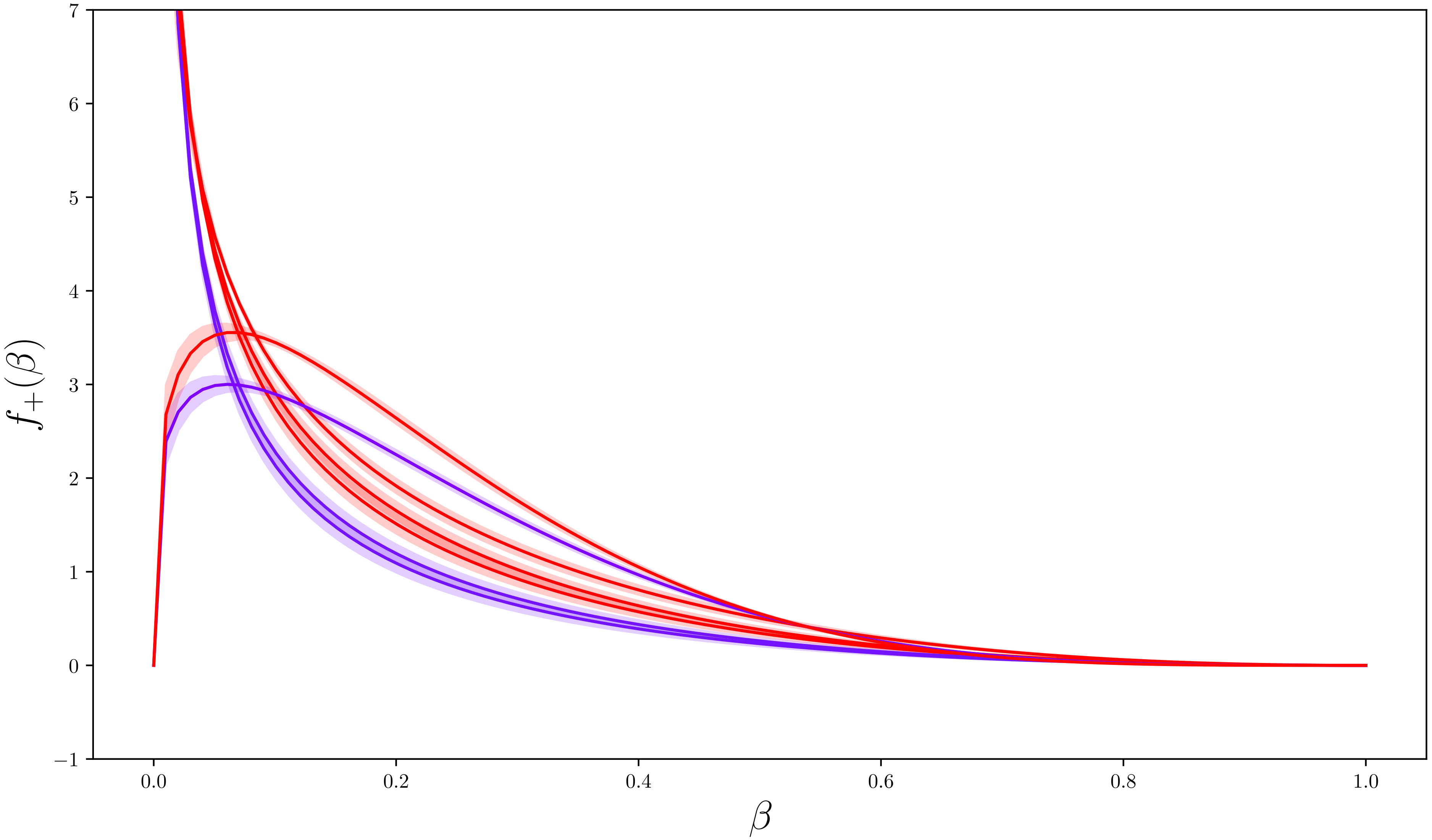

Future work and Conclusion

- Double Distributions are a natural choice to identify “lightcone” combination of amplitudes for lattice GPDs
- Also a good parameterization of lattice GPDs
- Using Distillation gives ready access to a wide range of momenta
- A scan of all 3 variables, with hundreds of matrix elements, are necessary to constrain the GPD
- Bayesian Model averaging will be necessary to tamper the biases of the inverse problem

Thank you and the organizers!