Nonperturbative renormalization of HQET operators in Position Space

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To connect bare lattice quantities to the continuum, we need to renormalize our operators. There are many different approaches:

- **Bare Lattice Matrix Elements**
- **Nonperturbative renormalisation**
- **Lattice Perturbation Theory**
- **Intermediate Scheme**
  - RI-(s)MOM Schrödinger Functional
  - Position-space scheme
- **Matching in the continuum**
- **MS-renormalized matrix elements**

**Goal:** Matching of HQET operators in position-space
Position Space Schemes (X-space)

- Renormalization conditions in momentum space
  (RI-(s)MOM)
  \[ Z_Γ(p | δ_Γ | p) \bigg|_{p_1^2=p_2^2=q^2=-μ^2} = \langle p | δ | p \rangle \bigg|_\text{tree} \]

- Renormalization conditions in position space
  \[ Z_Γ^2(δ_Γ(x)δ_Γ(0)) \bigg|_{x^2=-μ^2} = \langle δ_Γ(x)δ_Γ(0) \rangle \bigg|_\text{tree} \]

The good

- Easy to implement!
- Gauge invariant
  - no gauge fixing required
  - no Gribov copies
  - no gauge non-invariant operators

The bad

- Similar window problem to other schemes:
  \[ a \ll x_0 \ll Λ_{\text{QCD}}^{-1} \]
- Mixing often requires three (or more) point-function calculations to have enough constraints
- Perturbative calculations contain more loops.
  e.g. \( O(α_S) \) matching for four-quark operators requires 3-loop calculations
HQET four-quark operators of interest

When studying inclusive decay rates of B-hadrons, an OPE leads to four-quark operators:

\[ O_1^q = \left( \bar{q} \gamma_\mu P_L q \right) \left( q \gamma_\mu P_L \bar{q} \right) \]
\[ O_2^q = \left( \bar{q} P_L q \right) \left( q P_R \bar{q} \right) \]
\[ O_3^q = \left( \bar{Q} \gamma_\mu P_L q \right) \left( q \gamma_\mu P_L \bar{Q} \right) \]
\[ O_4^q = \left( \bar{Q} P_L q \right) \left( q P_R \bar{Q} \right) \]

We’ll focus for now on isospin non-singlet operators, \( O_u - O_d \). These are protected from power-divergent mixings arising from eye contractions of the light quark.

(These are the enhanced “Spectator Effects”)

The four-quark operators showing up in B-meson mixing are similar, \( \Delta B = 2 \) rather than \( \Delta B = 0 \). The matching of these operators in the continuum will not be a lot of additional work.
Survey

X-space dimension-3 light bilinears

- $O(\alpha_s^3)$ matching for scalar, vector bilinears
  K. G. Chetyrkin, A. Maier [hep-ph/1010.1145]

- Wilson fermions
  V. Gimenez et al [hep-lat/0406019]
  S. Calì et al [hep-lat/2003.05781]

- $N_f = 2$, twisted-mass fermions
  ETMC [hep-lat/1207.0628]

- $N_f = 2 + 1$, Domain-wall fermions
  JLQCD [hep-lat/1604.08702]

X-space other bilinears

- Energy Momentum Tensor (dimension-4)
  Cyprus group [hep-lat/2102.00858, hep-lat/2212.07730]

- Staple-shaped operators
  C. Alexandrou et al [hep-lat/2305.11824]

- Heavy-light Bilinears
  P. Korcyl et al [hep-lat/1512.00069]

X-space Four-quark operators

- $\Delta s = 1$, $K \to \pi\pi$ operators
  matching between 3 and 4-flavor theories.
  M. Tomii et al [hep-lat/1811.11238,1901.04107]

Matching for X-space ↔ MS has not yet been calculated for four-quark operators
The scheme

* Requires three-point renormalization condition:

\[
\frac{\langle J_a(x_0) \sigma_j^{(X)}(\mu = x_0^{-1}, 0) J_a^\dagger(-x_0) \rangle}{\langle J_a(x_0) J_a^\dagger(-x_0) \rangle} = \frac{\langle J_a(x_0) \sigma_j^{(0)}(0) J_a^\dagger(-x_0) \rangle}{\langle J_a(x_0) J_a^\dagger(-x_0) \rangle} =: T_{j,a}
\]

Measured on the lattice

\[
M_{i,a} = \frac{\langle J_a(x_0) \sigma_i^{(0)}(0) J_a^\dagger(-x_0) \rangle}{\langle J_a(x_0) J_a^\dagger(-x_0) \rangle} = Z_{ij}(\mu = x_0^{-1}) \frac{\langle J_a(x_0) \sigma_j^{(X)}(\mu = x_0^{-1}, 0) J_a^\dagger(-x_0) \rangle}{\langle J_a(x_0) J_a^\dagger(-x_0) \rangle} = Z_{ij}(x_0) T_{j,a}
\]

\[
Z_{ij} = M_{i,a} T_{a,j}^{-1}
\]

* Ratios are nice because Z-factors of source cancels, and linear divergence of Wilson line cancels.

* Sources need to be chosen so that the tree level matrix is invertible, otherwise don’t have enough constraints on the Z-matrix. Need baryonic sources, otherwise the colour-mixed operators will have vanishing matrix elements.

\[
O_q^3 = \left( \bar{q} T_a \gamma_\mu P_L q \right) \left( \bar{q} T_a \gamma_\nu P_L Q \right) \quad O_q^4 = \left( \bar{q} T_a P_L q \right) \left( \bar{q} T_a P_R Q \right)
\]
How to integrate?

* $O(\alpha_s)$ position-space correlation functions break up into building blocks:

\[
\int d^d p_L \, d^d p_R \, d^d k \, e^{i p_L \cdot x_L - i p_R \cdot x_R} \, p_R^\beta (p_R - k)^\beta (p_L - k)^\mu p_L^\mu \\
\int d^d p_L \, d^d p_R \, d^d k \, e^{i k \cdot x_L - i k \cdot x_R} \, p_R^\beta (p_R - k)^\beta (p_L - k)^\mu p_L^\mu \\
\int d^d k \left( \frac{\partial}{\partial k_\mu} \cdot p_\sigma \right) f(k) = 0
\]

Integration by parts

Two-loop master integrals

* In practice, in-house codes written in Mathematica to handle this


TRACER; M. Jamin, M. Lautenbacher
Master integrals and blocks

★ Master Integrals look like:

\[ T_{LL}(x_L, x_R; n_1, n_2, n_3) = \int \frac{d^d p_L d^d p_R}{(2\pi)^{2d}} \frac{e^{i p_L x_L} e^{-i p_R x_R}}{(-p_L^2)^{n_1} (-p_R^2)^{n_2} (-p_L - p_R)^2)^{n_3}} \] (agrees with hep-lat/2102.00858)

\[ = -\Gamma(\frac{d}{2} - n_1) \Gamma(d - n_1 - n_2 - n_3) \Gamma(n_2) \Gamma(n_3) \frac{2 F_1}{(2\pi)^{d/2} 4^{n_1 + n_2 + n_3} \pi^d} \int_0^1 dx (1 - x_1)^{-d/2 + n_1 + n_2 - 1} x_1^{-d/2 + n_1 + n_3 - 1} \frac{d}{2} - n_1, d - n_1 - n_2 - n_3, \frac{d}{2} - \frac{(x_L - x_R)^2}{x_1 (1 - x_1) x_R^2} \]

\[ T_{HH}(x_L, x_R; n_1, n_2, n_3) = \int \frac{d^d p_L d^d p_R}{(2\pi)^{2d}} \frac{e^{i p_L x_L} e^{-i p_R x_R}}{(v \cdot p_L)^{n_1} (v \cdot p_R)^{n_2} (-p_L - p_R)^2)^{n_3}} \]

\[ = \frac{1}{4^{n_1 + n_2 + n_3}} \frac{\Gamma(\frac{d}{2} - n_3)}{\Gamma(n_1 + n_2)} 2 F_1 \left(n_1, d - 2n_3, n_1 + n_2, \frac{v \cdot (x_L - x_R)}{v \cdot x_L}\right) (-iv \cdot (x_R - x_L))^{-1 + n_1 + n_2} (-v \cdot x_L)^{-d/2 + n_1 + n_3} \delta_1(v \cdot (x_L - x_R)) \theta(v \cdot (x_R - x_L) > 0) \]

★ Building blocks look like:

\[ \sum_{-t_0}^{t_0} \frac{k}{\gamma} \frac{n_1}{\gamma} \frac{m_2}{\gamma} \left[ \begin{array}{c} -1 \\ 32\pi^2 \end{array} \right] + \left[ \begin{array}{c} 10 \log 2 - 3 \log \pi - 3 \gamma_E \\ 64\pi^2 \end{array} \right] (\gamma_a \gamma_b) + \left[ \begin{array}{c} -1 \\ 128\pi^2 \end{array} \right] (\gamma_a \gamma_b) + \left[ \begin{array}{c} 1 \\ -1 - 2 \log 2 + 3 \log \pi + 3 \gamma_E \\ 256\pi^2 \end{array} \right] (\gamma_a \gamma_b) \]

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In the MS scheme at $O(\alpha_s)$, the operators mix in 2x2 sub-blocks. So as a check of our machinery, we can work in a scheme that forces this mixing structure, meaning we only need two sources in our X-space scheme:

$$Z_{2\times 2}^{(X)} = 1 + g^2(-\mu^2 x_0^2)^{\frac{1}{2}} \left( \frac{1}{2\pi c} + \frac{1}{-1} - \frac{1}{12\pi^2 c} \right) \left( \frac{24 + 9\gamma_E + 4\pi^2 - 18 \log 2 + 9 \log \pi}{36\pi^2} - \frac{51 + 18\gamma_E - 8\pi^2 - 36 \log 2 + 18 \log \pi}{432\pi^2} \right) \left( \frac{-3}{8\pi^2 c} + \frac{27 + 9\gamma_E + 28\pi^2 + 9 \log \pi - 18 \log 2}{288\pi^2} \right)$$

Important check: divergent parts match!

This scheme exploits the divergence structure at $O(\alpha_s)$, a full treatment requires four sources (in progress).
Future Prospects

Completed

☑ Master integrals and building blocks computed
   ☑ Cross-checks that three-point renormalizations of bilinears has correct divergent pieces

In progress

☐ Writing down matching coefficients for explicit choices of sources for $\Delta B = 0$ four-quark operators
☐ Investigating the window problem on RBC/UKQCD 2+1 Domain Wall fermion ensembles

Wishlist

☐ Writing down matching coefficients for all light four-quark operators
Thanks!