Studies on finite-volume effects in the inclusive semileptonic decays of charmed mesons

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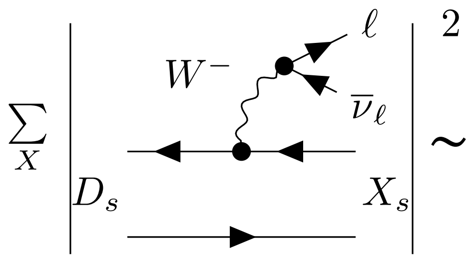
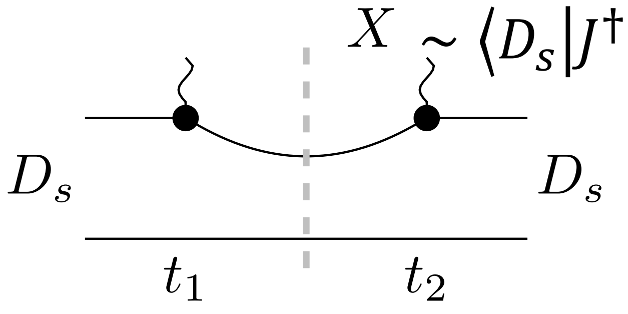
Motivation

Inclusive semileptonic decay rate $D_s \to X_s \ell \nu_\ell$

Lattice: **4Pt function**



$$\sim \int d\omega \, K(\omega) \, \langle D_s | J^\dagger \, \delta(\omega - \omega_X) J | D_s \rangle$$



$$\sim \langle D_s | J^\dagger \, e^{-\hat{H}(t_2 - t_1)} J | D_s \rangle$$

Contributions from all possible states

$K(\omega_X)$ is determined by kinematics

$$\int d\omega_X \, K(\omega_X) \, [\ \ ]_{\text{Lattice}}$$

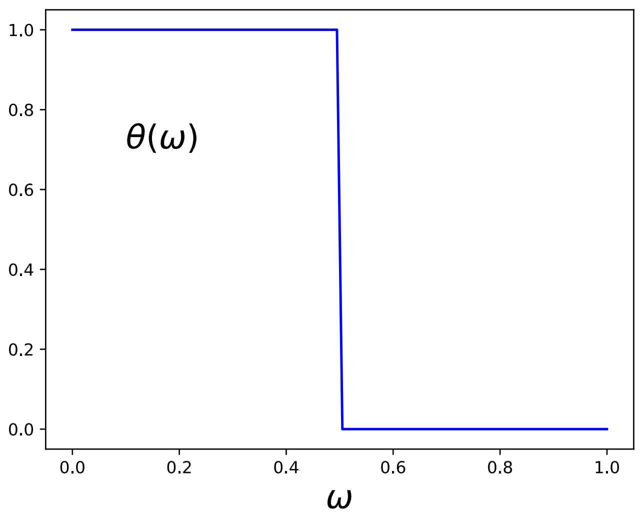
Problems:

1. Upper limit of the energy integral $\theta(\omega_{th} - \omega)$

2. In a finite volume we deal with a discrete set of states

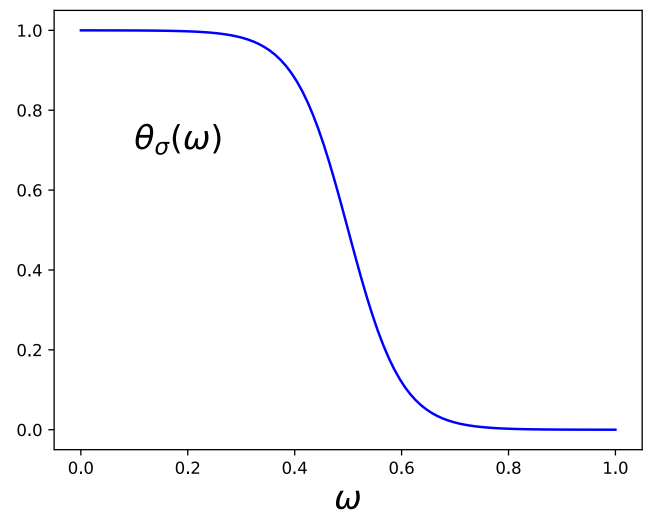
➡ Sources of systematic errors

1. Upper limit of the energy integral



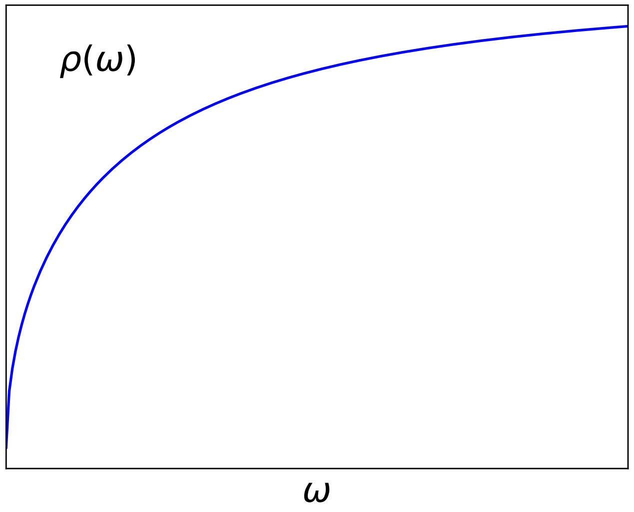
Direct approximation with $e^{-\omega(t_2-t_1)}$ not possible

Apply smearing

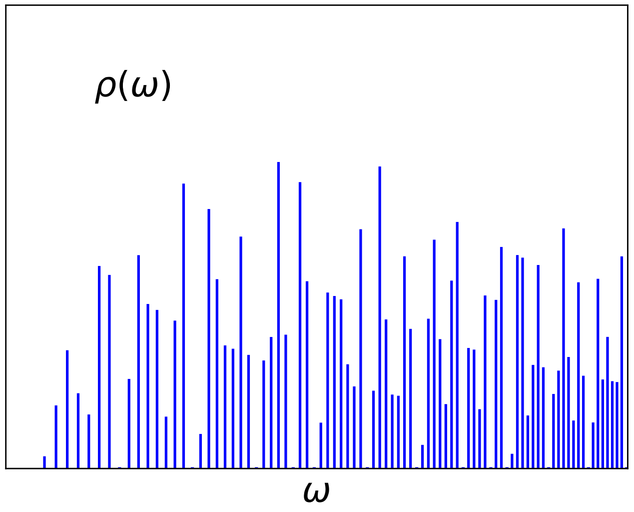


Talk from A. Barone [A. Barone et al., arXiv:2305.14092] Monday 07/31 5 PM

2. Discrete set of states



Finite volume



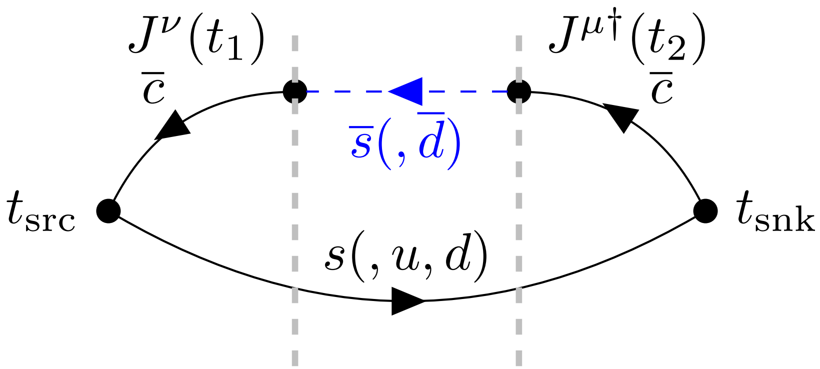
This talk

Develop and verify a modelling strategy to estimate the infinite volume limit

Introduction

Continuation of the project presented last year
[arXiv:2211.16830]

$$\Gamma \sim \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \bar{X}(\boldsymbol{q}^2)$$



$$C(t) \sim \langle D_s | \tilde{J}^{\dagger}_{\mu}(-\boldsymbol{q}) e^{-\hat{H}(t_2 - t_1)} \tilde{J}_{\nu}(\boldsymbol{q}) | D_s \rangle$$

$\bar{X}(\boldsymbol{q}^2)$ contains the energy integral and can be written as

$$\bar{X}(\boldsymbol{q}^2) \sim \int_{\omega_0}^{\infty} d\omega\, K^{(l)}(\omega, \boldsymbol{q}^2) \langle D_s(\boldsymbol{0}) | \tilde{J}^{\dagger}_{\mu}(-\boldsymbol{q}) \delta(\hat{H} - \omega) \tilde{J}_{\nu}(\boldsymbol{q}) | D_s(\boldsymbol{0}) \rangle$$

Kernel function

contains terms of power ω_X^l, with $l = 0,1,2$

Lattice data used for reconstruction of $\bar{X}(\boldsymbol{q}^2)$ using Chebyshev polynomials

The Kernel function

The shifted Chebyshev polynomials $T_j^*(e^{-\omega})$ allow an approximation of $K(\omega)$ in the range $[\omega_0, \infty]$, with $0 \leq \omega_0 < \omega_{min}$

$$K(\omega) \simeq \sum_j c_j^* T_j^*(e^{-\omega})$$

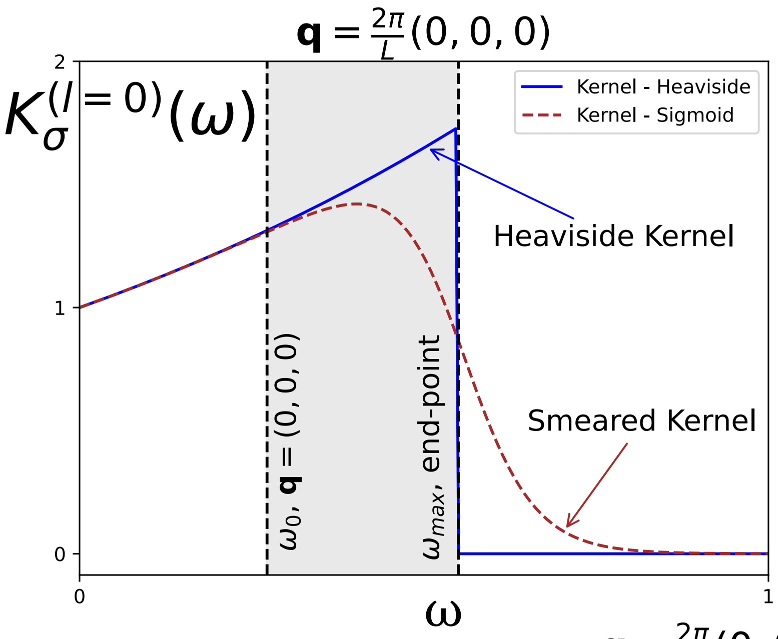
$$T_0^*(x) = 1, T_1^*(x) = 2x - 1, T_2^*(x) = 8x^2 - 8x + 1, \dots$$

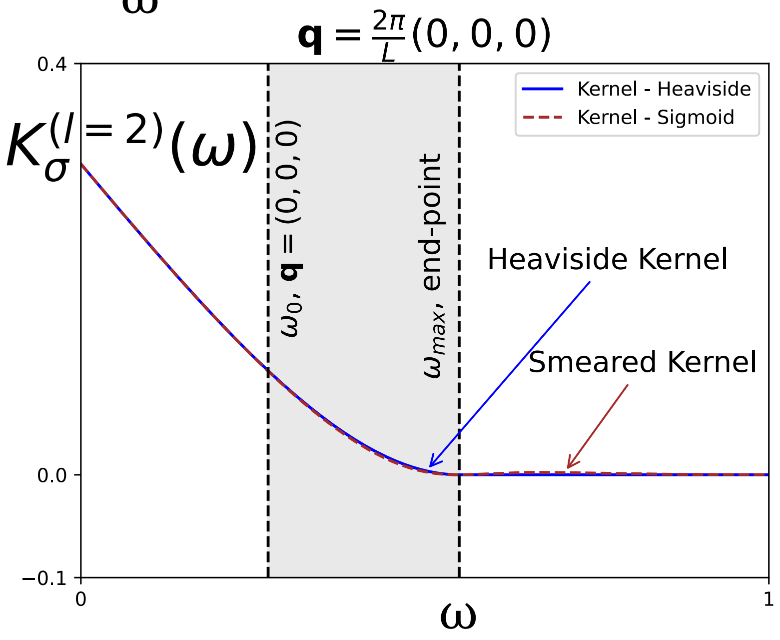
Kernel we wish to approximate

$$K_\sigma^{(l)}(\omega) = e^{2\omega t_0} \left(\sqrt{\boldsymbol{q}^2}\right)^{2-l} \left(m_{D_s} - \omega\right)^l \theta_\sigma\left(m_{D_s} - \sqrt{\boldsymbol{q}^2} - \omega\right),$$

$$\theta_\sigma(x) = \frac{1}{1 + e^{-x/\sigma}}$$

Momentum and energy of hadronic final state

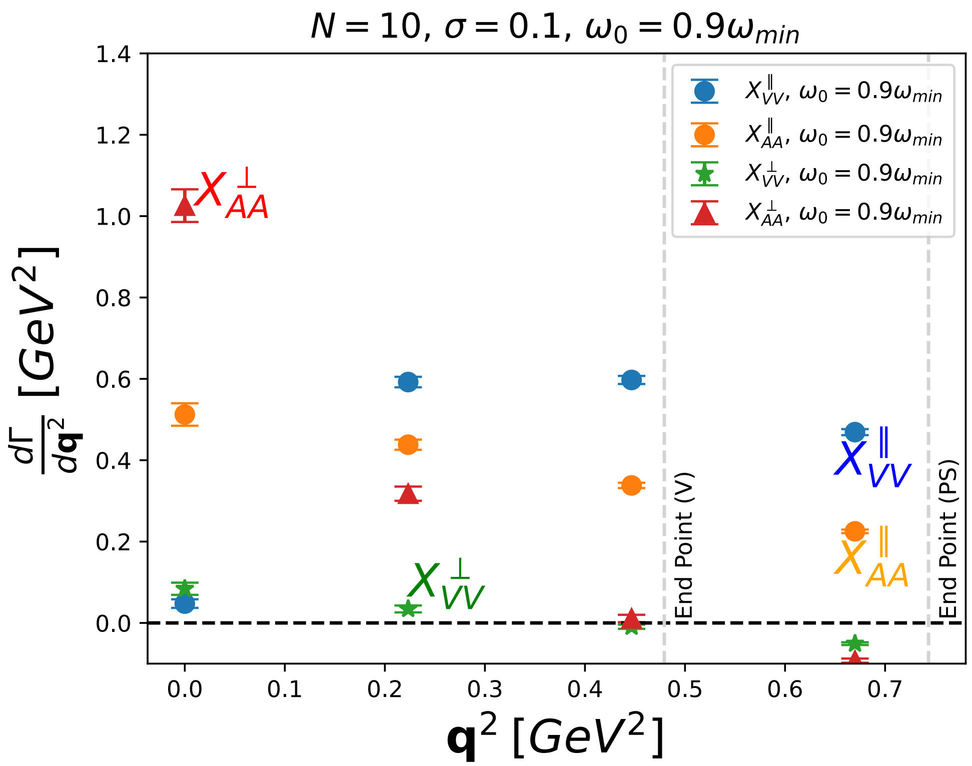




The differential rate $\bar{X} \sim \frac{d\Gamma}{d\boldsymbol{q}^2}$

$$\bar{X} = \langle D_s(\mathbf{0}) | \tilde{J}_\mu^\dagger(-\boldsymbol{q}) K(\hat{H}, \boldsymbol{q}^2) \tilde{J}_\nu(\boldsymbol{q}) | D_s(\mathbf{0}) \rangle$$

Using the smeared kernel we obtain



Decomposed $\bar{X}$ into channels of V and A; $\parallel$ and $\perp$

Questions

- Error due to approximation? [arXiv:2211.16830]
- Infinite volume limit?

Model for the infinite volume limit

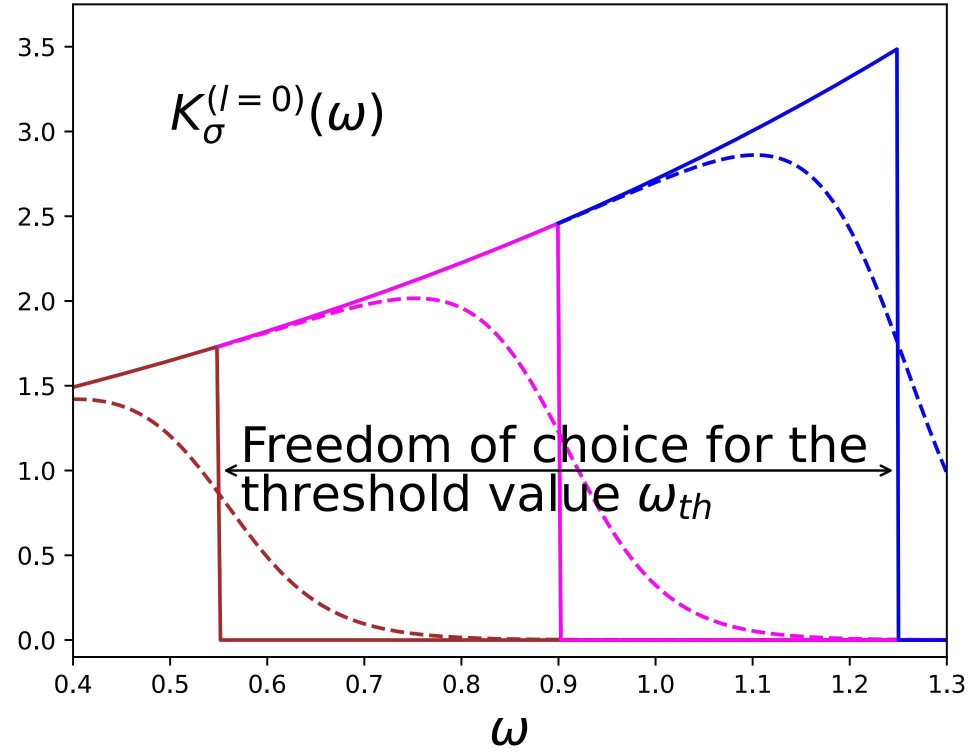
The remaining two problems
- $\sigma \to 0$
- $V \to \infty$

Proper estimate requires

$$\lim_{\sigma\to 0}\lim_{V\to\infty} \bar{X}(\boldsymbol{q}^2)$$

Necessary data not available

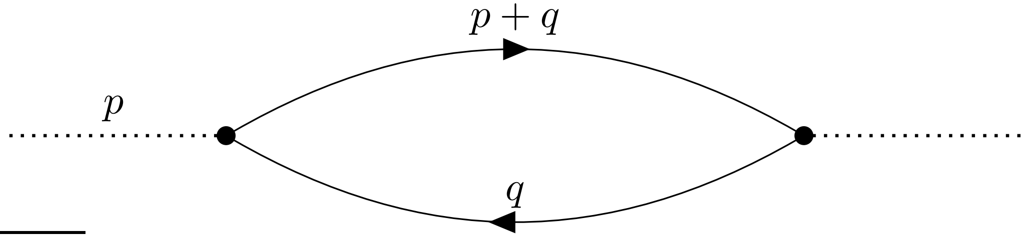
Introduce a model
- Include two-body final states
- Freely vary the upper limit of the energy integral ω_{th}



- Verify if the model reproduces the correct dependence on ω_{th}
- Estimate the $V \to \infty$ limit

The Model - Spectral reconstruction

Assume a vacuum-polarization ansatz

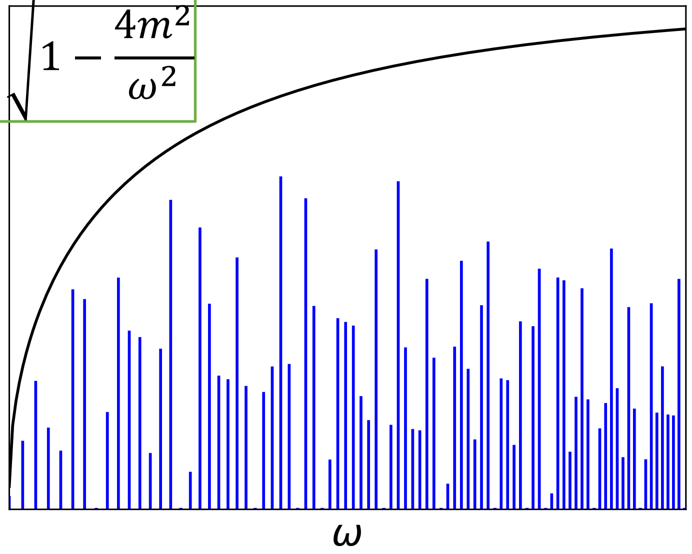


$$[\,]_{\text{Diagramm}} \sim i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(p+q)^2 - m^2 + i\epsilon} \frac{1}{q^2 - m^2 + i\epsilon}$$

$$\xrightarrow{\text{Im}} \pi \int \frac{d^3 \boldsymbol{q}}{(2\pi)^3} \frac{1}{\left(2\sqrt{m^2 + \boldsymbol{q}^2}\right)^2} \delta\left(p_0 - 2\sqrt{m^2 + \boldsymbol{q}^2}\right)$$

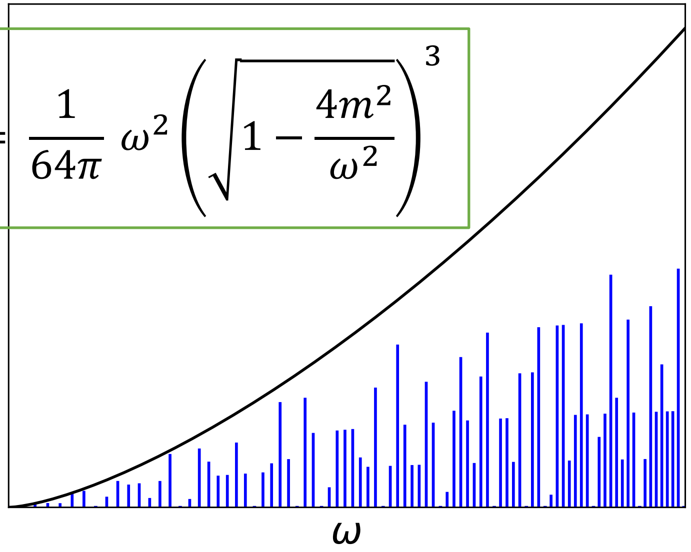
$$\rho(\omega) = \frac{1}{16\pi} \sqrt{1 - \frac{4m^2}{\omega^2}}$$

$J = 0$ (A4A4)



$$\rho(\omega) = \frac{1}{64\pi} \omega^2 \left(\sqrt{1 - \frac{4m^2}{\omega^2}}\right)^3$$

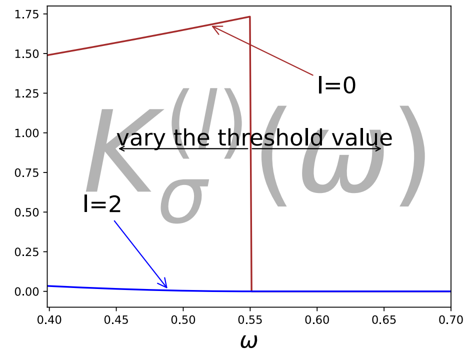
$J = 1$ (A3A3)



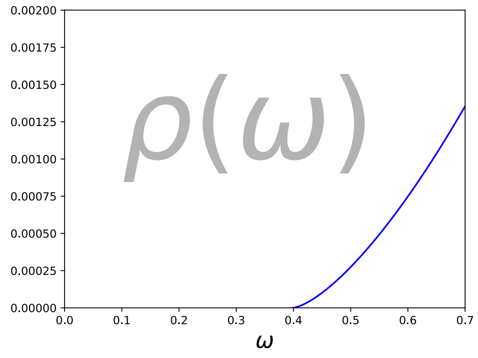
The Model – Infinite volume reconstruction

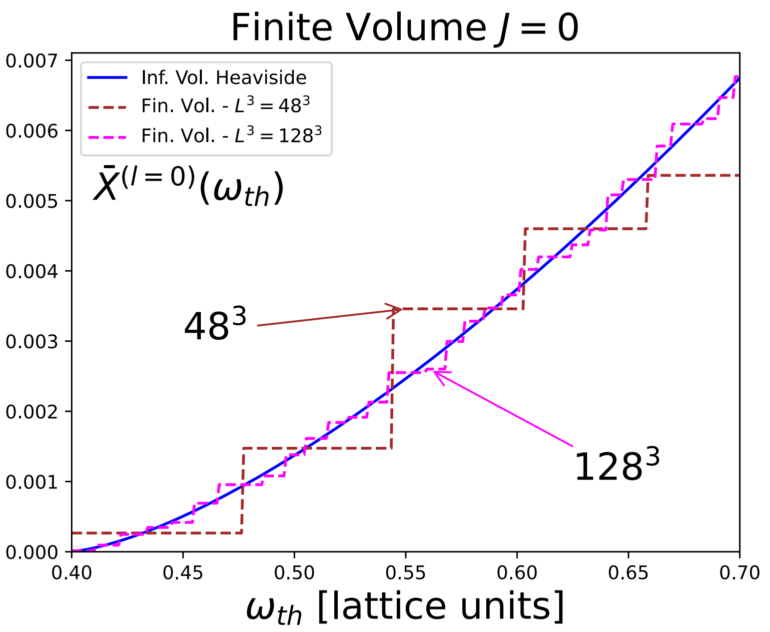
To verify the infinite volume behavior we can look at

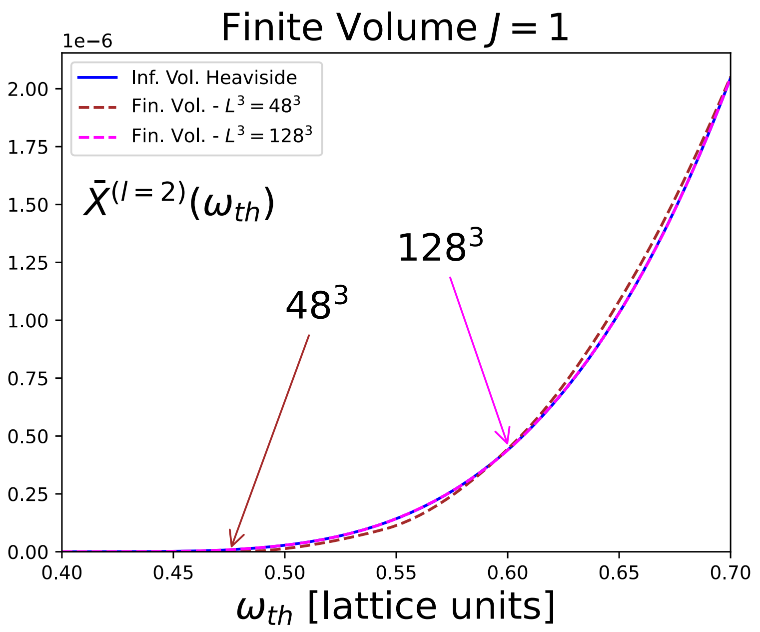
$$\bar{X}^{(l)}(\omega_{th}) \sim \int_0^{\omega_{th}} d\omega$$



$\times$

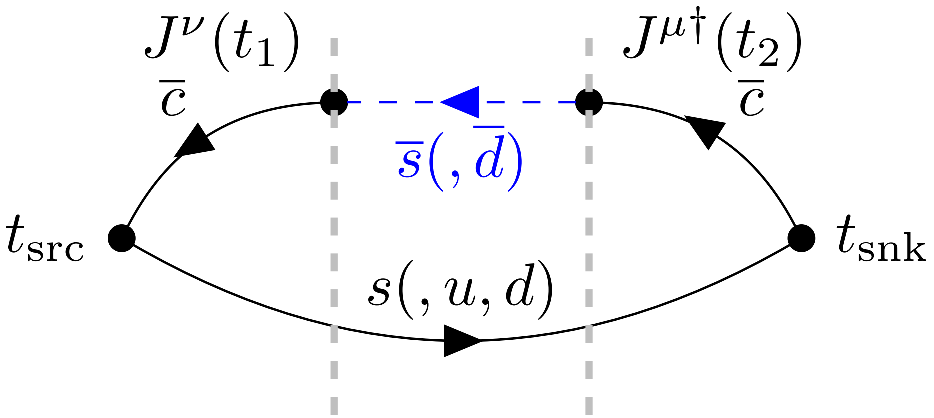






Lattice Setup

- Lattice Size: $48^3 \times 96$
- Lattice Spacing: $a = 0.055$ fm
- 2+1 Möbius domain-wall fermions
- u, d quarks at $m_\pi \simeq 300$ MeV
- s, c quarks at near-physical values
- 4 choices of momentum insertion corresponding to $q = (0,0,0) \to (1,1,1)$
- Numerical computation on Fugaku
- Used Grid/Hadrons [P. Boyle et al., https://github.com/paboyle/Grid; A. Portelli et al., https://github.com/aportelli/Hadrons]

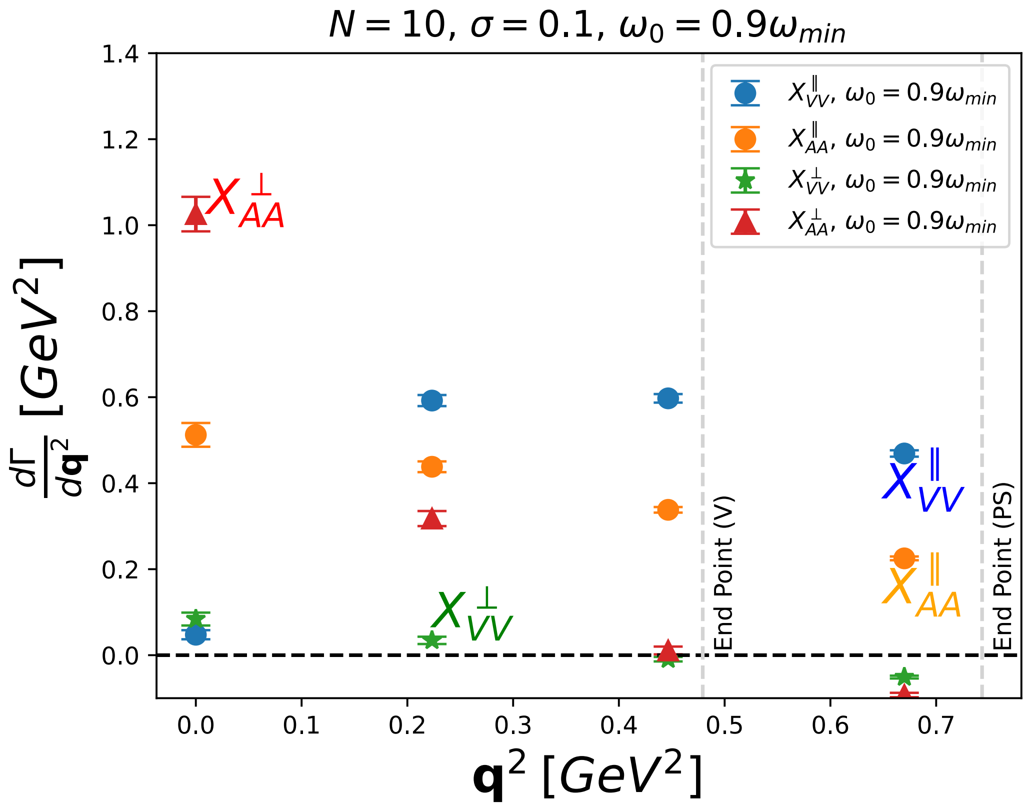


[Colquhoun et al., arXiv:2203.04938]

ID	a (fm)	β	$L^3 \times N_T \times L_s$	$N_{\rm cfg}$	am_l	am_s	am_Q
M-ud5-sa	0.055	4.35	$48^3 \times 96 \times 8$	50	0.012	0.025	0.27287
							0.42636
							0.68808
M-ud4-sa	0.055	4.35	$48^3 \times 96 \times 8$	50	0.008	0.025	0.27287
							0.42636
							0.68808
M-ud3-sa	0.055	4.35	$48^3 \times 96 \times 8$	42	0.0042	0.025	0.27287
							0.42636
							0.68808

Numerical analyis – Preliminaries

This analysis focuses on the contribution of $\bar{X}_{AA}^{\perp}(\mathbf{0})$ to the total $\bar{X}(\mathbf{0})$



Only contribution to $\bar{X}_{AA}^{\perp}(\mathbf{0})$ comes from the spatial Axial-Vector current insertions

$$K_{\sigma}^{(l)}(\omega) = \left(\sqrt{\boldsymbol{q}^2}\right)^{2-l}\ \bar{K}_{\sigma}^{(l)}(\omega)$$

$\bar{K}$: kernel up to a trivial factor of $\left(\sqrt{\boldsymbol{q}^2}\right)^{2-l}$

Temporal component only contribute with $l = 0$

Spatial compenents contribute with $l = 2$ (for $\boldsymbol{q}^2 = \mathbf{0}$)

Numerical Analysis – Procedure

The idea is to use the information from out model and fit this to our lattice data and then perform the infinite volume extrapolation based on the fitted data

Fit prescription

$$C(t) = A_0 e^{-E_0 t} + s\left[\sum_i A_i e^{-E_i t} F(E_i)\right]$$

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Ground state

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$$C(t) = A_0 e^{-E_0 t} + s\left[\sum_i A_i e^{-E_i t} F(E_i)\right]$$

Ground state

Excited states

E_i and A_i extracted from the model

Numerical Analysis – Procedure

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Fit prescription

$$C(t) = A_0 e^{-E_0 t} + s \left[\sum_i A_i e^{-E_i t} F(E_i) \right]$$

Ground state

Overall contribution of excited states

Excited states

E_i and A_i extracted from the model

Numerical Analysis – Procedure

The idea is to use the information from out model and fit this to our lattice data and then perform the infinite volume extrapolation based on the fitted data

Fit prescription

$$C(t) = A_0 e^{-E_0 t} + s\left[\sum_i A_i e^{-E_i t} F(E_i)\right]$$

Ground state

Overall contribution of excited states

Excited states

E_i and A_i extracted from the model

Form factor motivated by the timelike kaon form factor

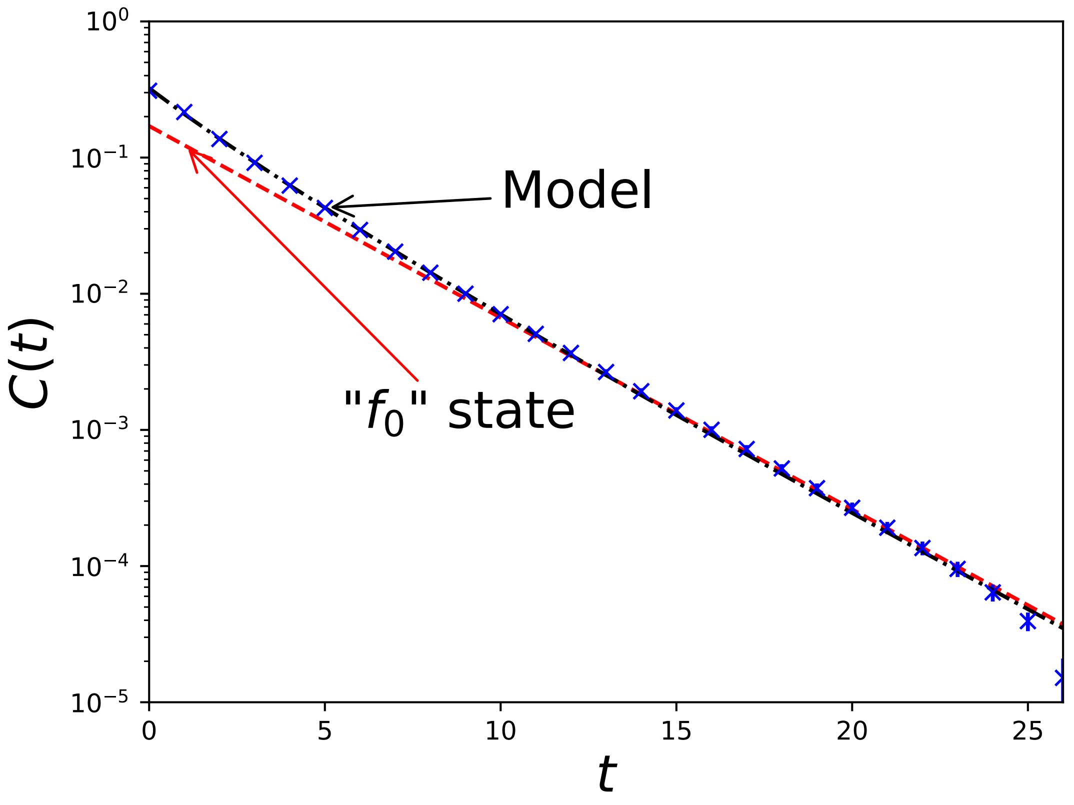
$$F(E_i) = \frac{1}{E_i^2 - m_J^2}$$

$J = 0$: "f_0" state (0^+)

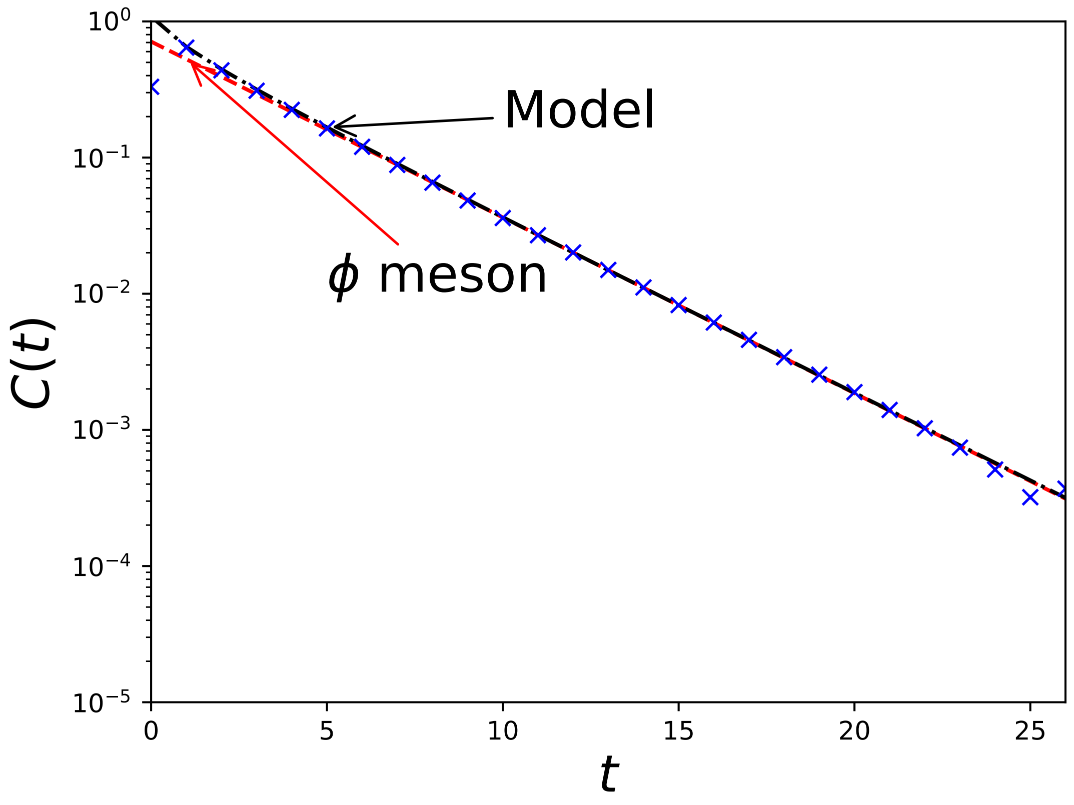
$J = 1$: ϕ meson (1^-)

Numerical Analysis – Correlator fit

$\langle D_s | A_4 A_4 | D_s \rangle$



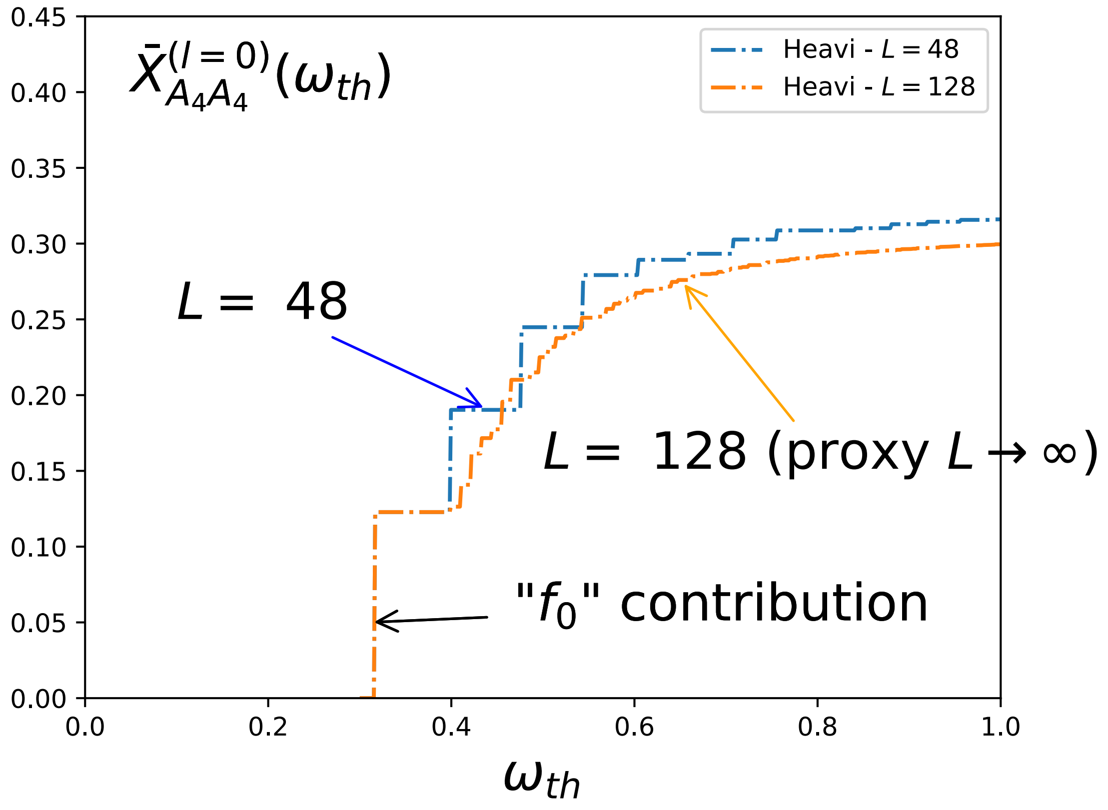
$\langle D_s | A_3 A_3 | D_s \rangle$



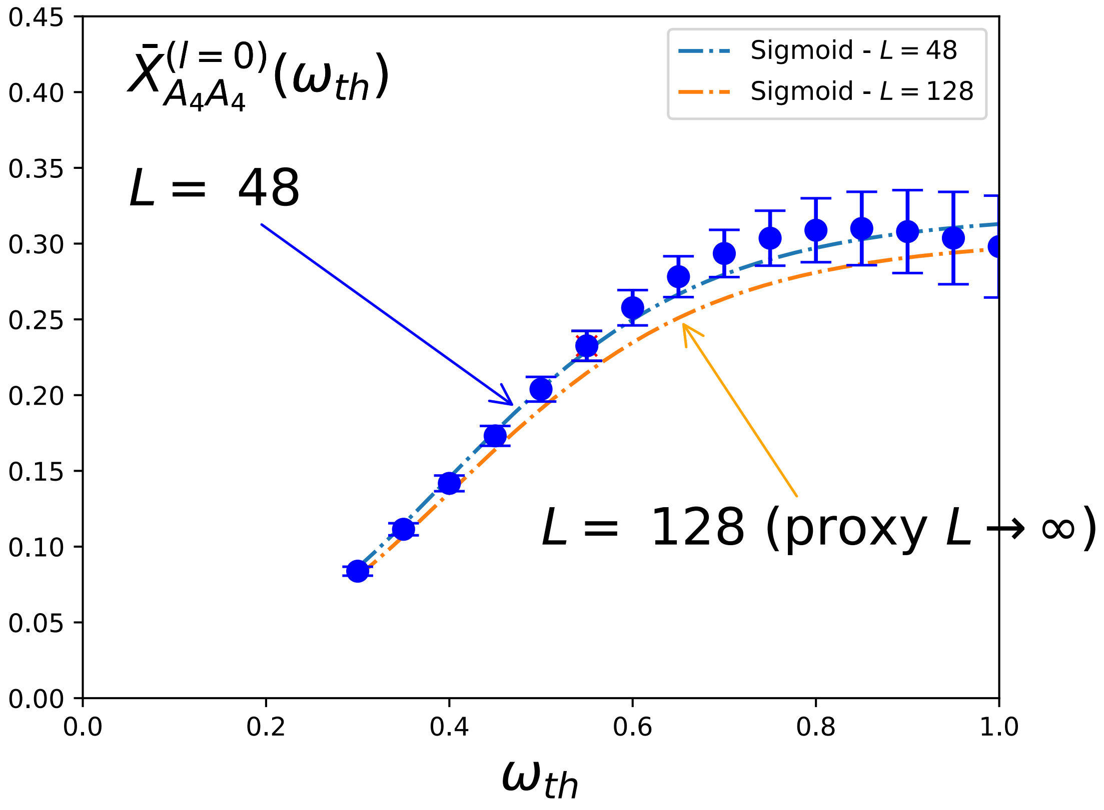
Numerical Analysis – Result

$$\langle D_s | A_4 A_4 | D_s \rangle$$

Heaviside



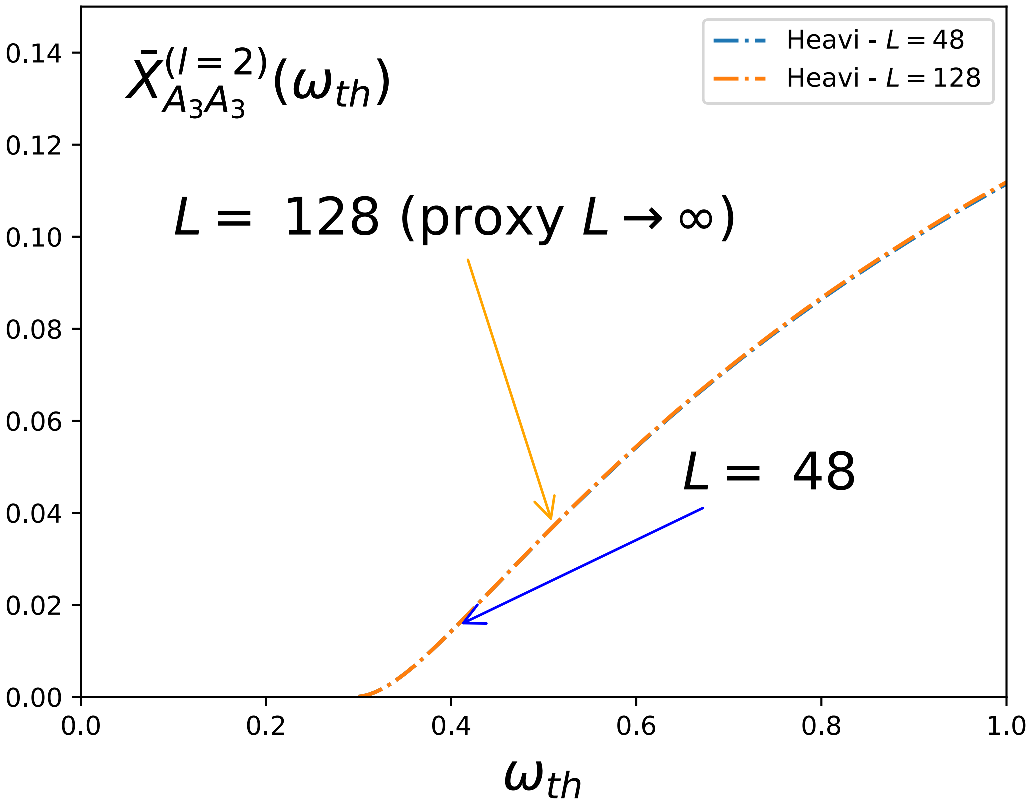
Smearing($\sigma = 0.1$) + lattice data



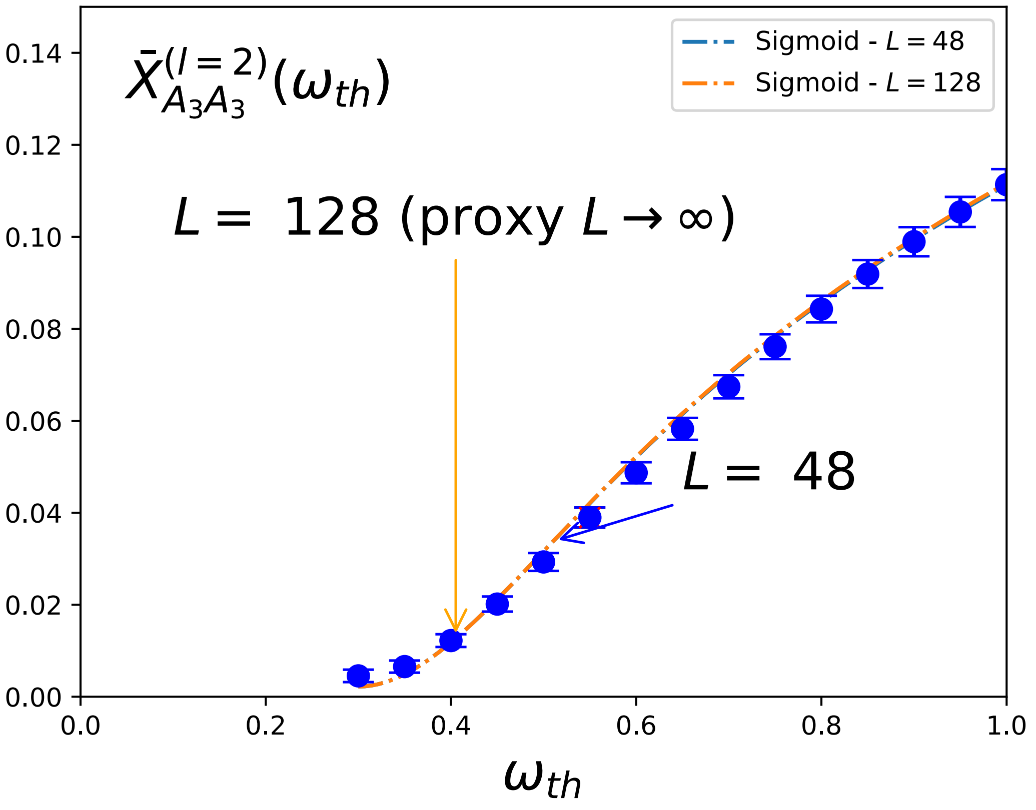
Numerical Analysis – Result

$$\langle D_s | A_3 A_3 | D_s \rangle$$

Heaviside



Smearing($\sigma = 0.1$) + lattice data



Infinite Volume limit

We can now estimate the limit $V \to \infty$, followed by $\sigma \to 0$ (at $\omega_{th} = \omega_{th}^{Phys}$)

Obtained from Model

$$\bar{X}_{AA}^{\perp}(\mathbf{0}) \sim \text{Data} + \text{Sigmoid}(\text{L} = 128) - \text{Sigmoid}(\text{L} = 48) + \text{Sigmoid}(\sigma = 0) - \text{Sigmoid}\left(\sigma = \frac{1}{N}\right)$$

Infinite volume limit

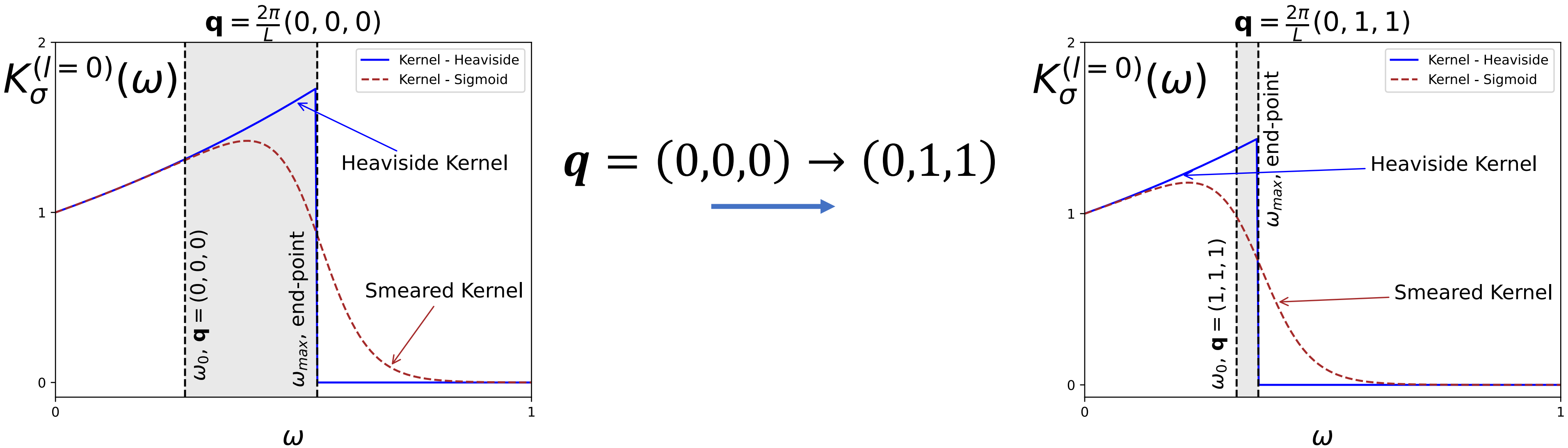
$\sigma \to 0$ limit

$$0.0389(22) + 0.0001(0) + 0.0028(1) = 0.0418(22)$$

For the safest choice of $\boldsymbol{q}^2 = \mathbf{0}$:

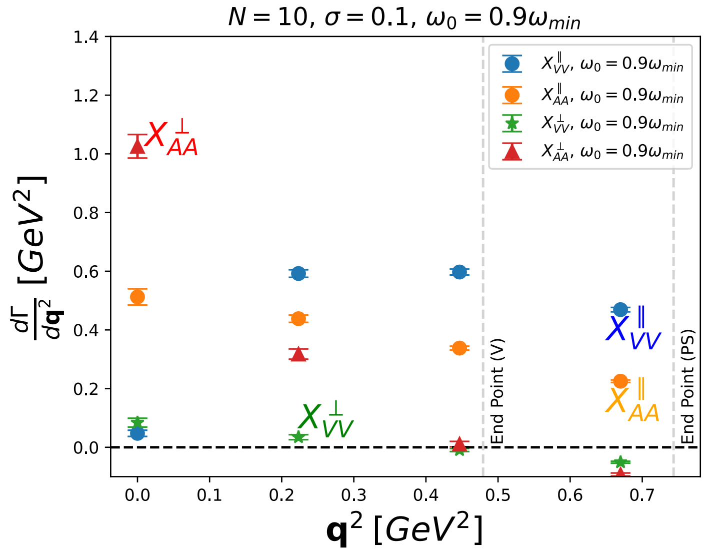
- Negligible corrections from finite volume effects
- $\sigma \to 0$ limit gives a $\sim 7\%$ correction

Small corrections for $q^2 = 0$, BUT



Approximation becomes harder; expect larger errors

At the same time



Total contribution to $\bar{X}(q^2)$ becomes smaller

Total contribution to error budget?

Summary and Outlook

Preliminary results on the systematic error induced due to finite volume corrections for the inclusive semileptonic decay rate for $D_s \to X_s \ell \nu_\ell$

- Presented a modelling strategy based on the assumption of two-body final states
  - Good approximation of the data
- Results for the simplest case $\boldsymbol{q}^2 = \mathbf{0}$
  - Allows a good estimate for the infinite volume limit
  - Small corrections due to finite volume effects and $\sigma \to 0$ limits
- Going forward:
  - Repeat for different current contributions and higher momentum insertions
  - Proper estimate of systematic error requires further studies on the model
    - Interactions, initial state, …