The Thirring Model in 2+1d with Optimised Domain Wall Fermions

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The Thirring Model in 2+1d

$$\mathscr{L} = \bar{\psi}_i(\partial\!\!\!/ + m)\psi_i + \frac{g^2}{2N}(\bar{\psi}_i\gamma_\mu\psi_i)^2$$

Covariant quantum field theory of
N flavors of interacting fermion in 2+1 dimensions.
Fermions are spinor fields $\psi, \bar{\psi}$ acted on by 4x4 Dirac matrices γ_μ

Interaction between conserved currents:
like charges ***repel***, opposite charges ***attract***

$$\partial\!\!\!/ \equiv \partial_\mu\gamma_\mu \quad \mu = 0, 1, 2 \qquad i = 1, \ldots, N$$

$$\mathrm{tr}(\gamma_\mu\gamma_\mu) = 4 \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$$

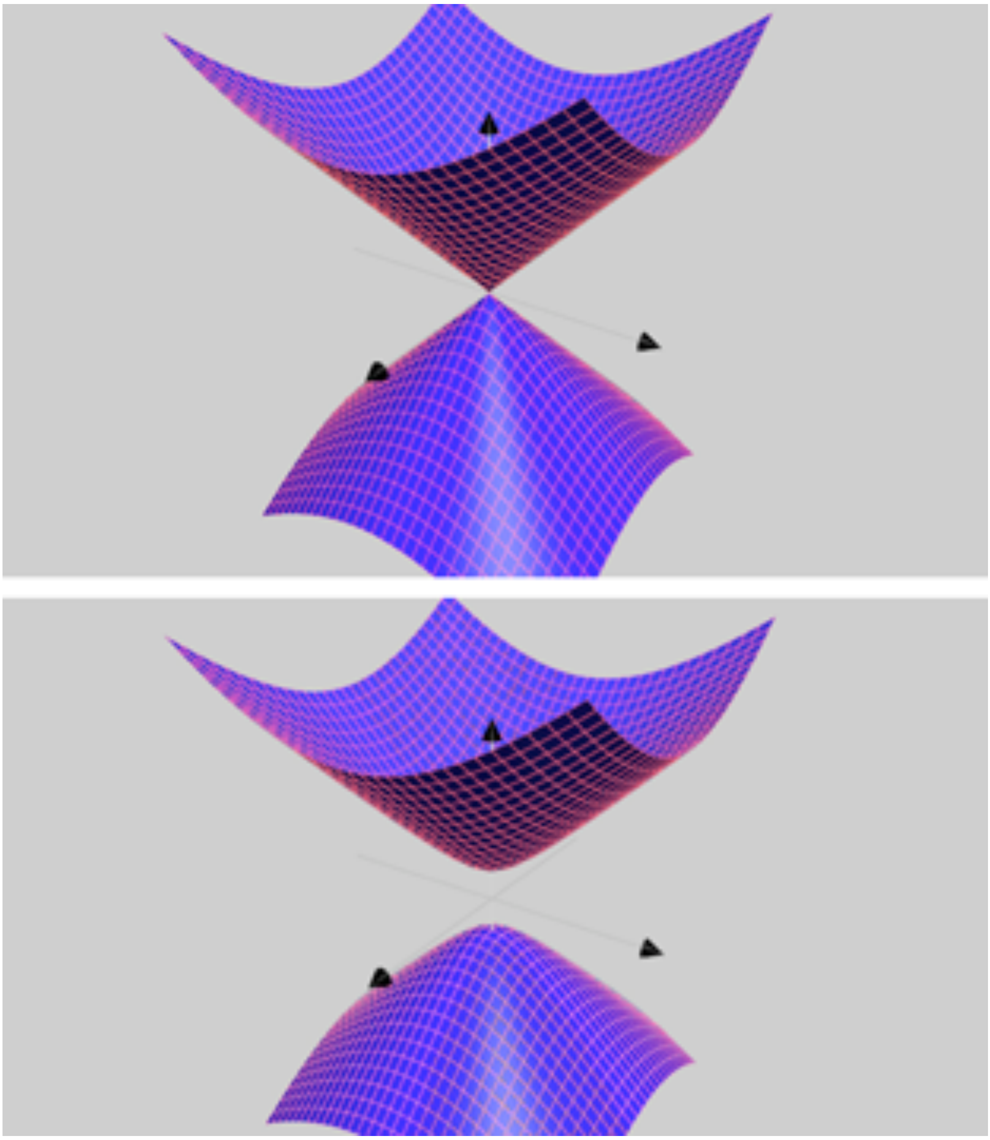
$$\mu, \nu = 0,1,2,3$$

For sufficiently large self-interaction g^2, and sufficiently small N, the Fock vacuum is conceivably disrupted by a particle-hole **bilinear condensate**

$$\langle\bar{\psi}\psi\rangle \equiv \frac{\partial \ln Z}{\partial m} \neq 0$$

resulting in a dynamically-generated mass gap at the Dirac point

semi-metal → insulator



Cf. chiral symmetry breaking in QCD

Hypothesis:

the transition at $g_c^2(N)$ defines a **Quantum Critical Point** whose universal properties perhaps characterise low-energy excitations in graphene…

D.T. Son, Phys. Rev. B**75** (2007) 235423

Corresponds to a new strongly-interacting QFT…

…a priori no small dimensionless parameters

Continuum Symmetries in $d = 2 + 1$

$$\mathcal{S} = \int d^3x \; \bar{\Psi}(\gamma_\mu \partial_\mu)\Psi \; + \; m\bar{\Psi}\Psi$$

For $m=0$ S is invariant under global U(2N) symmetry generated by

(i) $\Psi \mapsto e^{i\alpha}\Psi \, ; \qquad \bar{\Psi} \mapsto \bar{\Psi}e^{-i\alpha},$ (ii) $\Psi \mapsto e^{i\alpha\gamma_5}\Psi \, ; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{i\alpha\gamma_5}$

(iii) $\Psi \mapsto e^{\alpha\gamma_3\gamma_5}\Psi \, ; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{-\alpha\gamma_3\gamma_5}$, (iv) $\Psi \mapsto e^{i\alpha\gamma_3}\Psi \, ; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{i\alpha\gamma_3}$

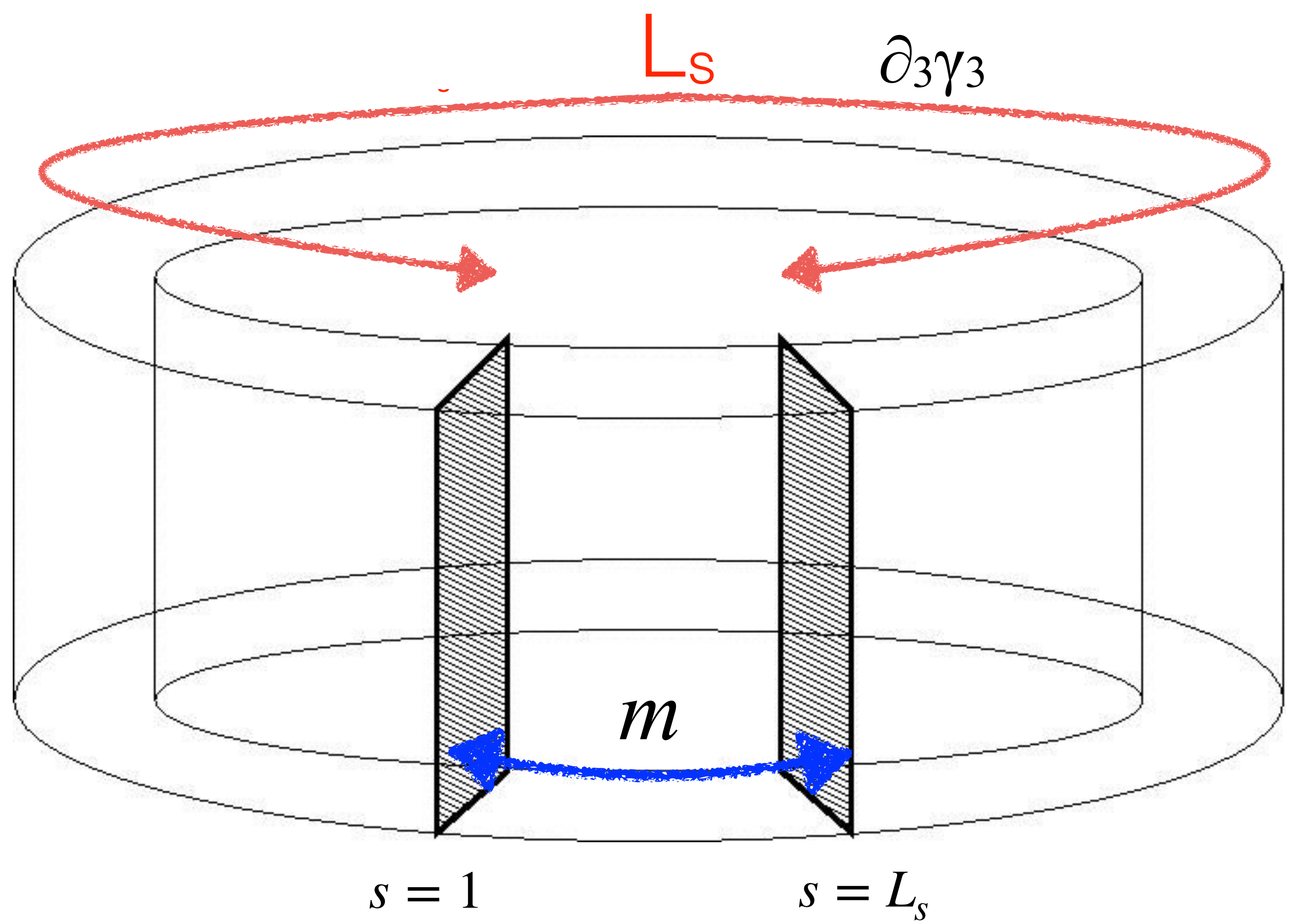
For $m \neq 0$, γ_3 (iv) and γ_5 (ii) rotations are no longer symmetries

$\Rightarrow$ U(2N) → U(N)⊗U(N)

Cf. models based on staggered/Kähler-Dirac formulations:

$\Rightarrow$ U(N)⊗U(N) → U(N)

Domain Wall Fermions



$$\mathscr{L} = \bar{\Psi}(x,s) D_{DWF} \Psi(y,s')$$

Fermions propagate freely along a fictitious third direction of extent L_s with open boundaries

Basic idea as $L_s \to \infty$:

- zero-modes of D_{DWF} localised on walls are ± eigenmodes of γ_3
- Modes propagating in bulk can be decoupled (with cunning)

"Physical" fields in 2+1d target space

$$\psi(x) = P_- \Psi(x,1) + P_+ \Psi(x,L_s);$$
$$\bar{\psi}(x) = \bar{\Psi}(x,L_s) P_- + \bar{\Psi}(x,1) P_+;$$

with projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_3)$

The closest we can get to U(2) symmetry is articulated by the **Ginsparg-Wilson** relations:

$$\{\gamma_3, D\} = 2D\gamma_3 D \quad \{\gamma_5, D\} = 2D\gamma_5 D \qquad [\gamma_3\gamma_5, D] = 0$$

satisfied by the 2+1d *overlap* operator

$$D_{ov} = \frac{1}{2}\left[(1+m_h) + (1-m_h)\frac{\mathscr{A}}{\sqrt{\mathscr{A}^\dagger \mathscr{A}}}\right]$$

with, eg.

Shamir kernel $\mathscr{A} = [2 + D_W - M]^{-1}[D_W - M]$ D_W local; $Ma = O(1)$

locality of D_{ov} not manifest but confirmed numerically

SJH, Mesiti, Worthy PRD **102** (2020) 094502

DWF provide a regularisation of overlap with a *local* kernel in 2+1+1d

ie. $$\frac{\det D_{\rm DWF}(m_i)}{\det D_{\rm DWF}(m_h = 1)} = \det D_{L_s}(m_i)$$ with $$\lim_{L_s\to\infty} D_{L_s} = D_{ov}$$

SJH PLB **754** (2016) 264

Formulational issues

By analogy with QCD, formulate auxiliary field $A_\mu(x)$ throughout bulk and 3-static ie. $\partial_3 A_\mu = 0$:

⇔ A_μ couples to conserved DWF fermion current

$$\mathcal{S} = \bar{\Psi}\mathcal{D}\Psi = \bar{\Psi}D_W\Psi + \bar{\Psi}D_3\Psi + m_i S_i \quad \text{with} \quad \begin{aligned} D_W &= \gamma_\mu D_\mu - (\hat{D}^2 + M); \\ D_3 &= \gamma_3\partial_3 - \hat{\partial}_3^2, \end{aligned}$$

NB $D_\mu \propto (1 + iA_\mu)$, not e^{iA_μ},

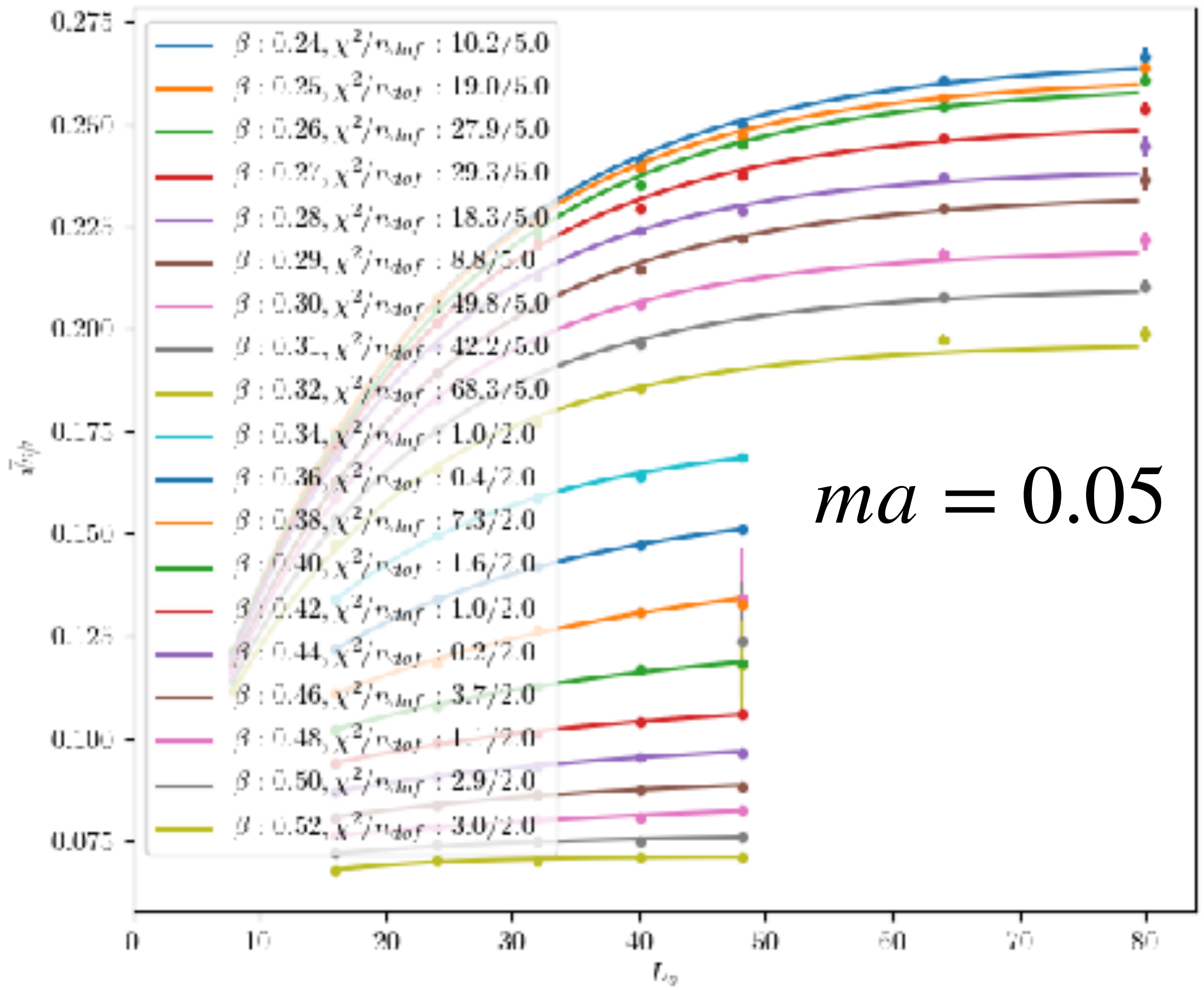
ie. links are ***non-compact*** and ***non-unitary***

$[\partial_3, D_\mu] = [\partial_3, \hat{D}^2] = 0$ but $[\partial_3, \hat{\partial}_3^2] \neq 0$ on walls

obstruction to proving $\det\mathscr{D} > 0$

RHMC with measure $\sqrt{\det(\mathscr{D}^\dagger\mathscr{D})}$ **for** $N = 1$

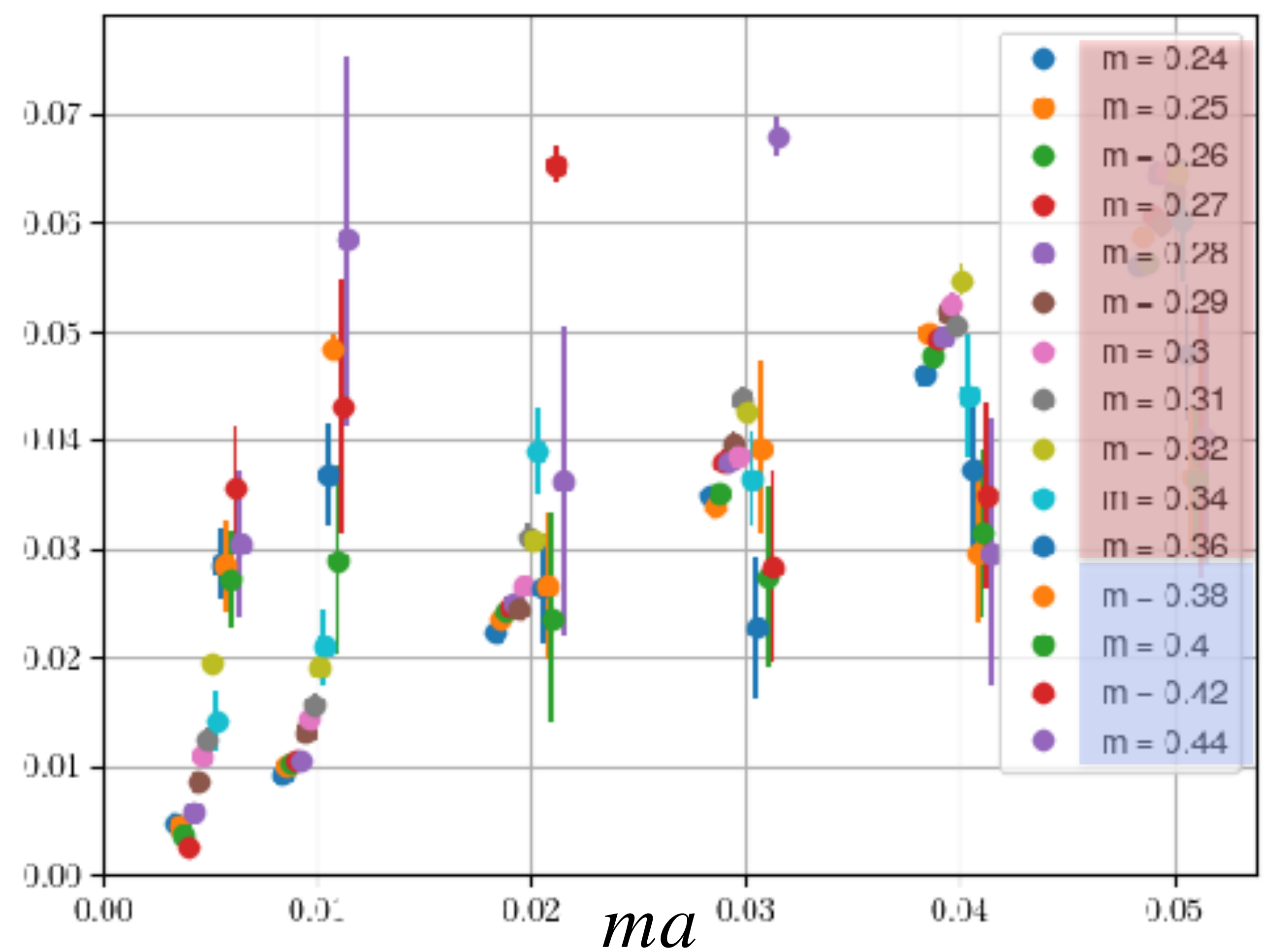
Stress-testing DWF…



$$\langle\bar\psi\psi\rangle_\infty - \langle\bar\psi\psi\rangle_{L_s} = A(\beta,m)e^{-\Delta(\beta,m)L_s}$$

Have $L_s = 8,16,\ldots,80$

Decay constant $\Delta(\beta,m)$:



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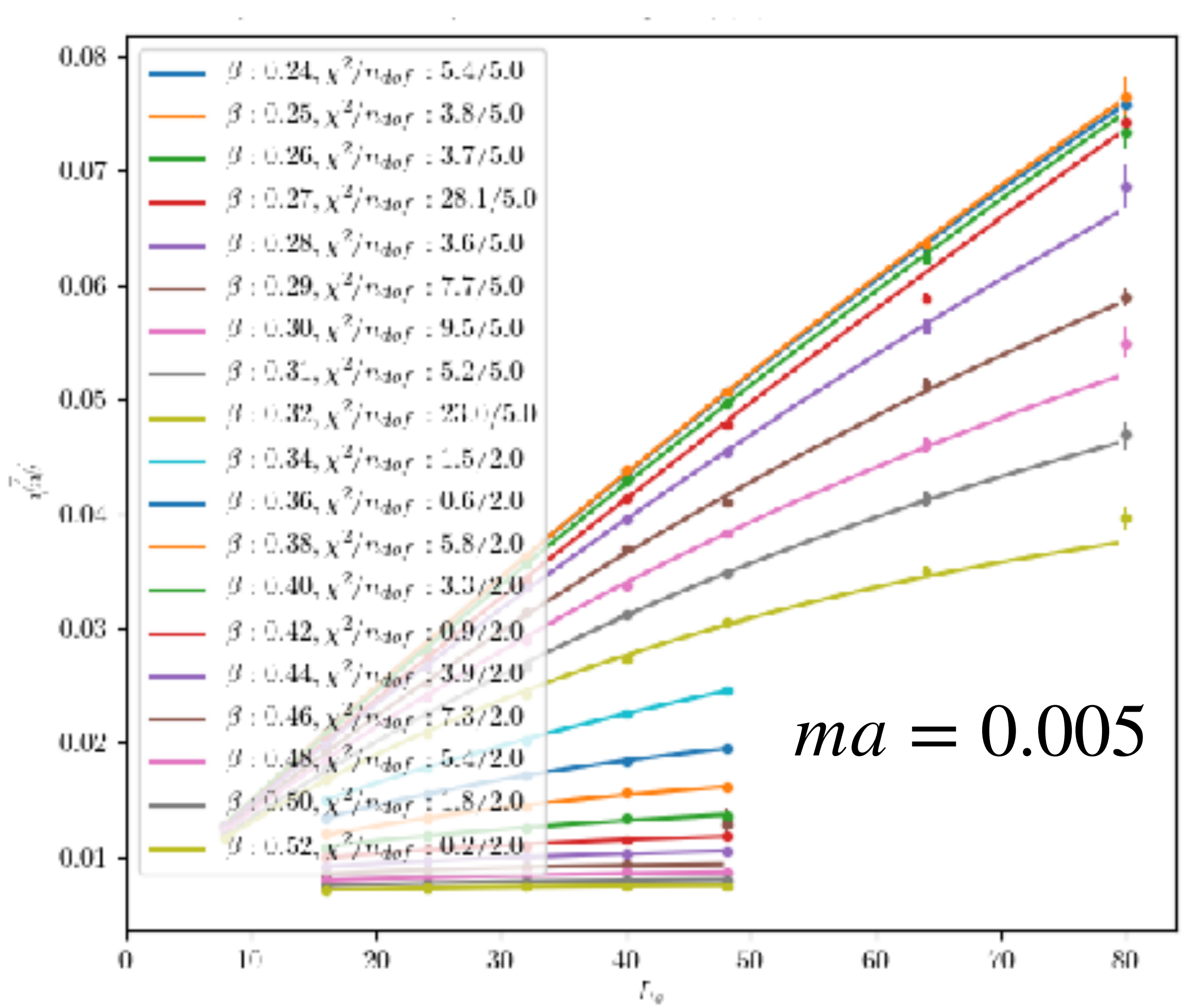
$L_s \to \infty$ not yet under control
at lightest masses, strongest couplings



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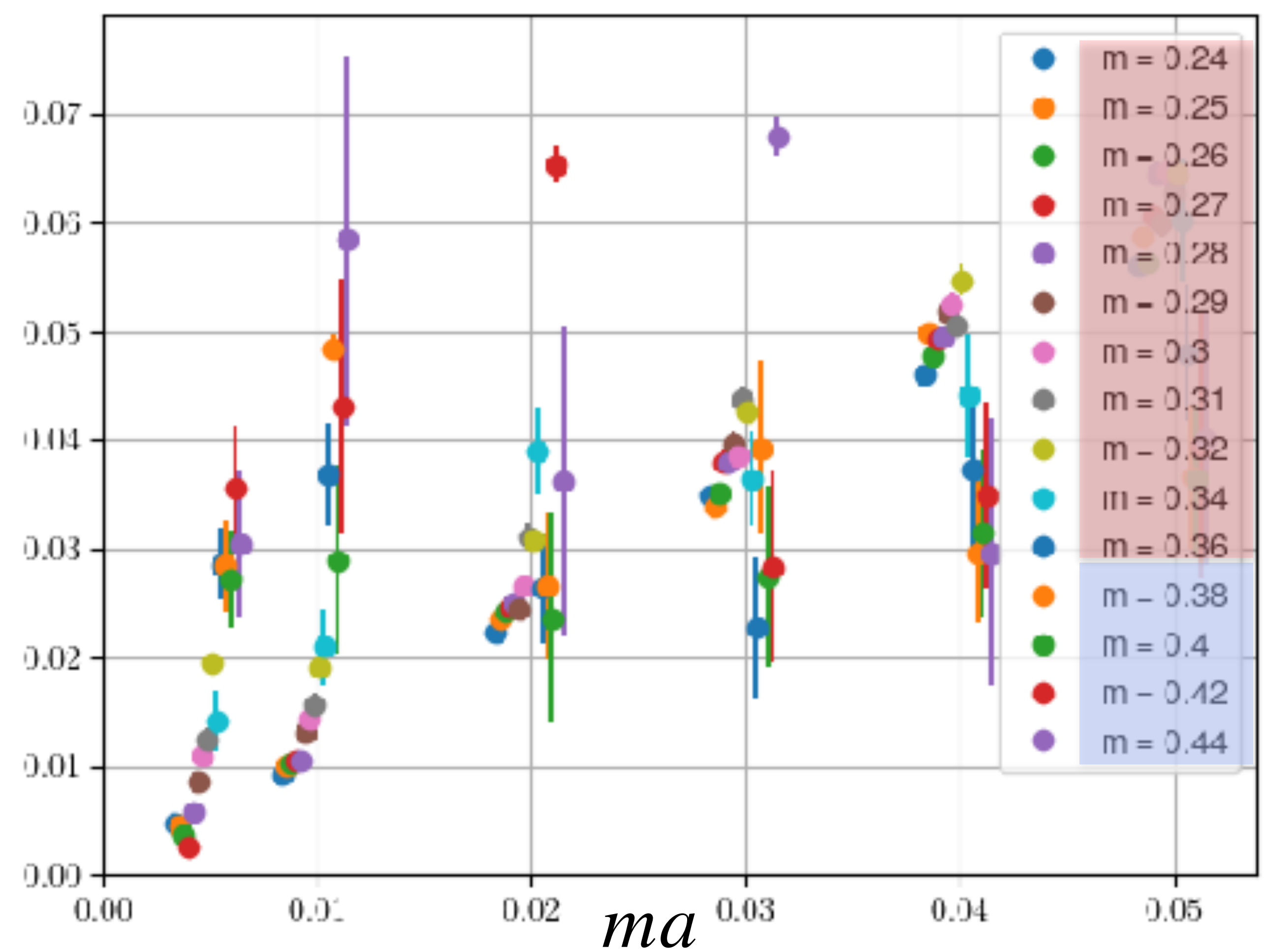

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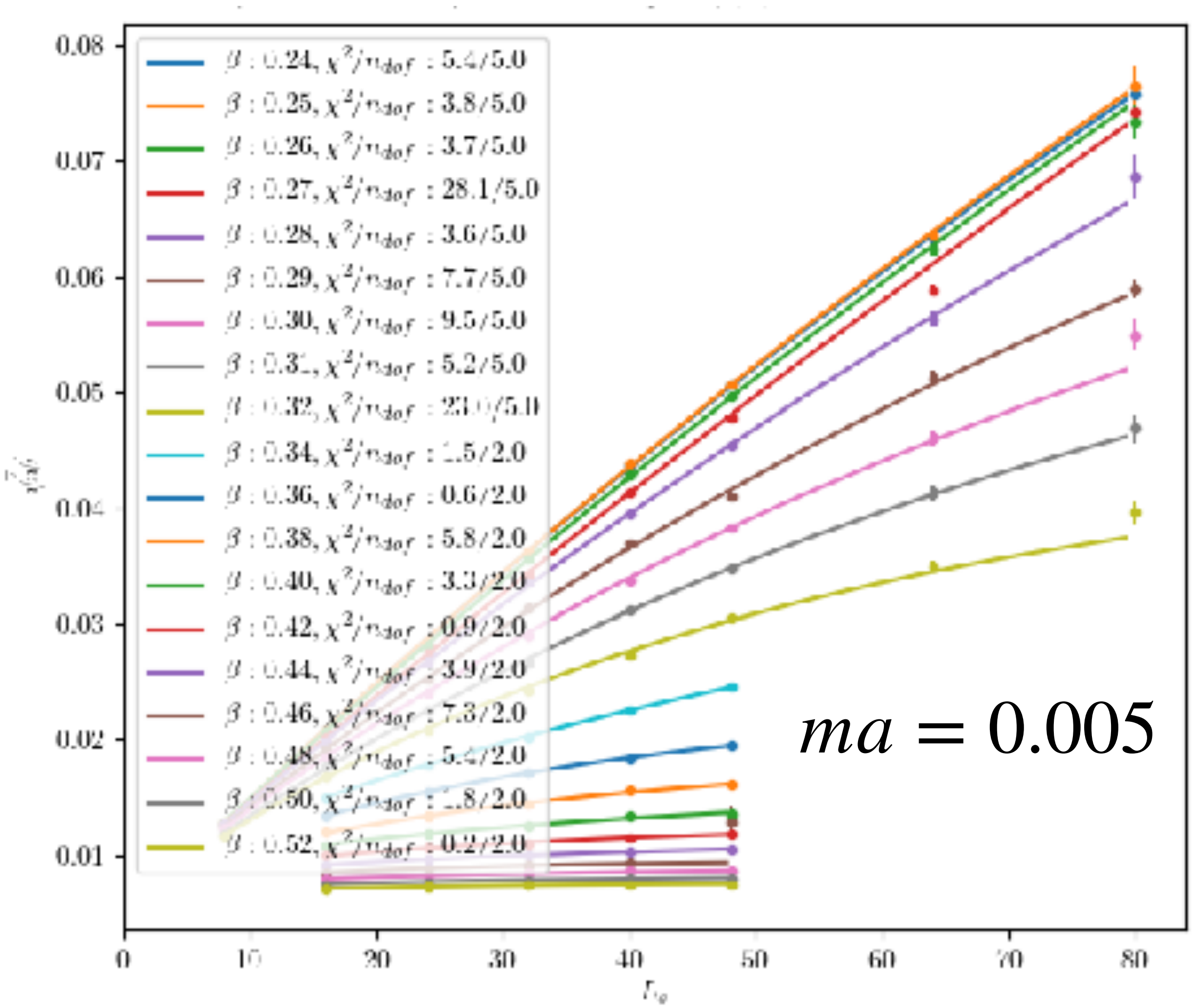

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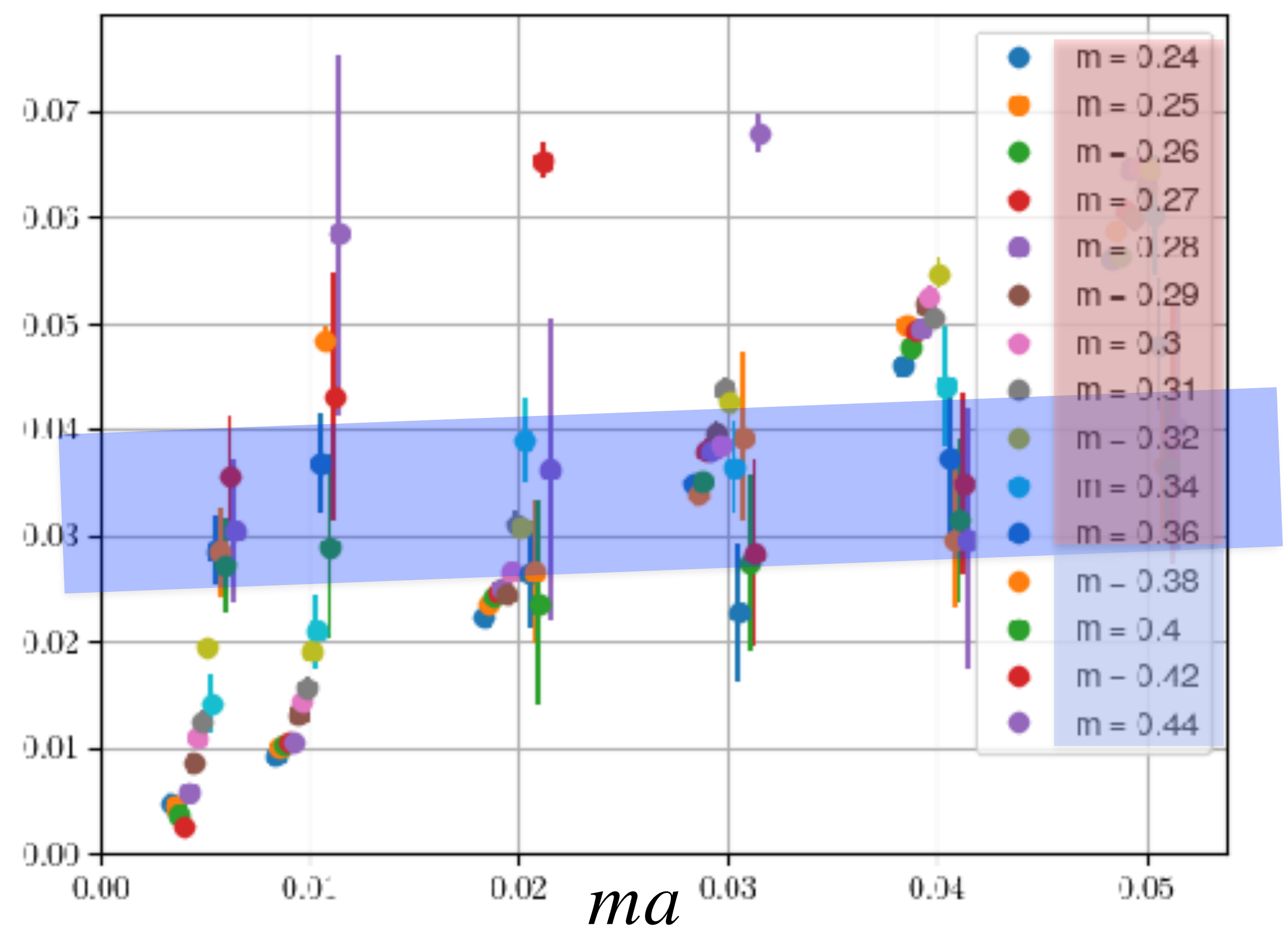
$L_s \to \infty$ not yet under control at lightest masses, strongest couplings

Decay constant $\Delta(\beta, m)$:

$\sim \propto m^0$ at weak coupling



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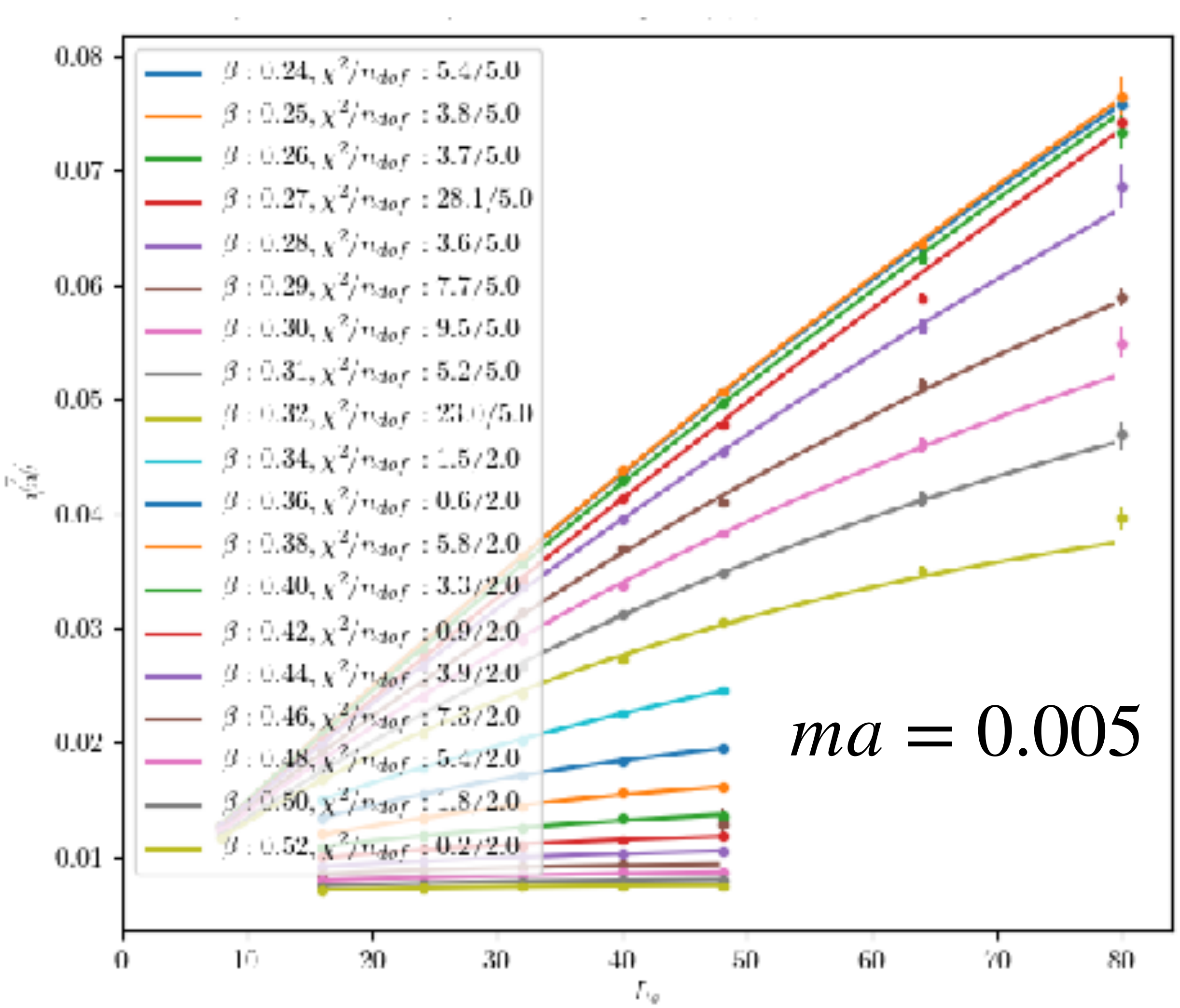

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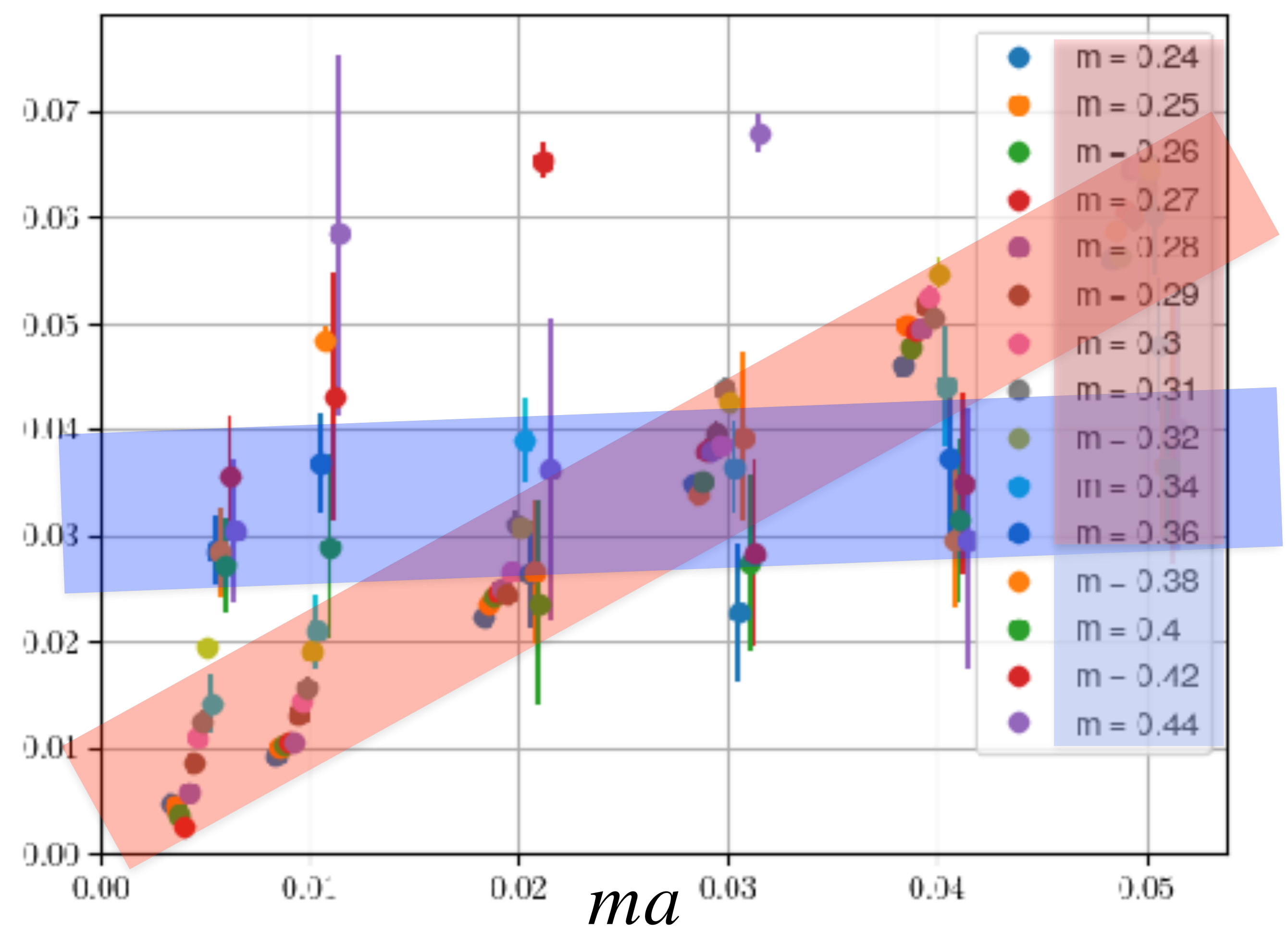
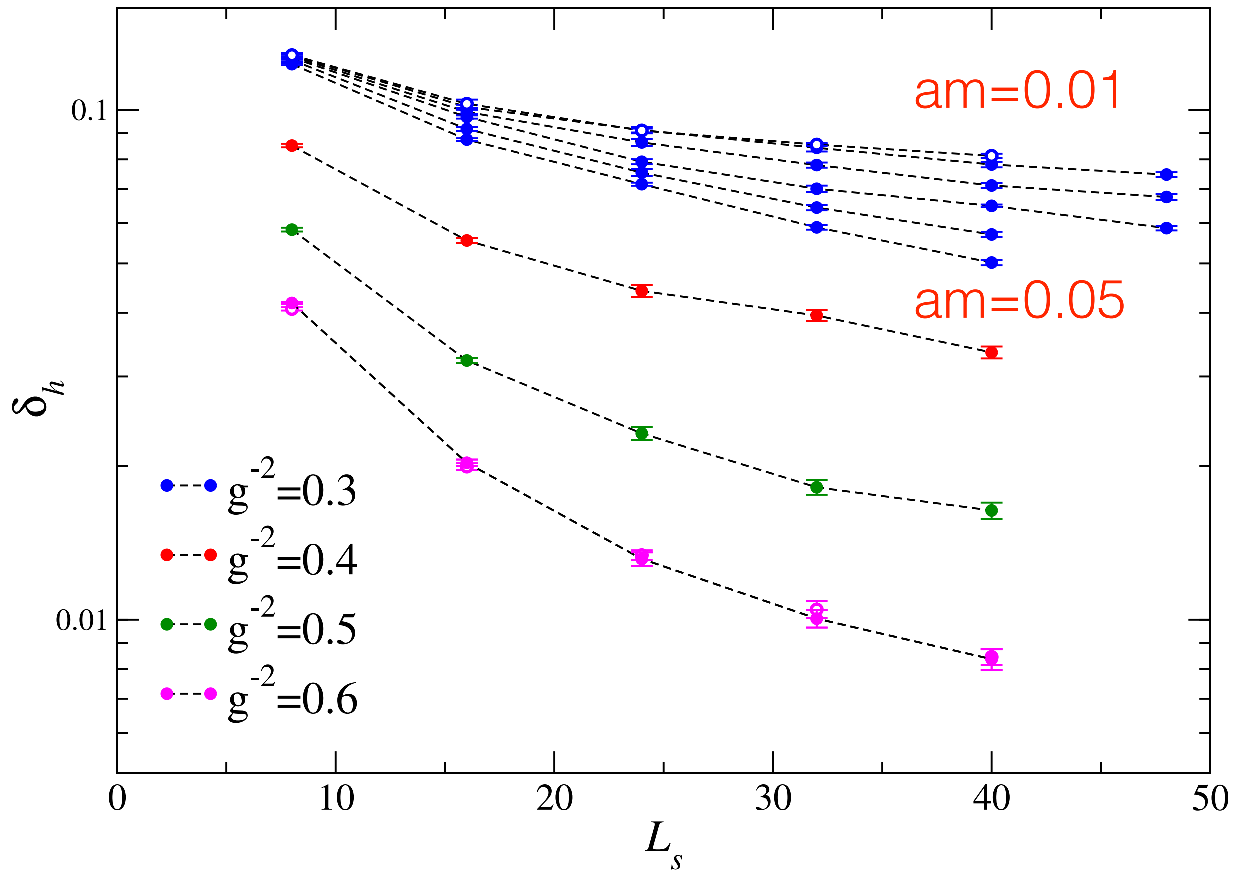
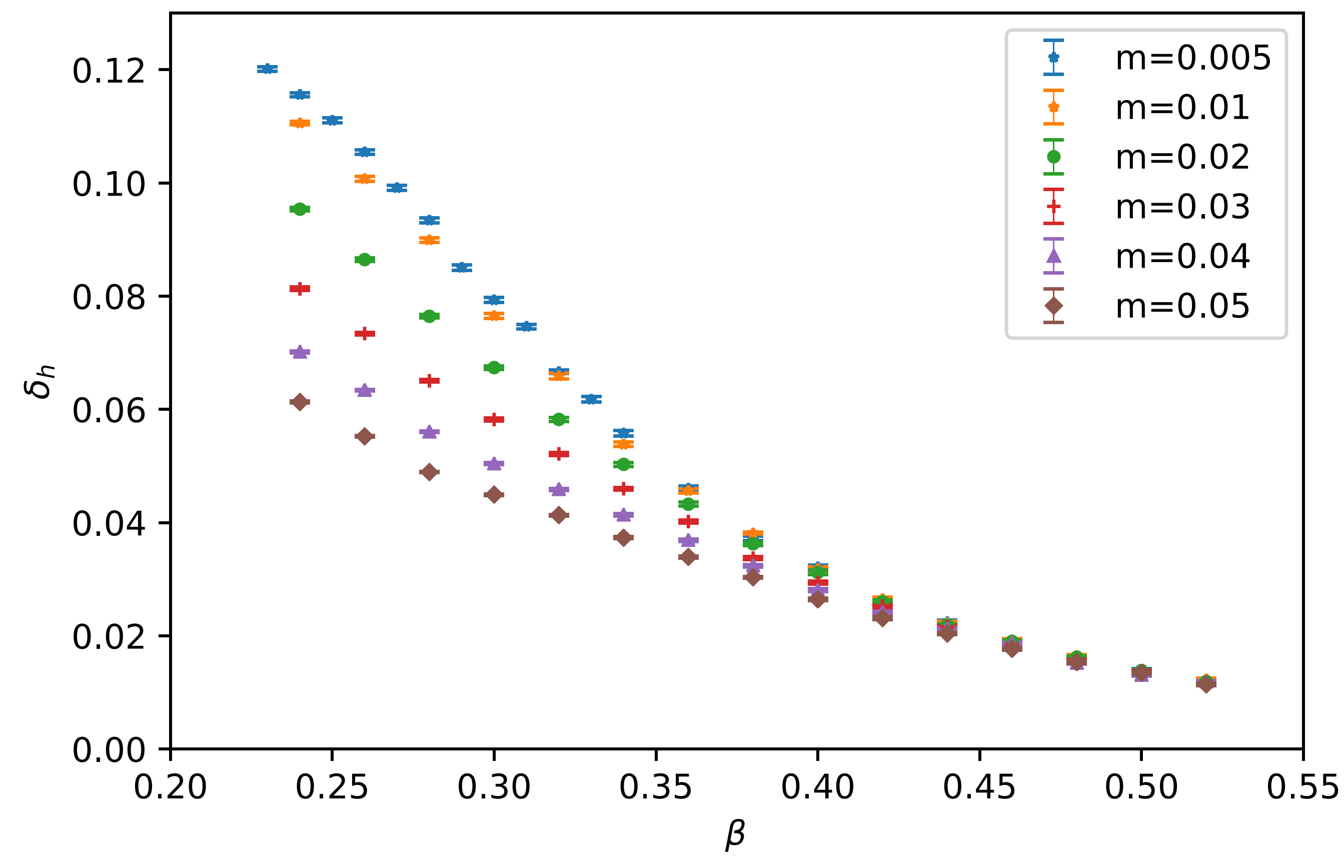


U(2) symmetry restoration requires residual $\delta_h \to 0$



on $12^3 \times L_s \to \infty$



as a function of $\beta \equiv a g^{-2}$ on $16^3 \times 48$

Qualitatively different at strong and weak coupling, and *slow*…

$$\delta_h = \mathrm{Im}\langle \bar{\Psi}(1)\gamma_3\Psi(L_s)\rangle \approx \frac{1}{2}\left(\langle\bar{\psi}\psi\rangle - i\langle\bar{\psi}\gamma_3\psi\rangle\right)$$

Need to improve control over $L_s \to \infty$

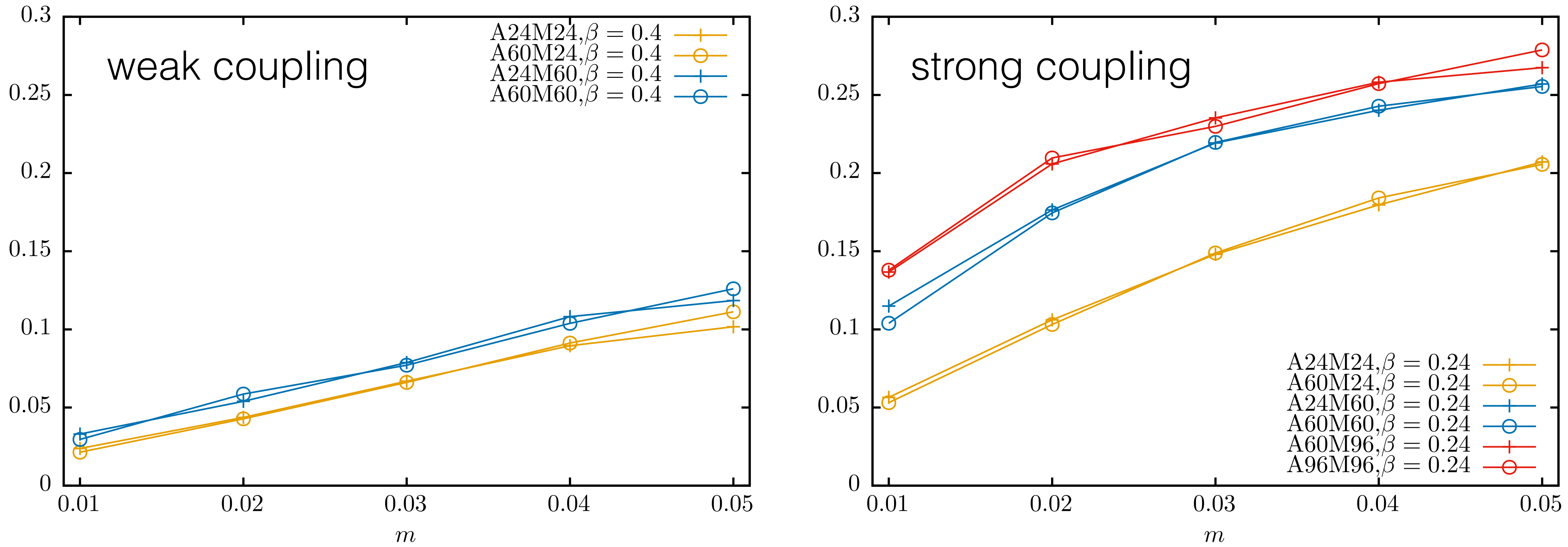
- Partial Quenching (PQ): $L_s(\mathrm{sea}) < L_s(\mathrm{valence})$
- Use a better kernel in D_{ov}: Shamir → Wilson
- better rational approximation to sgn:

  hyperbolic tangent (HT)→ Zolotarev (Z)

Partial Quenching

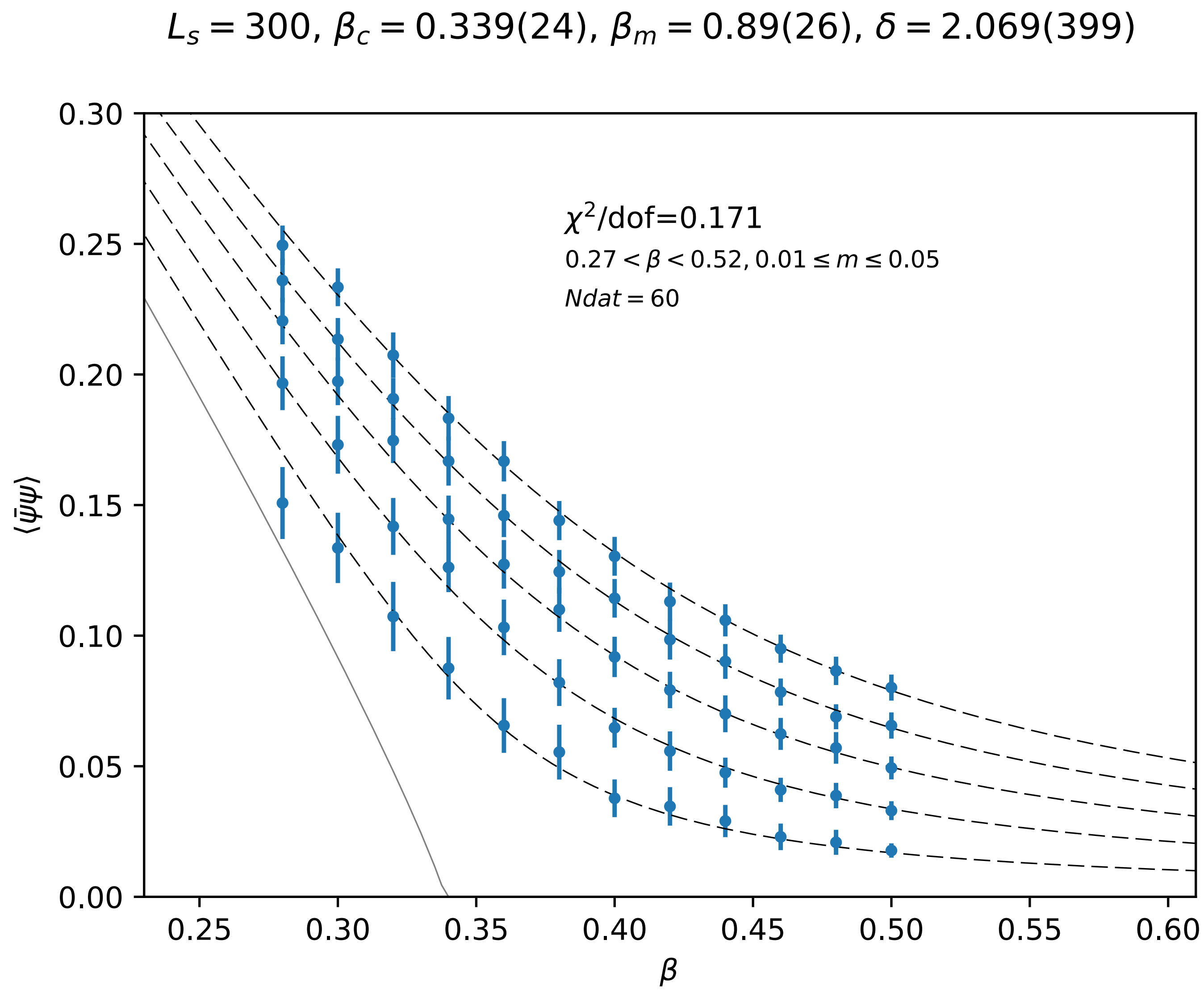
Shamir kernel, 12^3,

$L_s(\text{sea}) = 24{,}60{,}96 \;\; L_s(\text{valence}) = 24,\ 60,\ 96$



Bilinear condensate signal determined by $L_s(\text{valence})$

12^3 PQ, Shamir, HT $\quad L_s(\text{sea}) = 96, \quad L_s(\text{valence}) = 300$



order parameter requires $D^{-1}[A]$ hence measurements taken with $m > 0$

$\Rightarrow$ Equation of State

$$m = A(g^{-2} - g_c^{-2})\langle\bar{\psi}\psi\rangle^{\delta - 1/\beta} + B\langle\bar{\psi}\psi\rangle^{\delta}$$

fitted critical parameters

$$g_c^{-2} = 0.339(24); \quad \beta = 0.89(26); \quad \delta = 2.07(40)$$

$$\nu = 0.91(28); \quad \eta = 0.96(18)$$

Better kernel $D_{ov}(\mathscr{A})$

Replace *Shamir kernel*

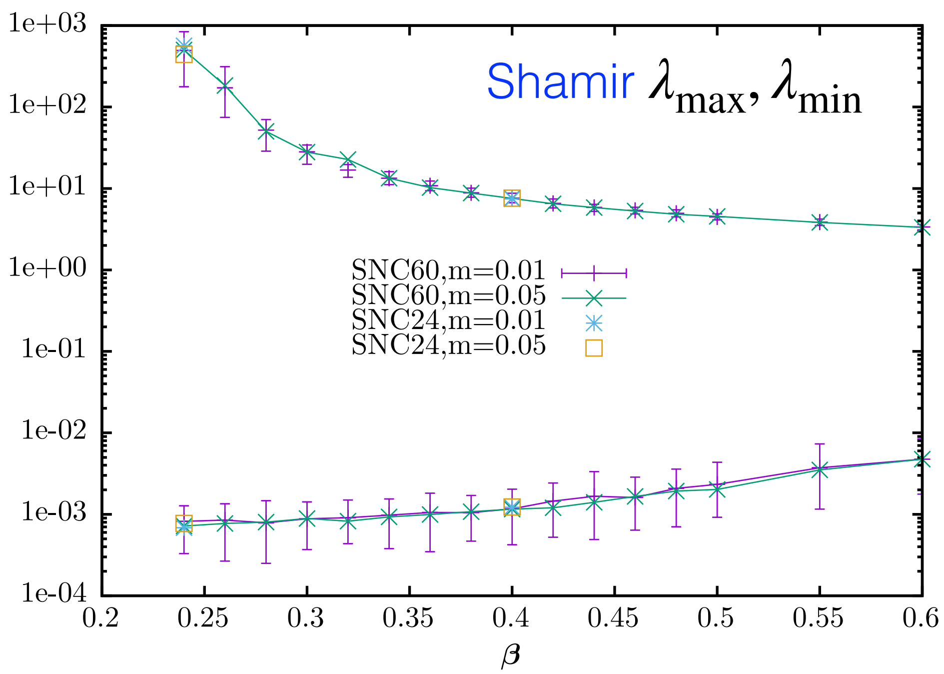
$$\mathscr{A} = (2 + D_W)^{-1} D_W$$

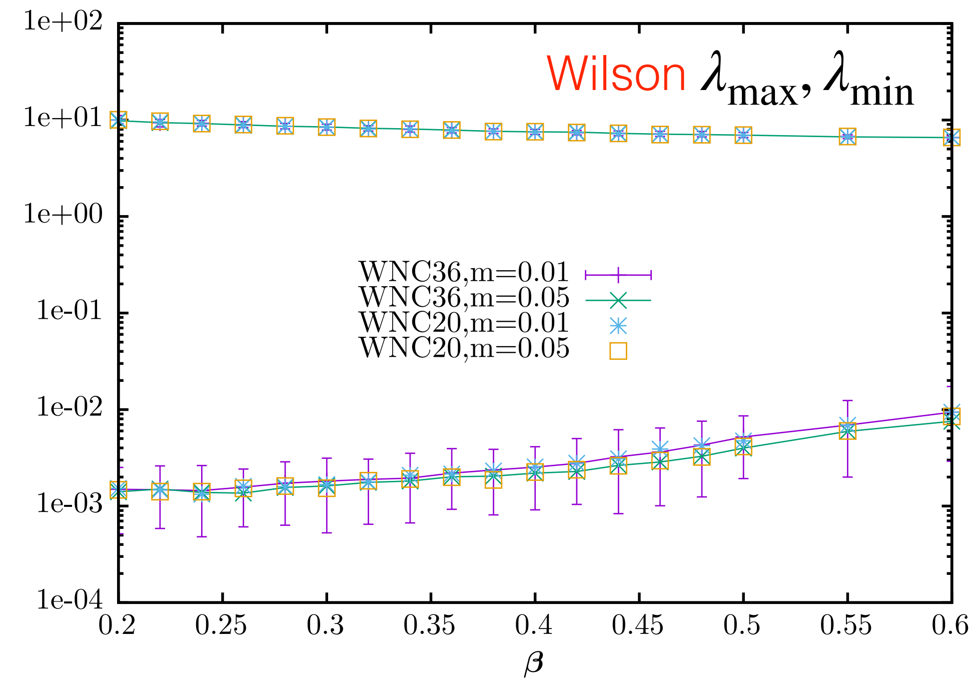
$$D_{SHT} = \begin{pmatrix} D_W + I & -P_- & 0 & imP_+ \\ -P_+ & D_W + I & -P_- & 0 \\ 0 & -P_+ & D_W + I & -P_- \\ -imP_- & 0 & -P_+ & D_W + I \end{pmatrix}$$

with *Wilson kernel* $\mathscr{A} = D_W$

$$D_W[A_\mu] = \gamma_\mu D_\mu - \hat{D}^2 - M$$

$$D_{WHT} = \begin{pmatrix} D_W + I & (D_W - I)P_- & 0 & -im(D_W - I)P_+ \\ (D_W - I)P_+ & D_W + I & (D_W - I)P_- & 0 \\ 0 & (D_W - I)P_+ & D_W + I & (D_W - I)P_- \\ +im(D_W - I)P_- & 0 & (D_W - I)P_+ & D_W + I \end{pmatrix}$$





Better rational approximation of $\mathrm{sgn}(\mathscr{A})$

Replace $\mathrm{sgn}(x) \approx \tanh(L_s \tanh^{-1} x) = \dfrac{1 - \mathcal{T}_{HT}}{1 + \mathcal{T}_{HT}}$

$$\mathcal{T}_{HT} = \left(\frac{1-x}{1+x}\right)^{L_s}$$

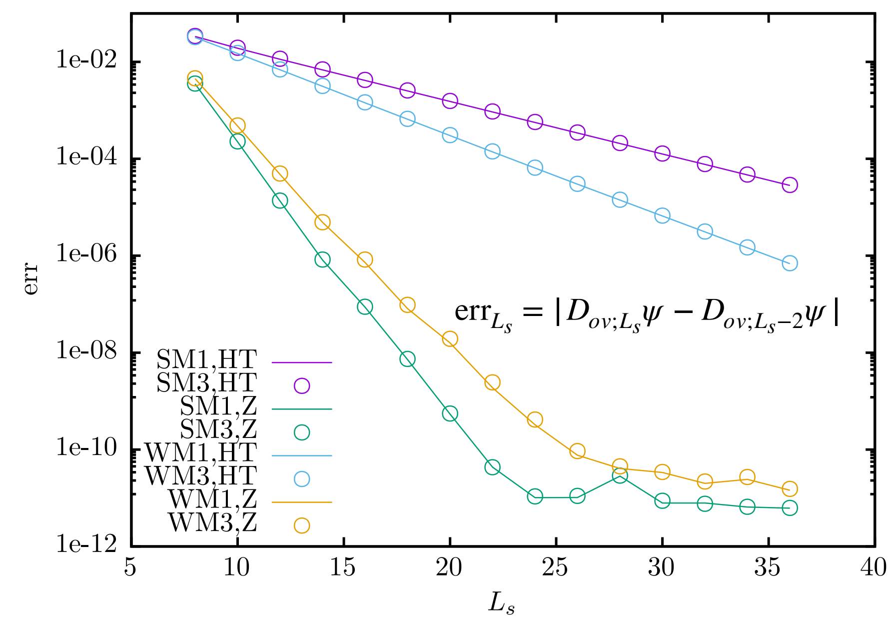
Euclidean Cayley transform

with $\mathrm{sgn}(x) \approx \dfrac{1 - \mathcal{T}_Z}{1 + \mathcal{T}_Z} \equiv dx \dfrac{\prod_{m=1}^{L_s/2-1} (a_m - x^2)}{\prod_{m=1}^{L_s/2} (d_m - x^2)}$

Zolotarev approximation

coefficients a_m, d_m, d depend on range of applicability of approximation and are given in terms of Jacobi elliptic functions

With $\mathcal{T}_Z = \displaystyle\prod_{s=1}^{L_s} \frac{1 - \omega_s x}{1 + \omega_s x}$



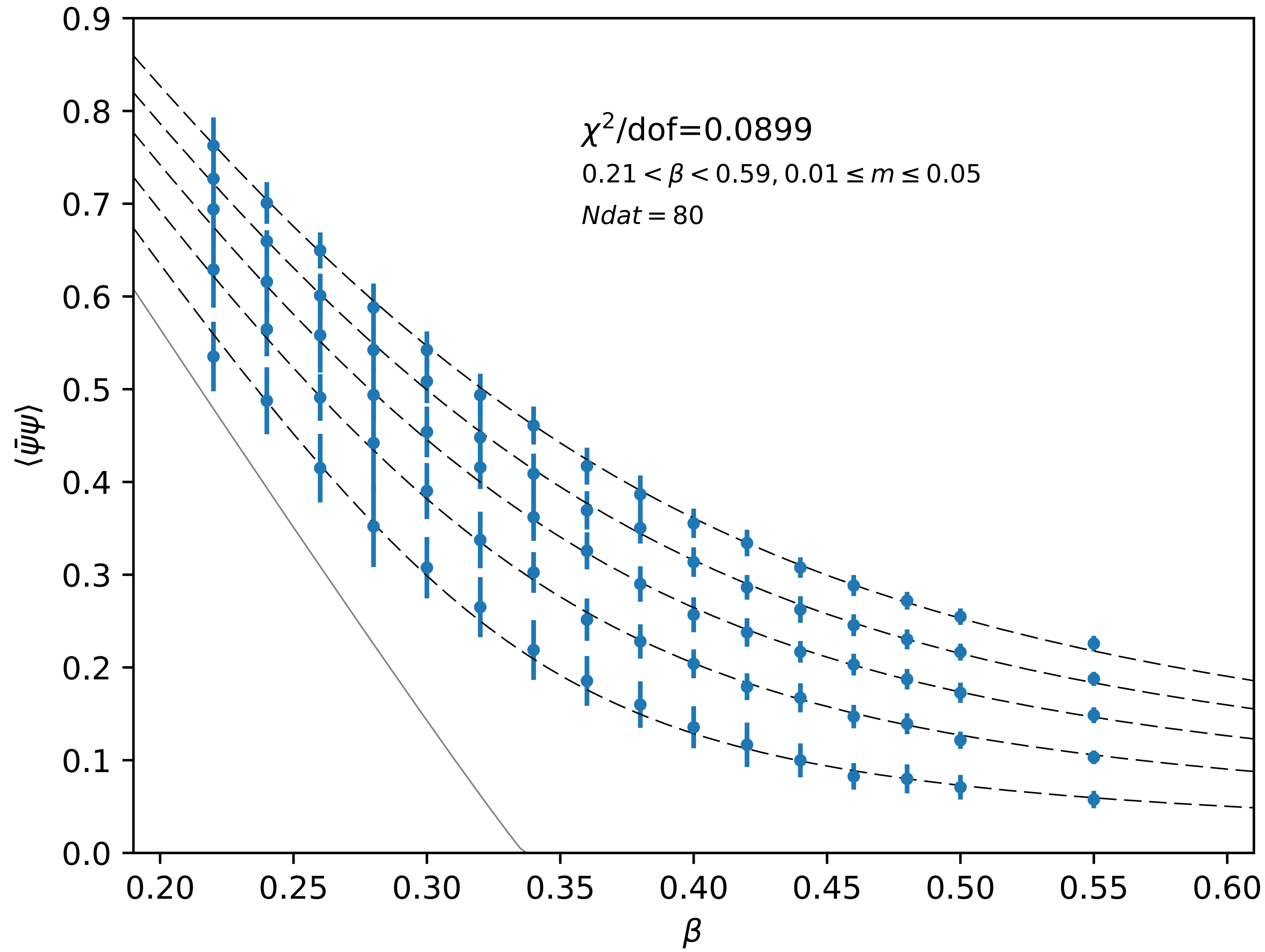
Optimal DWF

T-W Chiu PRL90 (2003) 071601

$$\Rightarrow \quad D_{WZ} = \begin{pmatrix} \omega_1 D_W + I & (\omega_1 D_W - I)P_- & 0 & -im(\omega_1 D_W - I)P_+ \\ (\omega_2 D_W - I)P_+ & \omega_2 D_W + I & (\omega_2 D_W - I)P_- & 0 \\ 0 & (\omega_3 D_W - I)P_+ & \omega_3 D_W + I & (\omega_3 D_W - I)P_- \\ +im(\omega_4 D_W - I)P_- & 0 & (\omega_4 D_W - I)P_+ & \omega_4 D_W + I \end{pmatrix}$$

12^3 Wilson, L_s(sea)=30HT, L_s(valence)=30Z

$L_s = 30$, $\beta_c = 0.336(33)$, $\beta_m = 1.04(29)$, $\delta = 2.078(325)$



EoS: $$m = A(g^{-2} - g_c^{-2})\langle\bar{\psi}\psi\rangle^{\delta-1/\beta} + B\langle\bar{\psi}\psi\rangle^{\delta}$$

fitted critical parameters

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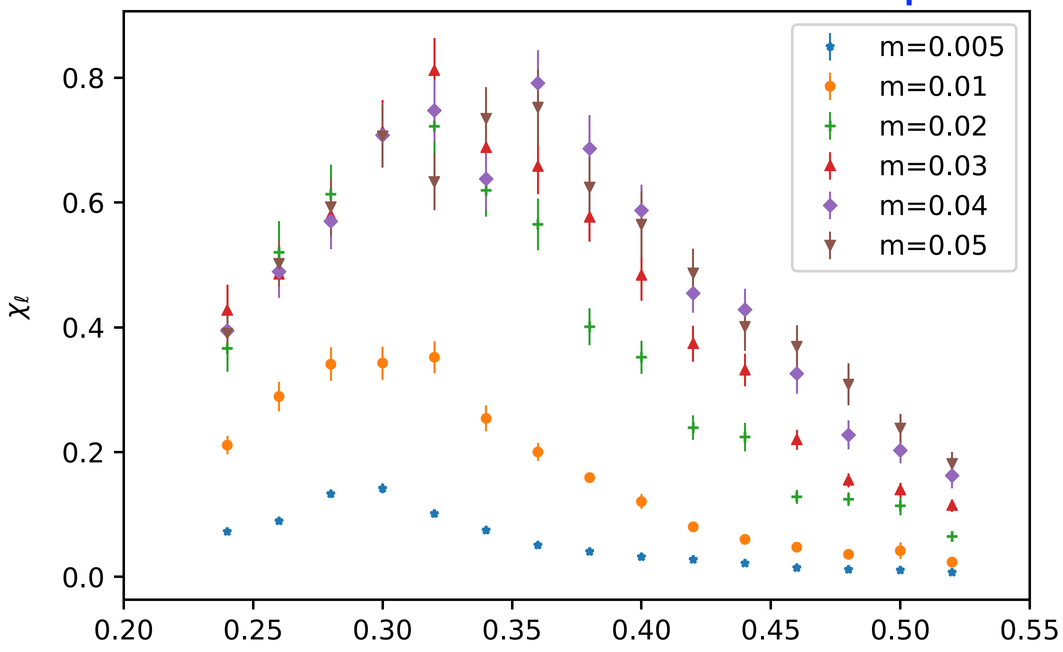
$$\nu = 1.1(3); \quad \eta = 0.95(15)$$

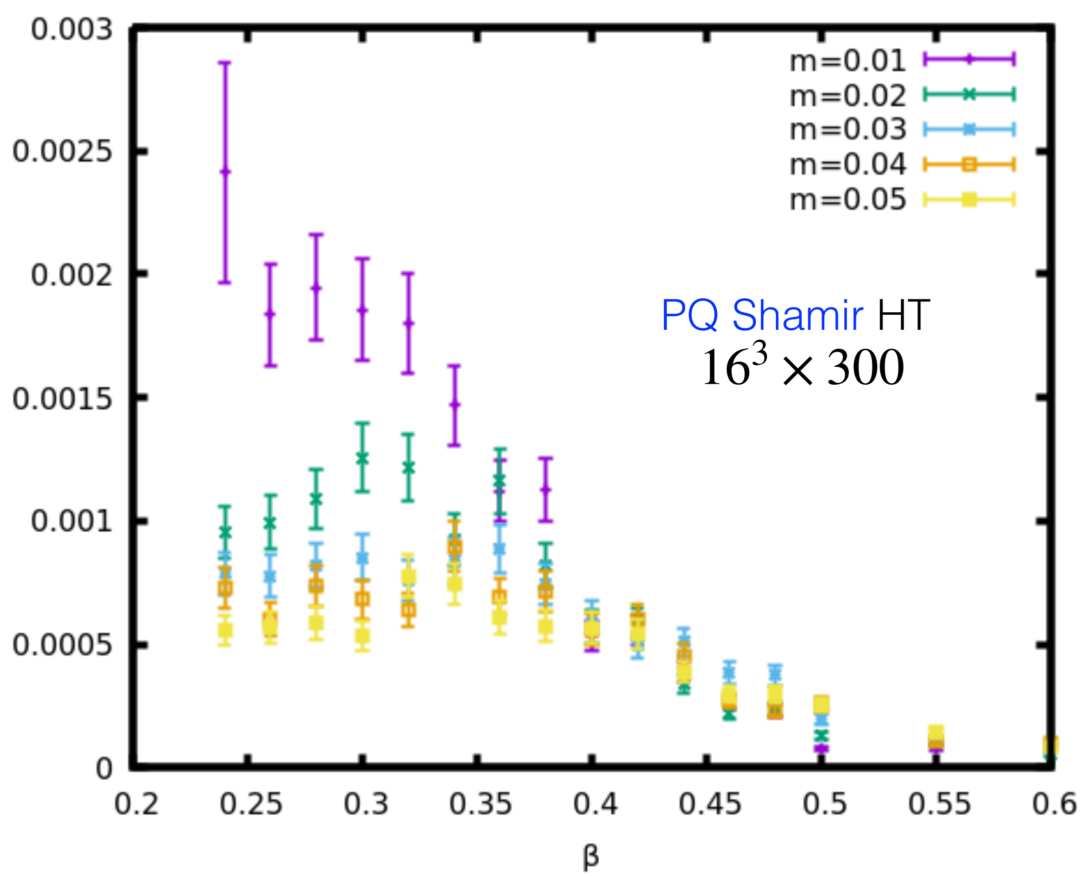
Funnies….

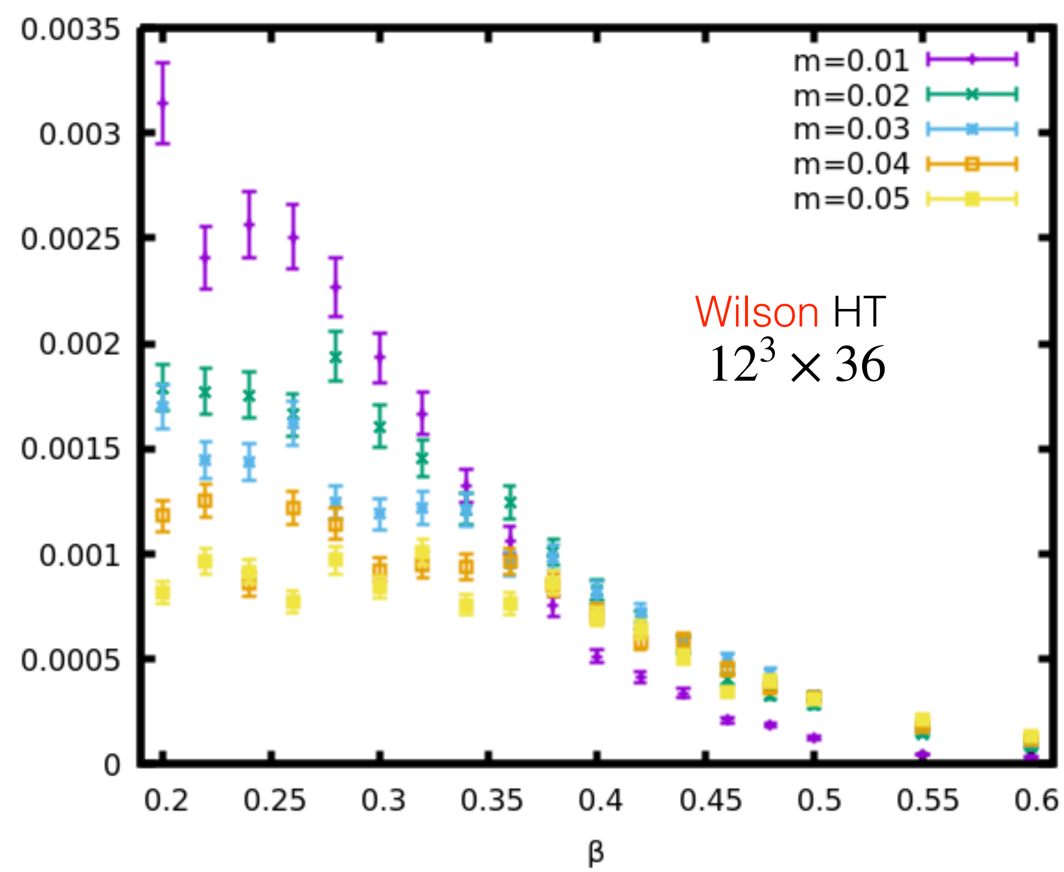
- Susceptibility $\chi_\ell = \langle(\bar{\psi}\psi)^2\rangle - \langle\bar{\psi}\psi\rangle^2$ previously showed inverted mass hierarchy



Shamir HT $16^3 \times 48$

It looks as if order parameter fluctuations are particularly L_s-sensitive





Summary

	12^3 Shamir L_s=300 (HTv)	12^3 Wilson L_s=30 (HTs&Zv)	16^3 Shamir HT L_s=8,…,80	16^3 staggered HMC	staggered FSS (Bag)
g_c^{-2}	0.339(24)	0.336(33)	0.283(1)	-	-
β	0.89(26)	1.04(29)	0.320(5)	0.57(2)	0.70(1)
δ	2.07(40)	2.08(33)	4.17(5)	2.75(9)	2.63(2)
ν	0.91(28)	1.1(3)	0.55(1)	0.71(3)	0.85(1)
η	0.96(18)	0.95(15)	0.16(1)	0.60(4)	0.65(1)

- Two different DWF regularisations (Shamir, Wilson) of N=1 give **compatible results**
- Results clearly **distinct** from previous based on uncontrolled $L_s \to \infty$
- Is $\mathrm{U}(2) \to \mathrm{U}(1) \otimes \mathrm{U}(1)$ (Dirac, DWF) distinct from $\mathrm{U}(1) \otimes \mathrm{U}(1) \to \mathrm{U}(1)$ (Kähler-Dirac, staggered) ?
- Promising, by need larger volumes, higher statistics…

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