Chebyshev and Backus-Gilbert reconstruction for inclusive semileptonic $B_{(s)}$-meson decays from Lattice QCD

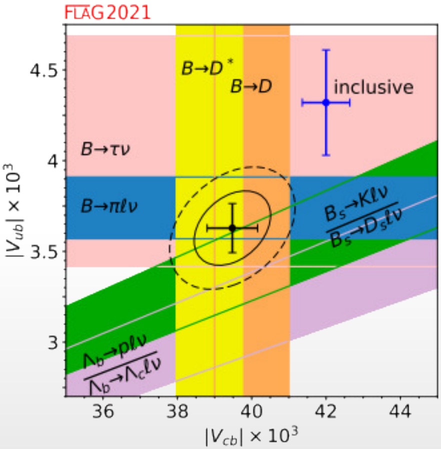
Alessandro Barone

in collaboration with

Andreas Jüttner, Shoji Hashimoto,
Takashi Kaneko, Ryan Kellermann

Lattice2023, 31st July 2023

Introduction and motivations



[Aoki et al. (2021)[1]]

- $\sim 3\sigma$ discrepancy (in the plot) between inclusive (blue cross)/exclusive (black cross) determination;
- lattice QFT represents a fully nonperturbative theoretical approach to QCD;
- no current predictions from lattice QCD for the inclusive decays.

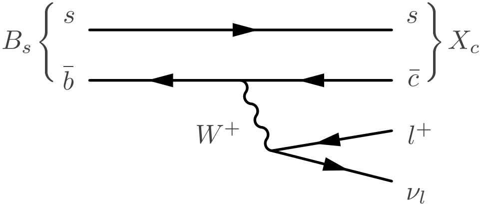
This talk: Pilot study $B_s \to X_c l\nu_l$

[Barone et al. (2023)[2]]

- improve existing strategies for inclusive decays on the lattice;
- compare two different methods for the analysis.

→ see also **Ryan Kellermann**'s talk, Mon 17:20

Differential decay rate



Differential decay rate:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2\mathrm{d}q_0\mathrm{d}E_l} = \frac{G_F^2|V_{cb}|^2}{8\pi^3}\boxed{L_{\mu\nu}}\,\boxed{W^{\mu\nu}},$$

Hadronic tensor

Leptonic tensor
(analytically known)

$$\boxed{W^{\mu\nu}} = \sum_{X_c}(2\pi)^3\delta^{(4)}(p-q-r)\frac{1}{2E_B}\langle B_s(\boldsymbol{p})|\, J^{\mu\dagger}(\boldsymbol{q})\,|X_c(\boldsymbol{r})\rangle\,\langle X_c(\boldsymbol{r})|\, J^{\nu}(\boldsymbol{q})\,|B_s(\boldsymbol{p})\rangle\,.$$

contains all the **nonperturbative** QCD

Observables

The total decay rate is

$$\Gamma \propto \int \mathrm{d}\boldsymbol{q}^2 \mathrm{d}\omega \mathrm{d}E_l \, L_{\mu\nu} W^{\mu\nu} \,, \qquad \omega = E_{X_c} \,.$$

Other interesting observables are *moments* such as

$$\begin{aligned} &q^2 \textbf{ moments} && \langle (q^2)^n \rangle && \propto \frac{1}{\Gamma} \int \mathrm{d}\boldsymbol{q}^2 \mathrm{d}\omega \mathrm{d}E_l \, \boxed{(q^2)^n} \, L_{\mu\nu} W^{\mu\nu} \,, \\ &\textbf{hadronic mass moments} && \langle (M_{X_c}^2)^n \rangle && \propto \frac{1}{\Gamma} \int \mathrm{d}\boldsymbol{q}^2 \mathrm{d}\omega \mathrm{d}E_l \, \boxed{(M_{X_c}^2)^n} \, L_{\mu\nu} W^{\mu\nu} \,, \\ &\textbf{lepton moments} && \langle E_l^n \rangle && \propto \frac{1}{\Gamma} \int \mathrm{d}\boldsymbol{q}^2 \mathrm{d}\omega \mathrm{d}E_l \, \boxed{E_l^n} \, L_{\mu\nu} W^{\mu\nu} \,. \end{aligned}$$

Observables

After integrating over E_l, all of them can be rewritten as

$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} \mathrm{d}\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \boxed{\bar{X}}\,,$$

and similarly

$$\langle (q^2)^n \rangle \propto \frac{1}{\Gamma} \int_0^{\boldsymbol{q}^2_{\max}} \mathrm{d}\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \boxed{\bar{X}_Q^{(n)}}\,,$$

$$\langle (M^2_{X_c})^n \rangle \propto \frac{1}{\Gamma} \int_0^{\boldsymbol{q}^2_{\max}} \mathrm{d}\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \boxed{\bar{X}_H^{(n)}}\,,$$

$$\langle E_l^n \rangle \propto \frac{1}{\Gamma} \int_0^{\boldsymbol{q}^2_{\max}} \mathrm{d}\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \boxed{\bar{X}_L^{(n)}}\,,$$

know kinematics

$$\boxed{\bar{X}} = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega\, \boxed{k_{\mu\nu}} \times \boxed{W^{\mu\nu}}\,.$$

from lattice?

portal to compute the observables

Inclusive decays on the lattice

[Hansen et al. (2017)[3], Hashimoto (2017)[4], Gambino and Hashimoto (2020)[5]]

We need the nonperturbative calculation of the hadronic tensor

$$W^{\mu\nu}(\boldsymbol{q},\omega) \sim \sum_{X_c} \langle B_s| J^{\mu\dagger} |X_c\rangle \langle X_c| J^{\nu} |B_s\rangle \,.$$

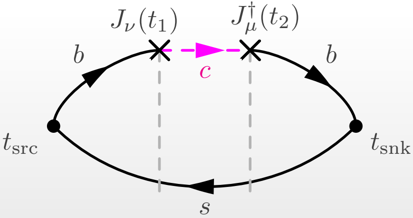
Inclusive decays on the lattice

[Hansen et al. (2017)[3], Hashimoto (2017)[4], Gambino and Hashimoto (2020)[5]]

We need the nonperturbative calculation of the hadronic tensor

$$W^{\mu\nu}(\boldsymbol{q},\omega) \sim \sum_{X_c} \langle B_s| J^{\mu\dagger} |X_c\rangle \langle X_c| J^{\nu} |B_s\rangle \, .$$

On the lattice, this is achieved with a **4pt correlation function**:



- $t_{\rm src}$, t_2, $t_{\rm snk}$ fixed
- $t_{\rm src} \leq t_1 \leq t_2$
- $t = t_2 - t_1$
- t small → signal-to-noise ratio deteriorate with t

$$\boxed{C^{\mu\nu}(t) \quad \leftrightarrow \quad \langle B_s|\tilde{J}^{\mu\dagger}(\boldsymbol{q},0)\, e^{-t\hat{H}}\, \tilde{J}^{\nu}(\boldsymbol{q},0)|B_s\rangle} \, .$$

Inclusive observables from lattice data

For the inclusive case we compute $\bar{X}$

$$\bar{X} = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega \, W^{\mu\nu} \underset{\text{kinematics factors}}{\underset{\downarrow}{\boxed{k_{\mu\nu}(\boldsymbol{q}, \omega)}}} , \qquad \overset{\text{lattice data for inclusive}}{\overset{\uparrow}{\boxed{C_{\mu\nu}(t)}}} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \underset{\sum_{X_c} \langle B_s | J_\mu^\dagger | X_c \rangle \langle X_c | J_\nu | B_s \rangle \delta(\omega - E_{X_c})}{\underset{\downarrow}{\boxed{W_{\mu\nu}(\boldsymbol{q}, \omega)}}} e^{-\omega t}$$

where the hadronic tensor acts now as a spectral function.

Inclusive observables from lattice data

For the inclusive case we compute $\bar{X}$

$$\bar{X} = \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega \, W^{\mu\nu} \underbrace{\boxed{k_{\mu\nu}(\boldsymbol{q},\omega)}}_{\text{kinematics factors}} , \qquad \overbrace{\boxed{C_{\mu\nu}(t)}}^{\text{lattice data for inclusive}} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \underbrace{\boxed{W_{\mu\nu}(\boldsymbol{q},\omega)}}_{\sum_{X_c} \langle B_s| J_\mu^\dagger |X_c\rangle \langle X_c| J_\nu |B_s\rangle \, \delta(\omega - E_{X_c})} e^{-\omega t}$$

where the hadronic tensor acts now as a spectral function.
To make use of $C_{\mu\nu}(t)$ we need to extend the range

$$\begin{aligned} \bar{X} &= \int_{\omega_{\min}}^{\omega_{\max}} \mathrm{d}\omega \, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)} \\ 0 \leq \omega_0 \leq \omega_{\min} \longleftarrow \quad &= \int_{\boxed{\omega_0}}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \boxed{k_{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max} - \omega)} \longrightarrow \text{kernel operator} \\ &= \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} K_{\mu\nu}(\boldsymbol{q},\omega) \end{aligned}$$

Inclusive observables from lattice data

For the inclusive case the observable we compute is $O \equiv \bar{X}$

$$\bar{X} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \underset{\substack{\downarrow \\ \text{Kernel}}}{\boxed{K_{\mu\nu}(\boldsymbol{q}, \omega)}} \, ,$$

$$\overset{\substack{\text{lattice data for inclusive} \\ \uparrow}}{\boxed{C_{\mu\nu}(t)}} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \underset{\substack{\downarrow \\ \sum_{X_c} \langle B_s | J_\mu^\dagger | X_c \rangle \langle X_c | J_\nu | B_s \rangle \, \delta(\omega - E_{X_c})}}{\boxed{W_{\mu\nu}(\boldsymbol{q}, \omega)}} e^{-\omega t}$$

Inclusive observables from lattice data

For the inclusive case the observable we compute is $O \equiv \bar{X}$

$$\bar{X} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \underset{\text{Kernel}}{\underset{\downarrow}{\boxed{K_{\mu\nu}(\boldsymbol{q},\omega)}}} , \qquad \overset{\text{lattice data for inclusive}}{\overset{\uparrow}{\boxed{C_{\mu\nu}(t)}}} = \int_{\omega_0}^{\infty} \mathrm{d}\omega \underset{\sum_{X_c} \langle B_s | J_\mu^\dagger | X_c \rangle \langle X_c | J_\nu | B_s \rangle \, \delta(\omega - E_{X_c})}{\underset{\downarrow}{\boxed{W_{\mu\nu}(\boldsymbol{q},\omega)}}} e^{-\omega t}$$

Approximating the Kernel in polynomials in $e^{-a\omega}$ ($a = 1$ in lattice units)

$$K_{\mu\nu} \simeq c_{\mu\nu,0} + c_{\mu\nu,1} e^{-\omega} + \cdots + c_{\mu\nu,N} e^{-\omega N} \, ,$$

$$\Rightarrow \; \bar{X} \simeq c_{\mu\nu,0} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu}}_{C^{\mu\nu}(0)} + c_{\mu\nu,1} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} e^{-\omega}}_{C^{\mu\nu}(1)} + \cdots + c_{\mu\nu,N} \underbrace{\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} e^{-\omega N}}_{C^{\mu\nu}(N)}$$

Polynomial approximation strategies

$$K(\omega):\ [\underset{\displaystyle \omega_0 \in [0,\omega_{\min})}{\underset{\downarrow}{\boxed{\omega_0}}},\infty)\to\mathbb{R},\qquad K(\omega)\simeq\sum_{j}^{N}c_j\,\underset{\displaystyle \text{family of (shifted) polynomials in } e^{-\omega},\ \omega\in[\omega_0,\infty)}{\underset{\downarrow}{\boxed{\tilde{P}_j(\omega)}}}.$$

Polynomial approximation strategies

$$K(\omega):\ [\underset{\omega_0 \in [0, \omega_{\min})}{\underset{\downarrow}{\boxed{\omega_0}}}, \infty) \to \mathbb{R}, \qquad K(\omega) \simeq \sum_j^N c_j \underset{\text{family of (shifted) polynomials in } e^{-\omega},\ \omega \in [\omega_0, \infty)}{\underset{\downarrow}{\boxed{\tilde{P}_j(\omega)}}}\,.$$

Chebyshev approach

Standard Chebyshev polynomials:

$$T_j(\omega):\ [-1, 1] \to [-1, 1]\,,$$

$$K(\omega) \simeq \sum_{j=0}^N \tilde{c}_j \overset{\text{generic shifted Chebyshev}}{\overset{\uparrow}{\boxed{\tilde{T}_j(\omega)}}}\,,$$

$$\tilde{c}_j = \langle K, \tilde{T}_j \rangle\,.$$

Polynomial approximation strategies

$$K(\omega) : [\underset{\substack{\downarrow \\ \omega_0 \in [0, \omega_{\min})}}{\boxed{\omega_0}}, \infty) \to \mathbb{R}, \quad K(\omega) \simeq \sum_{j}^{N} c_j \underset{\substack{\downarrow \\ \text{family of (shifted) polynomials in } e^{-\omega},\ \omega \in [\omega_0, \infty)}}{\boxed{\tilde{P}_j(\omega)}} .$$

Chebyshev approach

Standard Chebyshev polynomials:

$$T_j(\omega) : [-1, 1] \to [-1, 1] \,,$$

$$K(\omega) \simeq \sum_{j=0}^{N} \tilde{c}_j \overset{\substack{\text{generic shifted Chebyshev} \\ \uparrow}}{\boxed{\tilde{T}_j(\omega)}} \,,$$

$$\tilde{c}_j = \langle K, \tilde{T}_j \rangle \,.$$

Backus-Gilbert approach

We minimise the functional (L_2-norm)

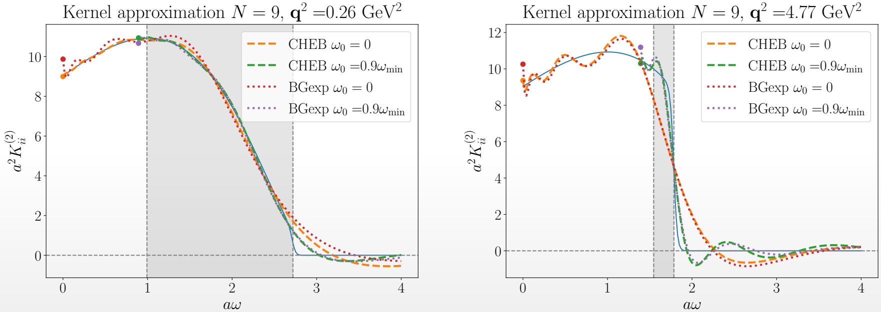
$$A[g] = \int_{\omega_0}^{\infty} \mathrm{d}\omega \underset{\substack{\downarrow \\ \text{weight factor to help numerical stability}}}{\boxed{\Omega(\omega)}} \left[K(\omega) - \sum_{j=0}^{N} g_j \tilde{P}_j(\omega) \right]^2 ,$$

$$g_j \quad \leftrightarrow \quad \frac{\partial A}{\partial g_j} = 0 \,.$$

Kernel: polynomial approximation

$$K_{\mu\nu}(\boldsymbol{q},\omega) = e^{2\omega t_0}\, k_{\mu\nu}(\boldsymbol{q},\omega)\, \boxed{\theta_\sigma\left(\omega_{\mathrm{max}} - \omega\right)} \rightarrow$$

smooth step-function (sigmoid):
cut the unphysical states above ω_{max}

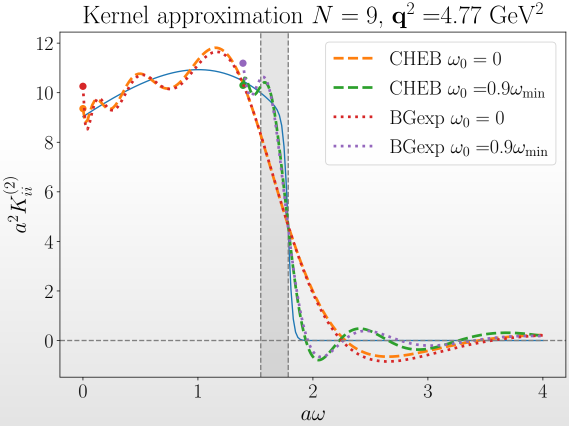


The kernel plotted here corresponds to the main contribution to the total decay rate from the $A_i A_i$ channel. The smearing parameter of the sigmoid $\theta_\sigma(x) = 1/(1+e^{-x/\sigma})$is kept fixed at $\sigma = 0.02$.

Kernel: polynomial approximation

$$K_{\mu\nu}(\boldsymbol{q},\omega) = e^{2\omega t_0}\, k_{\mu\nu}(\boldsymbol{q},\omega)\, \boxed{\theta_\sigma\left(\omega_{\max} - \omega\right)} \rightarrow$$

smooth step-function (sigmoid):
cut the unphysical states above $\omega_{\max}$



Analysis strategy

Problem: data too noisy, statistical errors add up with a “naive” approach!

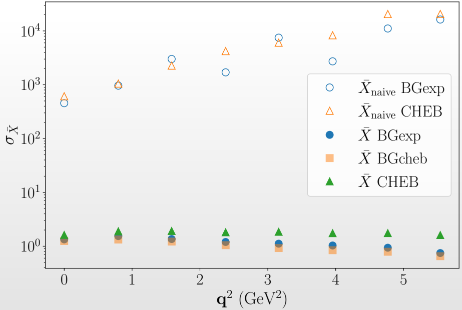
$$\bar{X}_{\mathrm{naive}} \simeq c_{\mu\nu,0} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{P}_0(\omega) + \cdots + c_{\mu\nu,N} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{P}_N(\omega) = \sum_{k=0}^{N} \bar{c}_{\mu\nu,k} C_{\mu\nu}(k) \, .$$

Analysis strategy

Problem: data too noisy, statistical errors add up with a “naive” approach!

$$\bar{X}_{\text{naive}} \simeq c_{\mu\nu,0} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{P}_0(\omega) + \cdots + c_{\mu\nu,N} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{P}_N(\omega) = \sum_{k=0}^{N} \bar{c}_{\mu\nu,k} C_{\mu\nu}(k) \,.$$

We need to add a correction term $\bar{X} = \bar{X}_{\text{naive}} + \delta\bar{X}$ (essentially a noisy zero) that takes care of the variance reduction.

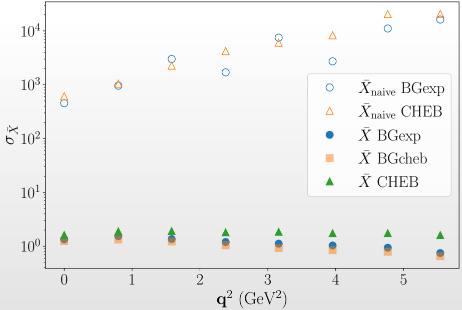


Analysis strategy

Problem: data too noisy, statistical errors add up with a “naive” approach!

$$\bar{X}_{\text{naive}} \simeq c_{\mu\nu,0}\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu}\,\tilde{P}_0(\omega) + \cdots + c_{\mu\nu,N}\int_{\omega_0}^{\infty} \mathrm{d}\omega\, W^{\mu\nu}\,\tilde{P}_N(\omega) = \sum_{k=0}^{N}\bar{c}_{\mu\nu,k}C_{\mu\nu}(k)\,.$$

We need to add a correction term $\bar{X} = \bar{X}_{\text{naive}} + \delta\bar{X}$ (essentially a noisy zero) that takes care of the variance reduction.



$$\begin{cases} \delta\bar{X}^{\text{CHEB}} = \sum_{k=0}^{N} c_{\mu\nu,k}\,\boxed{\delta C_{\mu\nu}(k)} \\ \delta\bar{X}^{\text{BG}} = \sum_{k=0}^{N} \boxed{\delta g_{\mu\nu,k}}\, C_{\mu\nu}(k) \end{cases}$$

acts on the data
(by imposing
rigourous bounds)

acts on the coefficients
(by including info
from the data)

Analysis strategy: Chebyshev

[Bailas et al. (2020)[6]]

For Chebyshev polynomials we have $\tilde{P}_k(\omega) \equiv \tilde{T}_k(\omega)$

$$\bar{X}_{\text{naive}} \simeq c_{\mu\nu,0} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_0(\omega) + \cdots + c_{\mu\nu,N} \int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_N(\omega) \,, \quad \tilde{T}_k(\omega) = \sum_{j=0}^{k} \tilde{t}_j^{(k)} e^{-j\omega}$$

Chebyshev polynomials are **bounded**, so we normalize

$$\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_k(\omega) \quad \rightarrow \quad -1 \leq \boxed{\frac{\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_k(\omega)}{\int_{\omega_0}^{\infty} \mathrm{d}\omega \, W^{\mu\nu} \tilde{T}_0(\omega)}} \leq 1 \,.$$

$\downarrow$

Chebyshev matrix element $\langle \tilde{T}_k \rangle_{\mu\nu} = \sum_{j=0}^{k} \tilde{t}_j^{(k)} C_{\mu\nu}(j)/C_{\mu\nu}(0)$

We extract $\langle \tilde{T}_k \rangle_{\mu\nu}$ from a **Bayesian fit with constraints** to the data such that

$$C_{\mu\nu}^{\text{fit}}(k) = C_{\mu\nu}(0) \sum_{j=0}^{k} \tilde{a}_j^{(k)} \langle \tilde{T}_j \rangle_{\mu\nu} \quad \Rightarrow \quad \delta C_{\mu\nu}(k) = C_{\mu\nu}^{\text{fit}}(k) - C_{\mu\nu}(k) \,.$$

Analysis strategy - Backus-Gilbert

[Hansen et al. (2019)[7], Bulava et al. (2021)[8], Alexandrou et al. (2023)[9]]

systematic error

$$\boxed{A_{\mu\nu}[g]} = \int_{\omega_0}^{\infty} \mathrm{d}\omega\, \Omega(\omega) \left[K_{\mu\nu}(\omega, \boldsymbol{q}) - \sum_{k=1}^{N} g_{\mu\nu,k} \tilde{P}_k(\omega)\right]^2 , \qquad \tilde{P}_k(\omega) = \sum_{j=0}^{k} \tilde{p}_j^{(k)} e^{-j\omega} ,$$

$$\boxed{B_{\mu\nu}[g]} = \sigma^2_{\bar{X}} = \sum_{i,j=1}^{N} g_{\mu\nu,i} \mathrm{Cov}[\bar{C}^P_{\mu\nu}(i), \bar{C}^P_{\mu\nu}(j)] g_{\mu\nu,j} , \qquad \bar{C}^P_{\mu\nu}(k) = \sum_{j=0}^{k} \tilde{p}_j^{(k)} \bar{C}_{\mu\nu}(j) .$$

statistical error

We minimise the functional

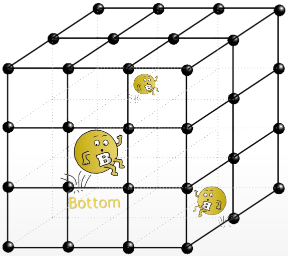
$$\boxed{F_{\mu\nu,\theta}[g] = A_{\mu\nu}[g] + \theta^2 B_{\mu\nu}[g]} , \qquad \frac{\partial F_{\mu\nu,\theta}}{\partial g_{\mu\nu,k}} = 0 .$$

and choose the value θ^{*2} that achieve optimal balance between statistical and systematical error, i.e. $A_{\mu\nu}[g^*] = B_{\mu\nu}[g^*]$

$$\delta g_{\mu\nu,j} = g_{\mu\nu,j}\Big|_{\theta^2 \neq 0} - g_{\mu\nu,j}\Big|_{\theta^2 = 0} .$$

Inclusive decays on the lattice: setup

Simulations carried out on the DiRAC Extreme Scaling service at the University of Edinburgh using the **Grid**[Boyle et al.[10]] and **Hadrons**[Portelli et al.[11]] software packages



Pilot study with RBC/UKQCD 2+1 flavour ensembles [Allton et al. (2008)[12]]:

- lattice size: $24^3 \times 64$;
- lattice spacing $a \simeq 0.11\,\text{fm}$;
- $M_\pi \simeq 330\,\text{MeV}$.

$$\Downarrow$$

Limited statistics/qualitative results!

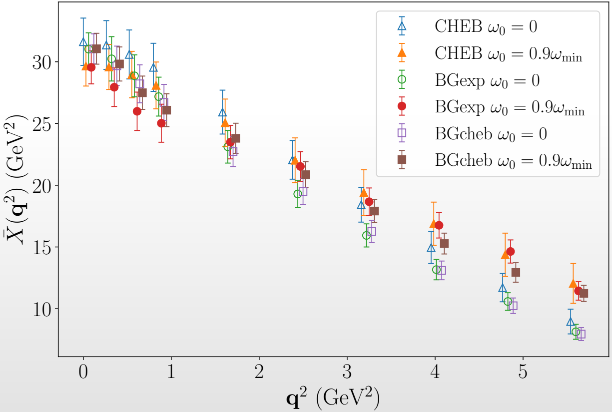
Simulation:

- b quark simulated at its **physical** mass (RHQ action [El-Khadra et al. (1997)[13], Christ et al. (2007)[14], Lin and Christ (2007)[15]]);
- s, c quarks simulated at **near-to-physical** mass (DWF action [Shamir (1993)[16], Furman and Shamir (1994)[17]]).

Results and comparison: decay rate

[Barone et al. (2023)[2]]

Comparison of Chebyshev (**CHEB**) and Backus-Gilbert (**BG**) approaches with different values of starting point of the approximation ω_0 and $\begin{cases} \textbf{exponential (BGexp)} & \tilde{P}_k(\omega) = e^{-k\omega} \\ \textbf{Chebyshev (BGcheb)} & \tilde{P}_k(\omega) = \tilde{T}_k(\omega) \end{cases}$ basis.

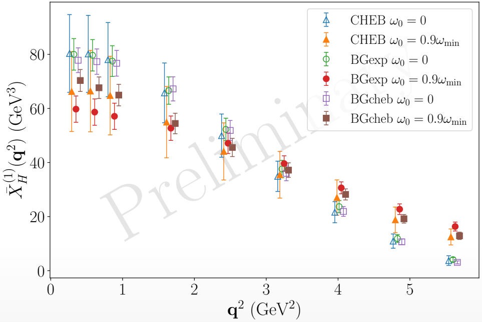


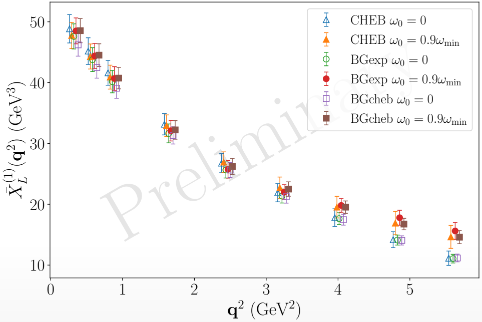
The smearing parameter of the sigmoid in the kernels is fixed at $\sigma = 0.02$.

Key points:

- Chebyshev and Backus-Gilbert approaches are fully compatible;
- pilot study:
  - values are in the right ballpark;
  - low statistics, roughly 5% statistical error.

Results and comparison: moments





Compatible results between the two approaches also for

- hadronic mass moments $\bar{X}_H^{(1)}$;
- lepton moments $\bar{X}_L^{(1)}$.

$\Rightarrow$ prospects for computing more observables (e.g. differential and central moments) and compare with continuum approaches.

Summary and outlook

Summary:

- full and flexible setup for studying inclusive semileptonic decays on the lattice;
- double approach for the analysis: Chebyshev and Backus-Gilbert approaches compatible within error;
- prospects for phenomenology studies and comparison with continuum approaches.

Coming next:

- continue to work towards understanding the systematics involved in solving the inverse problem;
- dedicated simulations to address the systematics for polynomial approximation, finite volume effects (→ see **Ryan Kellermann**'s talk, Mon 17:20), continuum limit,...;
- prepare for a full study B_s/B.

Summary and outlook

Summary:

- full and flexible setup for studying inclusive semileptonic decays on the lattice;
- double approach for the analysis: Chebyshev and Backus-Gilbert approaches compatible within error;
- prospects for phenomenology studies and comparison with continuum approaches.

Coming next:

- continue to work towards understanding the systematics involved in solving the inverse problem;
- dedicated simulations to address the systematics for polynomial approximation, finite volume effects (→ see **Ryan Kellermann**'s talk, Mon 17:20), continuum limit,...;
- prepare for a full study B_s/B.

THANK YOU!

References I

[1] Y. Aoki et al., "FLAG Review 2021", Chapter of the Flag Review 2021 **10**, 9849 (2021), arXiv:2111.09849.

[2] A. Barone et al., "Approaches to inclusive semileptonic $B_{(s)}$-meson decays from Lattice QCD", JHEP **07**, 145 (2023), arXiv:2305.14092 [hep-lat].

[3] M. T. Hansen et al., "From deep inelastic scattering to heavy-flavor semileptonic decays: Total rates into multihadron final states from lattice QCD", Phys. Rev. D **96**, 094513 (2017), arXiv:1704.08993 [hep-lat].

[4] S. Hashimoto, "Inclusive semi-leptonic B meson decay structure functions from lattice QCD", Progress of Theoretical and Experimental Physics **2017**, 53–56 (2017), arXiv:1703.01881.

[5] P. Gambino and S. Hashimoto, "Inclusive Semileptonic Decays from Lattice QCD", PHYSICAL REVIEW LETTERS **125**, 32001 (2020).

[6] G. Bailas et al., "Reconstruction of smeared spectral functions from Euclidean correlation functions", Progress of Theoretical and Experimental Physics **2020**, 43–50 (2020), arXiv:2001.11779.

[7] M. Hansen et al., "Extraction of spectral densities from lattice correlators", Physical Review D **99**, 10.1103/PhysRevD.99.094508 (2019).

References II

[8] J. Bulava et al., “Inclusive rates from smeared spectral densities in the two-dimensional O(3) non-linear σ-model”, `https://doi.org/10.48550/arXiv.2111.12774` (2021), arXiv:`2111.12774v1`.

[9] C. Alexandrou et al., “Probing the Energy-Smeared R Ratio Using Lattice QCD”, Phys. Rev. Lett. **130**, 241901 (2023), arXiv:`2212.08467 [hep-lat]`.

[10] P. Boyle et al., *Grid: Data parallel C++ mathematical object library*, `https://github.com/paboyle/Grid`.

[11] A. Portelli et al., *Hadrons: Grid-based workflow management system for lattice field theory simulations*, `https://github.com/aportelli/Hadrons`.

[12] C. Allton et al., “Physical results from 2+1 flavor domain wall QCD and SU(2) chiral perturbation theory”, Physical Review D - Particles, Fields, Gravitation and Cosmology **78**, `10.1103/PhysRevD.78.114509` (2008), arXiv:`0804.0473`.

[13] A. X. El-Khadra et al., “Massive fermions in lattice gauge theory”, Physical Review D - Particles, Fields, Gravitation and Cosmology **55**, 3933–3957 (1997), arXiv:`9604004 [hep-lat]`.

[14] N. H. Christ et al., “Relativistic heavy quark effective action”, `10.1103/PhysRevD.76.074505` (2007).

References III

[15] H. W. Lin and N. Christ, “Nonperturbatively determined relativistic heavy quark action”, Physical Review D - Particles, Fields, Gravitation and Cosmology **76**, `10.1103/PhysRevD.76.074506` (2007), arXiv:0608005 `[hep-lat]`.

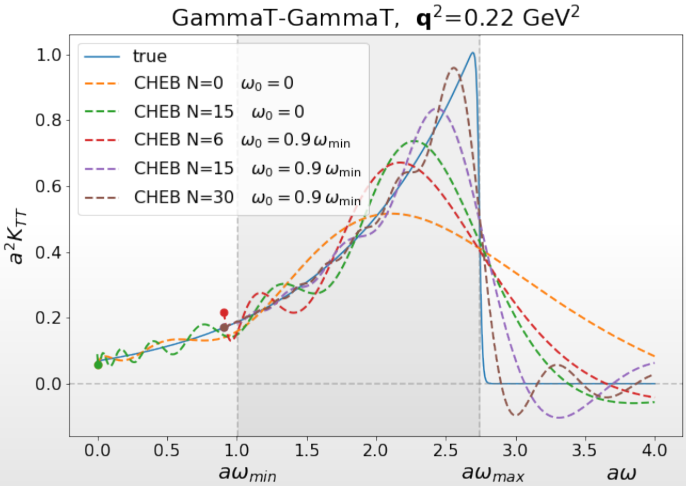
[16] Y. Shamir, “Chiral Fermions from Lattice Boundaries”, Nuclear Physics, Section B **406**, 90–106 (1993), arXiv:9303005v1 `[hep-lat]`.

[17] V. Furman and Y. Shamir, “Axial symmetries in lattice QCD with Kaplan fermions”, Nuclear Physics, Section B **439**, 54–78 (1994), arXiv:9405004v2 `[hep-lat]`.

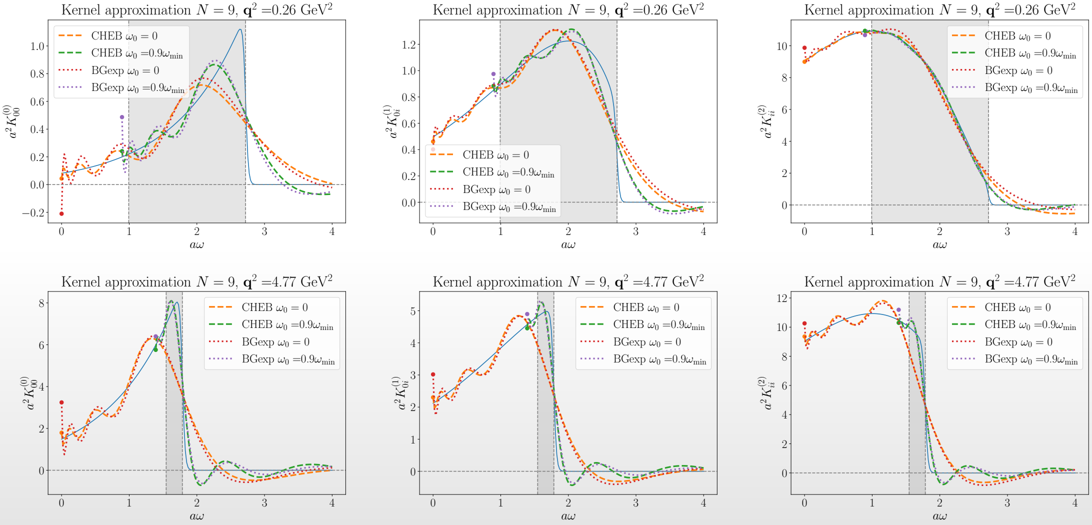
Backup

BACKUP

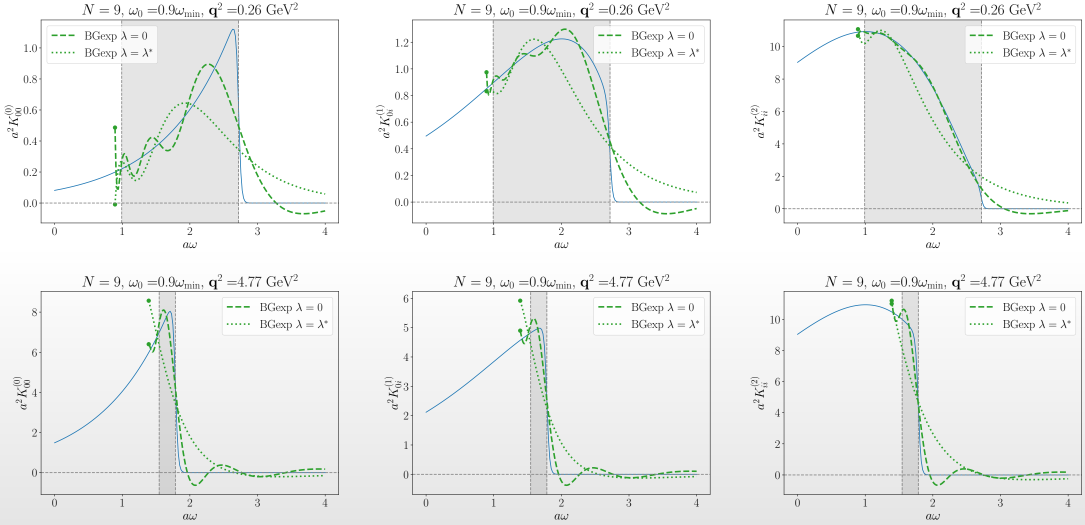
Chebyshev polynomial approximation: more



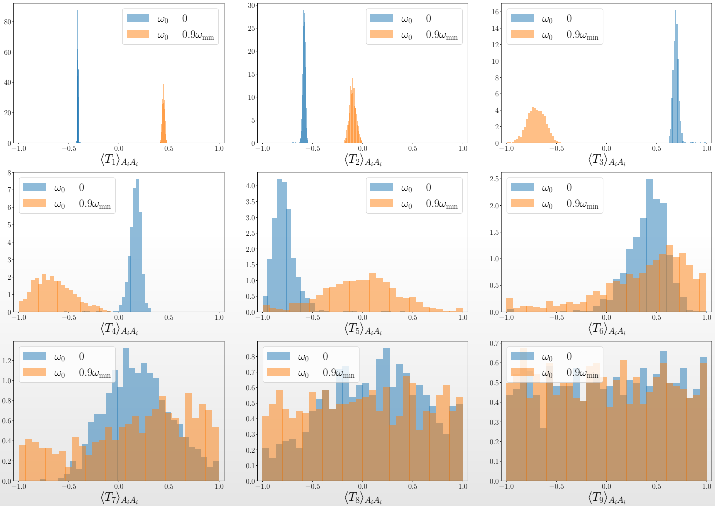
Kernels


Kernel approximation $N = 9$, $\mathbf{q}^2 = 0.26$ GeV^2
Kernel approximation $N = 9$, $\mathbf{q}^2 = 0.26$ GeV^2
Kernel approximation $N = 9$, $\mathbf{q}^2 = 0.26$ GeV^2
Kernel approximation $N = 9$, $\mathbf{q}^2 = 4.77$ GeV^2
Kernel approximation $N = 9$, $\mathbf{q}^2 = 4.77$ GeV^2
Kernel approximation $N = 9$, $\mathbf{q}^2 = 4.77$ GeV^2
CHEB $\omega_0 = 0$
CHEB $\omega_0 = 0.9\omega_{\min}$
BGexp $\omega_0 = 0$
BGexp $\omega_0 = 0.9\omega_{\min}$
$a^2 K_{00}^{(0)}$
$a^2 K_{0i}^{(1)}$
$a^2 K_{ii}^{(2)}$
$a\omega$
0
1
2
3
4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
6
8
10
12

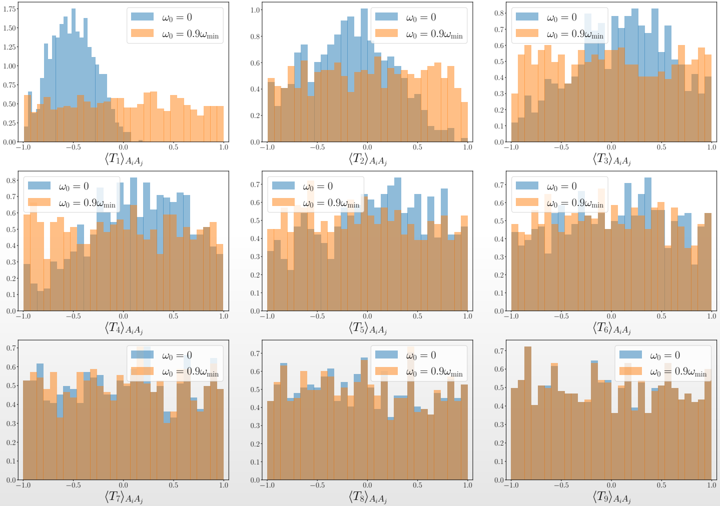
Kernels Backus-Gilbert



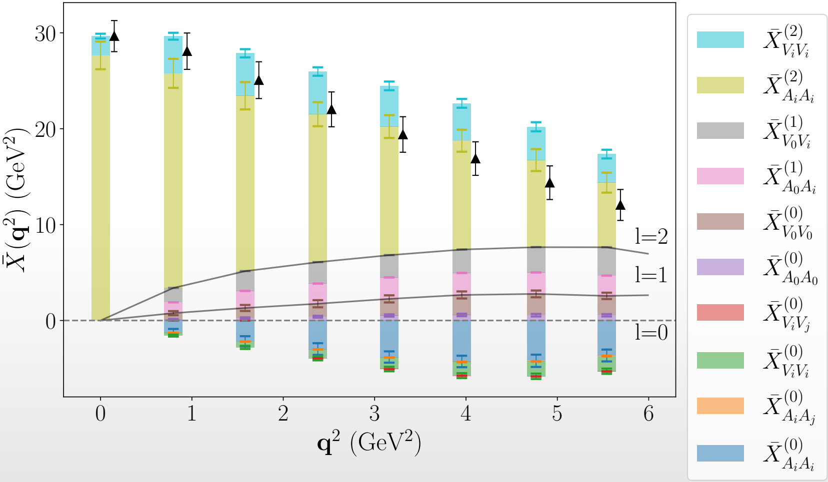
Chebyshev data reconstruction - A_iA_i distribution



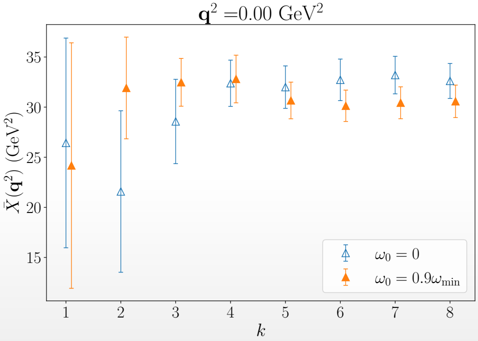
Chebyshev data reconstruction - A_iA_j distribution

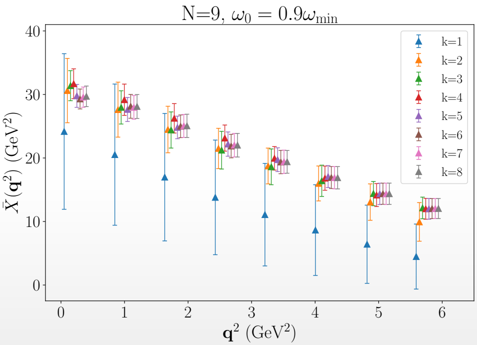


$\bar{X}$ contributions

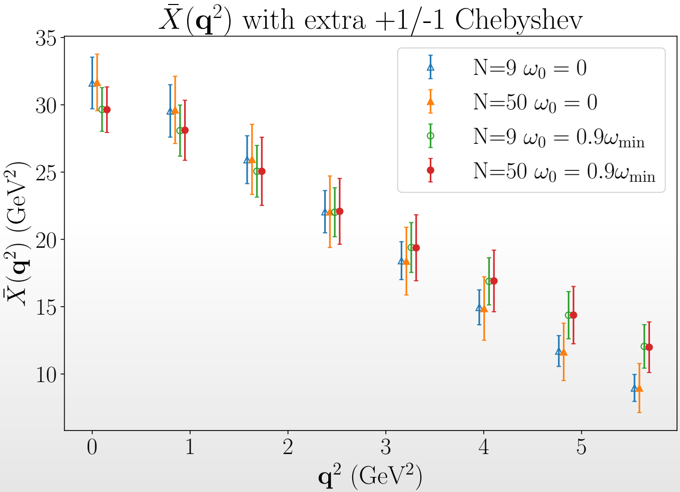


Systematics from Chebyshev

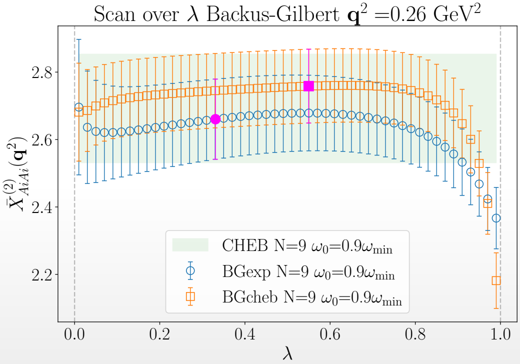


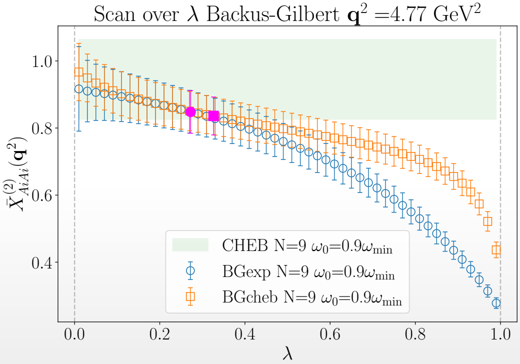


Systematics from Chebyshev



Scan over λ (Backus-Gilbert)





Ground state limit

As a useful cross-check of the inclusive analysis, we can consider the limit where only the ground state dominates, i.e.

$$W_{\mu\nu} \quad \to \quad \delta(\omega - E_{D_s}) \frac{1}{4 M_{B_s} E_{D_s}} \langle B_s | J_\mu^\dagger | D_s^{(*)} \rangle \langle D_s^{(*)} | J_\nu | B_s \rangle$$

If we decompose

$$\bar{X} = \bar{X}^{\parallel} + \bar{X}^{\perp}$$

and restrict to the **vector currents (VV)** the matrix element can be decomposed as

$$\langle D_s | V_\mu | B_s \rangle = f_+(q^2)(p_{B_s} + p_{D_s})_\mu + f_-(q^2)(p_{B_s} - p_{D_s})_\mu$$

and we can show that

$$\bar{X}_{VV}^{\parallel} \quad \to \quad \frac{M_{B_s}}{E_{D_s}} \boldsymbol{q}^2 |f_+(q^2)|^2$$

Exclusive decay limit

The matrix element and form factors can be extracted from **3pt-correlation functions**. From that we can generate mock data for the 4pt functions (where only the ground state contributes)

$$C^{G}_{\mu\nu} = \frac{1}{4M_{B_s}E_{D_s}} \langle B_s| V^{\dagger}_{\mu} |D_s\rangle\langle D_s| V_{\nu} |B_s\rangle e^{-E_{D_s}t}$$

and run the analysis!