Studies on finite-volume effects in the inclusive semileptonic decays of charmed mesons

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Motivation

Inclusive semileptonic decay rate $D_S \rightarrow X_S \ell \nu_\ell$

$$\sum_{X} \frac{W^-(\ell, \bar{\nu}_\ell)}{D_S} \sim \int d\omega K(\omega) \langle D_S | J^\dagger \delta(\omega - \omega_X) | D_S \rangle$$

$Lattice: 4Pt$ function

$X \sim \langle D_S | J^\dagger e^{-\beta(t_2-t_1)} | D_S \rangle$

$K(\omega_X)$ is determined by kinematics

$$\int d\omega_X K(\omega_X) [ ]_{Lattice}$$

Problems:

1. Upper limit of the energy integral $\theta(\omega_{th} - \omega)$

2. In a finite volume we deal with a discrete set of states

Sources of systematic errors
1. Upper limit of the energy integral

Direct approximation with $e^{-\omega(t_2-t_1)}$ not possible

Apply smearing

2. Discrete set of states

This talk

Develop and verify a modelling strategy to estimate the infinite volume limit

Talk from A. Barone [A. Barone et al., arXiv:2305.14092]
Monday 07/31 5 PM
Introduction

Continuation of the project presented last year
[arXiv:2211.16830]

\[ \Gamma \sim \int_0^{q_{\text{max}}} dq^2 \sqrt{q^2} \, \bar{X}(q^2) \]

\( \bar{X}(q^2) \) contains the energy integral and can be written as

\[ \bar{X}(q^2) \sim \int_0^\infty d\omega \, K^{(l)}(\omega, q^2) \langle D_s(0) | \bar{J}_\mu(-q) \delta(H - \omega) \bar{J}_\nu(q) | D_s(0) \rangle \]

Kernel function contains terms of power \( \omega_X^l \), with \( l = 0,1,2 \)

Lattice data used for reconstruction of \( \bar{X}(q^2) \) using Chebyshev polynomials
The Kernel function

The shifted Chebyshev polynomials $T_j^*(e^{-\omega})$ allow an approximation of $K(\omega)$ in the range $[\omega_0, \infty]$, with $0 \leq \omega_0 < \omega_{\text{min}}$

$$K(\omega) \simeq \sum_j c_j^* T_j^*(e^{-\omega})$$

$T_0^*(x) = 1, T_1^*(x) = 2x - 1, T_2^*(x) = 8x^2 - 8x + 1, \ldots$

Kernel we wish to approximate

$$K^{(l)}(\omega) = e^{2\omega t_0} \left( \sqrt{q^2} \right)^{2-l} (m_{D^*_s} - \omega)^l \theta( m_{D^*_s} - \sqrt{q^2} - \omega),$$

$$\theta(x) = \frac{1}{1 + e^{-x/\sigma}}$$

**Momentum and energy of hadronic final state**
The differential rate $\bar{X} \sim \frac{d\Gamma}{dq^2}$

$$\bar{X} = \langle D_s(0)|\bar{J}_\mu^+(\vec{q})K(\vec{H}, q^2)\bar{J}_\nu(q)|D_s(0)\rangle$$

Using the smeared kernel we obtain

Decomposed $\bar{X}$ into channels of $V$ and $A$; $\parallel$ and $\perp$

Questions

- Error due to approximation? [arXiv:2211.16830]
- Infinite volume limit?
Model for the infinite volume limit

The remaining two problems
- $\sigma \to 0$
- $V \to \infty$

Proper estimate requires

$$\lim_{\sigma \to 0} \lim_{V \to \infty} \bar{X}(q^2)$$

Necessary data not available

Introduce a model
- Include two-body final states
- Freely vary the upper limit of the energy integral $\omega_{th}$

- Verify if the model reproduces the correct dependence on $\omega_{th}$
- Estimate the $V \to \infty$ limit
The Model - Spectral reconstruction

Assume a vacuum-polarization ansatz

\[ [\text{Diagramm}] \sim i \int \frac{d^4q}{(2\pi)^4} \frac{1}{(p+q)^2 - m^2 + i\epsilon} \frac{1}{q^2 - m^2 + i\epsilon} \]

\[ \rho(\omega) = \frac{1}{16\pi} \sqrt{1 - \frac{4m^2}{\omega^2}} \]

\( J = 0 \) (A4A4)

\[ \rho(\omega) = \frac{1}{64\pi} \omega^2 \left( \sqrt{1 - \frac{4m^2}{\omega^2}} \right)^3 \]

\( J = 1 \) (A3A3)

\[ \pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{\left(2\sqrt{m^2 + q^2}\right)^2} \delta(p_0 - 2\sqrt{m^2 + q^2}) \]
The Model – Infinite volume reconstruction

To verify the infinite volume behavior we can look at

\[ \bar{X}^{(l)}(\omega_{th}) \sim \int_{0}^{\omega_{th}} d\omega \]

\[ K_{\sigma}^{(l)}(\omega) \]

\[ \rho(\omega) \]

Finite Volume \( J = 0 \)

\[ \bar{X}^{(l = 0)}(\omega_{th}) \]

Finite Volume \( J = 1 \)

\[ \bar{X}^{(l = 2)}(\omega_{th}) \]
Lattice Setup

- Lattice Size: $48^3 \times 96$
- Lattice Spacing: $a = 0.055$ fm
- 2+1 Möbius domain-wall fermions
- $u, d$ quarks at $m_\pi \approx 300$ MeV
- $s, c$ quarks at near-physical values
- 4 choices of momentum insertion corresponding to $q = (0,0,0) \rightarrow (1,1,1)$
- Numerical computation on Fugaku

![Diagram](image)

[Colquhoun et al., arXiv:2203.04938]

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This analysis focuses on the contribution of $X_{\Lambda A}^\perp(0)$ to the total $X(0)$

Only contribution to $X_{\Lambda A}^\perp(0)$ comes from the spatial Axial-Vector current insertions

$$K^{(l)}_{\sigma}(\omega) = (\sqrt{q^2})^{2-l} \bar{K}^{(l)}_{\sigma}(\omega)$$

$\bar{K}$: kernel up to a trivial factor of $(\sqrt{q^2})^{2-l}$

Temporal component only contribute with $l = 0$

Spatial components contribute with $l = 2$ (for $q^2 = 0$)
The idea is to use the information from our model and fit this to our lattice data and then perform the infinite volume extrapolation based on the fitted data.

**Fit prescription**

\[ C(t) = A_0 e^{-E_0 t} + s \left[ \sum_i A_i e^{-E_i t} F(E_i) \right] \]
Numerical Analysis – Procedure

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*Fit prescription*

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- **Ground state**
- **Excited states**

$E_i$ and $A_i$ extracted from the model
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- **Overall contribution of excited states**
- **Excited states**
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- **Ground state**
  - \( C(t) = A_0 e^{-E_0 t} \)

- **Overall contribution of excited states**
  - \( \sum_i A_i e^{-E_i t} \)

- **Excited states**
  - \( E_i \) and \( A_i \) extracted from the model

- **Form factor motivated by the timelike kaon form factor**
  - \( F(E_i) = \frac{1}{E_i^2 - m_j^2} \)
  - \( J = 0: "f_0" \) state (0⁺)
  - \( J = 1: \phi \) meson (1⁻)
Numerical Analysis – Correlator fit

\[ \langle D_s | A_4 A_4 | D_s \rangle \]

\[ \langle D_s | A_3 A_3 | D_s \rangle \]

Model

"f_0" state

Model

\( \phi \) meson

Graphs showing correlators for different states with models fitted to the data.
Heaviside

\[ \tilde{\chi}_{A_4A_4}^{(l=0)}(\omega_{th}) \]

Smearing (\( \sigma = 0.1 \)) + lattice data

\[ \tilde{\chi}_{A_4A_4}^{(l=0)}(\omega_{th}) \]
Numerical Analysis – Result

\[ \langle D_s | A_3 A_3 | D_s \rangle \]

Heaviside

\[ \tilde{X}_{A_3A_3}^{(l=2)}(\omega_{th}) \]

\begin{align*}
L = 128 \text{ (proxy } L \to \infty) \\
L = 48
\end{align*}

Smearing ($\sigma = 0.1$) + lattice data

\[ \tilde{X}_{A_3A_3}^{(l=2)}(\omega_{th}) \]

\begin{align*}
L = 128 \text{ (proxy } L \to \infty) \\
L = 48
\end{align*}
Infinite Volume limit

We can now estimate the limit $V \to \infty$, followed by $\sigma \to 0$ (at $\omega_{th} = \omega_{th}^{phys}$)

$$\bar{X}_{AA}^{\perp}(0) \sim \text{Data} + \text{Sigmoid}(L = 128) - \text{Sigmoid}(L = 48) + \text{Sigmoid}(\sigma = 0) - \text{Sigmoid} \left( \frac{1}{N} \right)$$

Infinite volume limit $\sigma \to 0$ limit

$$0.0389(22) + 0.0001(0) + 0.0028(1) = 0.0418(22)$$

For the safest choice of $q^2 = 0$:

- Negligible corrections from finite volume effects
- $\sigma \to 0$ limit gives a $\sim 7\%$ correction
Small corrections for $q^2 = 0$, BUT

Approximation becomes harder; expect larger errors

At the same time

Total contribution to $\bar{X}(q^2)$ becomes smaller

Total contribution to error budget?
Summary and Outlook

Preliminary results on the systematic error induced due to finite volume corrections for the inclusive semileptonic decay rate for $D_s \to X_s \ell \nu_\ell$

- Presented a modelling strategy based on the assumption of two-body final states
  - Good approximation of the data
- Results for the simplest case $q^2 = 0$
  - Allows a good estimate for the infinite volume limit
  - Small corrections due to finite volume effects and $\sigma \to 0$ limits
- Going forward:
  - Repeat for different current contributions and higher momentum insertions
  - Proper estimate of systematic error requires further studies on the model
    - Interactions, initial state, ...