Structure-dependent electromagnetic finite-volume effects through order $1/L^3$

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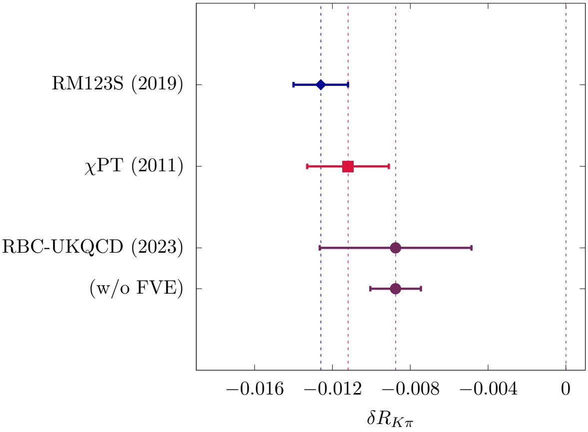
Leptonic decays: $P \to \ell\nu$



RBC-UKQCD 23:
$\delta R_{K\pi} = -0.0086(39)$

total	(39)
total (w/o FVE)	(13)
statistical	(3)
FVE	(37)
fit	(11)
QED quenching	(5)
discretisation	(5)

χPT : $\delta R_{K\pi} = -0.0112(21)$

RM123S 19: $\delta R_{K\pi} = -0.0126(14)$

Issue: Finite-volume effects not sufficiently understood

Pointlike $1/L^3$ very enhanced and structure dependence unknown

QED in a finite volume

- Gauss' law: Difficult to define charged states in finite volume with periodic boundary conditions
  (photon zero-momentum modes and absence of mass gap)
- Several prescriptions

1. $\mathrm{QED_M}$: Photon mass m_γ
   [Endres, Shindler, Tiburzi, Walker-Loud 2016; Bussone, Della Morte, Janowski 2018]
2. QED_∞: Do the QED part in infinite volume
   [Feng, Jin 2018; Christ, Feng, Jin, Sachrajda, Wang 2023]
3. $\mathrm{QED_C}$: Charge-conjugated boundary conditions
   [Kronfeld, Wiese 1991–1993; RC* 2019]
4. $\mathrm{QED_L^{IR}}$: Exclude/redistribute photon zero-mode
   [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
   - $\mathrm{QED_L}$: Exclude photon zero-mode [Hayakawa, Uno 2008]
   - $\mathrm{QED_r}$: Redistribute photon zero-mode [Di Carlo, Lattice 23]

- Each has advantages/drawbacks: $\mathrm{QED_L^{IR}}$ simple but non-local

Finite-size effects

- Massless photon + no zero-mode ($\mathrm{QED}_\mathrm{L}^\mathrm{IR}$ and QED_C)

  $V = \mathbb{R} \times L^3$: $\implies$ Finite-size effects in observable $\mathcal{O}(L)$:

$$\Delta\mathcal{O}(L) = \mathcal{O}(L) - \mathcal{O}(\infty) = \kappa_0 + \kappa_{\log} \log m_P L + \kappa_1 \frac{1}{m_P L} + \kappa_2 \frac{1}{(m_P L)^2} + \dots$$

- Scaling in L is observable-dependent:
  e.g. self-energy $\kappa_0 = \kappa_{\log} = 0$
- Coefficients depend on physical particle properties: masses, charges, structure (form-factors):
  Point-like + structure-dependent
- NB: Coefficients are prescription-dependent!

[Davoudi, Savage 2014; BMW 2015; RM-123/Soton 2017; Davoudi, Harrison, Jüttner, Portelli, Savage 2019; Bijnens, Harrison, H-T, Janowski, Jüttner, Portelli 2019; Di Carlo, Hansen, H-T, Portelli 2021]

- QED_M and QED_∞: no power-law effects

$P \to \ell\nu$: Finite-size effects in $\mathrm{QED}_{\mathrm{L}}^{\mathrm{IR}}$

$$
\begin{aligned}
Y^{(3)}(L) = \; & \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + \frac{c_3^{\mathrm{IR}} - 2\,c_3^{\mathrm{IR}}(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\log\left(\frac{m_W L}{4\pi}\right) \\
& - 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2^{\mathrm{IR}} - 4\,r_\ell^2\,c_2^{\mathrm{IR}}(\mathbf{v}_\ell)}{1-r_\ell^4}\right] \\
& + \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\,\left[(1+r_\ell^2)^2\,c_1^{\mathrm{IR}} - 4\,r_\ell^2\,c_1^{\mathrm{IR}}(\mathbf{v}_\ell)\right]}{1-r_\ell^4} + \frac{8\pi\,\left[(1+r_\ell^2)\,c_1^{\mathrm{IR}} - 2\,c_1^{\mathrm{IR}}(\mathbf{v}_\ell)\right]}{(1-r_\ell^4)}\right] \\
& + \frac{32\pi^2\,m_P}{f_P(1-r_\ell^4)\,(m_P L)^3}\left\{c_0^{\mathrm{IR}}(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\,A^{(0,1)}\left(0,-m_P^2\right)\right] + c_0^{\mathrm{IR}}\,\mathcal{C}_\ell\right\}
\end{aligned}
$$

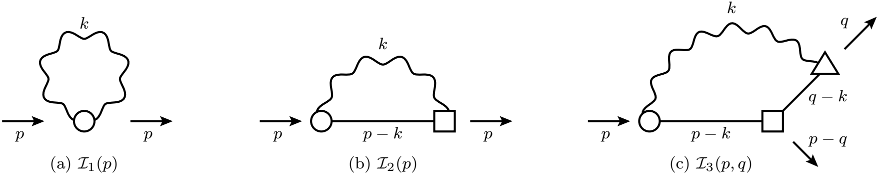
- $1/L^3$ calculated for the first time: Coefficients $c_0^{\mathrm{IR}}(\mathbf{v}_\ell)$ and c_0^{IR}
- Our problem: Only point-like part of $\mathcal{C}_\ell$ known (sizeable)
- F_A^P, F_V^P, $A^{(0,1)}\left(0,-m_P^2\right)$: From $P^- \to \ell^-\nu_\ell\gamma^{(*)}$
  Lattice [RM-123/Soton 2020/2022/2023, RBC/UKQCD 2023: Giusti, Lattice 23]
  chiral perturbation theory [Bijnens, Ecker, Gasser 1992], experiment [...].
- In the following: Explain how to get this and solve our problem

Finite-size effects

- Consider IR-safe quantity $\mathcal{O}$ at order α, with photon momentum $k = (k_0, \mathbf{k})$
- Finite-size effects in $\mathcal{O}(L)$ given by:

$$\Delta\mathcal{O}(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\in\Pi} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}\left(k_0, |\mathbf{k}|, \mathbf{p}\cdot\mathbf{k}\right)}{[(p-k)^2+m^2]} \frac{1+w_{|\mathbf{n}|^2}}{k^2}$$

- The function $f_{\mathcal{O}}\left(k_0, |\mathbf{k}|, \mathbf{p}\cdot\mathbf{k}\right)$ has no poles: Depends on observable, QED prescription and structure
- Weights $w_{|\mathbf{n}|^2}$:
  $\mathrm{QED}_\mathrm{L}^\mathrm{IR}$ and QED_r non-zero for finite number of $\mathbf{n} = \mathbf{k}\,L/(2\pi)$
  QED_L and QED_C $w_{|\mathbf{n}|^2} = 0$
- Example: Diagram with 2 propagators in self-energy or leptonic decays



(a) $\mathcal{I}_1(p)$ (b) $\mathcal{I}_2(p)$ (c) $\mathcal{I}_3(p,q)$

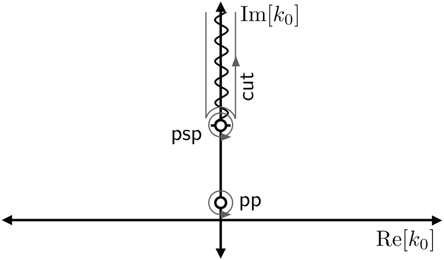
Finite-size effects

- Let us first consider the k_0 integral

$$g_{\mathcal{O}}(\mathbf{k}) = \int \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}\left(k_0, |\mathbf{k}|, \mathbf{p}\cdot\mathbf{k}\right)}{[(p-k)^2+m^2]} \frac{1}{k^2}$$
$$= \left(\int_{\text{poles}} + \int_{\text{rem}}\right) \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}\left(k_0, |\mathbf{k}|, \mathbf{p}\cdot\mathbf{k}\right)}{[(p-k)^2+m^2]} \frac{1}{k^2}$$

- 2 poles and remaining analytical structure (cut)

Example: Self-energy



Finite-size effects

- Once we have determined $g_{\mathcal{O}}(\mathbf{k}) = g_{\mathcal{O}}^{\text{poles}}(\mathbf{k}) + g_{\mathcal{O}}^{\text{rem}}(\mathbf{k})$ we have

$$\Delta\mathcal{O}(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\in\Pi} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \left(1 + w_{|\mathbf{n}|^2}\right) \left[g_{\mathcal{O}}^{\text{poles}}(\mathbf{k}) + g_{\mathcal{O}}^{\text{rem}}(\mathbf{k}) \right]$$

- Let us focus on the pole part for now with $\mathbf{k} = \frac{2\pi\mathbf{n}}{L}$

$$\Delta_{\text{poles}}\mathcal{O} = \left(\sum_{\mathbf{n}\in\Pi} - \int d^3\mathbf{n} \right) \frac{1}{L^3} \left(1 + w_{|\mathbf{n}|^2}\right) g_{\mathcal{O}}^{\text{poles}} \left(\frac{2\pi\mathbf{n}}{L} \right)$$

General expansion of $g_{\mathcal{O}}^{\text{poles}}(\mathbf{k})$

- Photon pole:

$$\frac{i}{L^3}\frac{f_{\mathcal{O}}\left(i|\mathbf{k}|,|\mathbf{k}|,\mathbf{v}\cdot\hat{\mathbf{k}}\right)}{2|\mathbf{k}|\ [(i\omega(\mathbf{p})-i|\mathbf{k}|)^2+\omega(\mathbf{p}-\mathbf{k})^2]}=\frac{1}{L}\frac{f_{\mathcal{O}}\left(i\frac{2\pi|\mathbf{n}|}{L},\frac{2\pi|\mathbf{n}|}{L},\mathbf{v}\cdot\hat{\mathbf{n}}\right)}{16\pi^2|\mathbf{n}|^2\ (1-\mathbf{v}\cdot\hat{\mathbf{n}})\omega(\mathbf{p})}$$

- Momentum: $p=(i\omega(\mathbf{p}),\mathbf{p})$, $p^2=-m^2$. Velocity $\mathbf{v}=\mathbf{p}/\omega(\mathbf{p})$
- Physical singularities: $1/|\mathbf{n}|$ (soft) and $1/(1-\mathbf{v}\cdot\hat{\mathbf{n}})$ (collinear)
- Expand in L: Define prescription-dependent coefficients

$$c_j^w(\mathbf{v})=\left(\sum_{\mathbf{n}\in\Pi}-\int d^3\mathbf{n}\right)\frac{1+w_{|\mathbf{n}|^2}}{|\mathbf{n}|^j\,(1-\mathbf{v}\cdot\hat{\mathbf{n}})}$$

Finite-size effects of $g_{\mathcal{O}}(\mathbf{k})$

$$\Delta_{\text{poles}}\mathcal{O} = \frac{f_{\mathcal{O}}\, c_2^w(\mathbf{v})}{16\pi^2 L\omega(\mathbf{p})} + \frac{f_{\mathcal{O}}^{(0,0,1)}\omega(\mathbf{p})\left(c_1^w(\mathbf{v}) - c_1^w\right) + \left(f_{\mathcal{O}}^{(0,1,0)} + i f_{\mathcal{O}}^{(1,0,0)}\right) c_1^w(\mathbf{v})}{8\pi L^2 \omega(\mathbf{p})}$$

$$\frac{1}{8L^3}\left\{ \frac{c_0^w(\mathbf{v})}{\omega(\mathbf{p})}\left[f_{\mathcal{O}}^{(0,2,0)} + \ldots\right] - c_0^w\left[f_{\mathcal{O}}^{(0,0,2)}\omega(\mathbf{p}) + \ldots\right]\right\}$$

$$\Delta_{\text{rem}}\mathcal{O} = \frac{c_0^w\, \mathcal{C}^{\text{rem}}}{L^3}$$

- The coefficients $c_j^w(\mathbf{v})$ and $f_{\mathcal{O}}$ vary in different QED prescriptions
- $f_{\mathcal{O}}$ not known for $\mathrm{QED_C}$ due to $V - A$ current and charge conjugation
- NB: $c_0^w(\mathbf{v})$ and c_0^w affect order $1/L^3$
- Branch-cut $\mathcal{C}^{\text{rest}}$: Unknown, but only affects c_0^w
- Next: focus on $c_j^w(\mathbf{v})$

Finite-volume coefficients

- $\mathbf{v} = |\mathbf{v}|\,(1,1,1)/\sqrt{3}$: $|\mathbf{v}| = 0.912401$ (corresponds to $K \to \mu\nu_\mu$)

$$\mathrm{QED_L},\, \mathrm{QED_r}: \quad \Pi_\mathrm{L} = \left\{\mathbf{n} \in \mathbb{Z}^3 \setminus \{0,0,0\}\right\} ,$$
$$\mathrm{QED_C}: \quad \Pi_\mathrm{C} = \left\{\mathbf{n} + \frac{\bar{\mathbf{n}}}{2} \,\middle|\, \mathbf{n} \in \mathbb{Z}^3 ,\, \bar{\mathbf{n}} = (1,1,1)\right\} ,$$

$\mathrm{QED_L}$: $c_j(\mathbf{v})$ $\quad$ $\mathrm{QED_r}$: $\bar{c}_j(\mathbf{v})$ $\quad$ $\mathrm{QED_C}$: $c_j^*(\mathbf{v})$

j	$c_j(\mathbf{v})$	$\bar{c}_j(\mathbf{v})$	$c_j^*(\mathbf{v})$	c_j	$\bar{c}_j$	c_j^*
2	-16.3454	-14.9613	-3.20674	-8.91363	-7.91363	-5.49014
1	-5.73018	-4.3461	3.51224	-2.8373	-1.8373	-0.80194
0	-2.12369	-0.7396	3.69273	-1	0	0

- Effects of non-locality at $1/L^3$ can be removed in $\mathrm{QED_r}$: $\bar{c}_0 = 0$
- Leptonic decays: Size of $f_\mathcal{O}$ important but $\mathrm{QED_r}$ solves it
- $c_0^*(\mathbf{v})$ can give a $1/L^3$ also in $\mathrm{QED_C}$ (but depends on $f_\mathcal{O}$)
- For generalities on coefficients: See talk by [Portelli, Lattice 23]

$P \to \ell\nu$: Finite-size effects in $\mathrm{QED}_\mathrm{L}^\mathrm{IR}$

$$
\begin{aligned}
Y^{(3)}(L) = {} & \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + \frac{c_3^{\mathrm{IR}} - 2\,c_3^{\mathrm{IR}}(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\log\left(\frac{m_W L}{4\pi}\right) \\
& - 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2^{\mathrm{IR}} - 4\,r_\ell^2\,c_2^{\mathrm{IR}}(\mathbf{v}_\ell)}{1-r_\ell^4}\right] \\
& + \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\,\left[(1+r_\ell^2)^2\,c_1^{\mathrm{IR}} - 4\,r_\ell^2\,c_1^{\mathrm{IR}}(\mathbf{v}_\ell)\right]}{1-r_\ell^4} + \frac{8\pi\,\left[(1+r_\ell^2)\,c_1^{\mathrm{IR}} - 2\,c_1^{\mathrm{IR}}(\mathbf{v}_\ell)\right]}{(1-r_\ell^4)}\right] \\
& + \frac{32\pi^2\,m_P}{f_P(1-r_\ell^4)\,(m_P L)^3}\left\{c_0^{\mathrm{IR}}(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\,A^{(0,1)}\left(0,-m_P^2\right)\right] + c_0^{\mathrm{IR}}\,\mathcal{C}_\ell\right\}
\end{aligned}
$$

- $1/L^3$ calculated for the first time
- $\mathcal{C}_\ell$ unknown, but c_0^{IR} can be put to zero with $\mathrm{QED}_\mathrm{L}^\mathrm{IR}$
- $c_0^{\mathrm{IR}}(\mathbf{v}_\ell)$ part can be removed with angle averaging too
- This solves our problem!

Conclusions

1. Generalities of finite-volume effects:
   - Depend on chosen QED prescription: Coefficients $c_j^w(\mathbf{v})$ and $f_{\mathcal{O}}$
   - Size of $f_{\mathcal{O}}$ important
   - $c_0^w(\mathbf{v})$ at $1/L^3$: Non-zero also in $\mathrm{QED_C}$
2. $\mathrm{QED_L^{IR}}$:
   - $1/L^3$ for leptonic decays (moving frame self-energy in back-up)
   - Life would be simpler without the $1/L^3$:

     $\implies$ Freedom of $\mathrm{QED_L^{IR}}$

     On-going work in Edinburgh, see $\mathrm{QED_r}$ [Di Carlo, Lattice 23]

Back-up slides

Moving frame self-energy in $\mathrm{QED}_{\mathrm{L}}^{\mathrm{IR}}$

- We can choose the moving frame with $\mathbf{p} \neq \mathbf{0}$:

$$\Delta m_P^2(L) = e^2 m_P^2 \Bigg\{ \frac{1}{\gamma(|\mathbf{v}|)} \frac{c_2^{\mathrm{IR}}(\mathbf{v})}{4\pi^2 \, m_P L} + \frac{c_1^{\mathrm{IR}}}{2\pi (m_P L)^2} + \frac{c_0^{\mathrm{IR}}}{(m_P L)^3} \Bigg[\frac{\left(\gamma(|\mathbf{v}|)^2 - 1\right) \left(1 - \frac{2 \langle r_P^2 \rangle m_P^2 \, \gamma(|\mathbf{v}|)^2}{3}\right)^2}{2\gamma(|\mathbf{v}|)^3} - \frac{2 \langle r_P^2 \rangle m_P^2}{3} \gamma(|\mathbf{v}|) + \mathcal{C} \Bigg] + \mathcal{O}\left[\frac{1}{(m_P L)^4}, e^{-m_P L}\right] \Bigg\}$$

- Lorentz contraction factor: $\gamma(|\mathbf{v}|) = (1 - |\mathbf{v}|^2)^{-1/2}$
- Pointlike limit agrees with $\mathrm{QED}_{\mathrm{L}}$ [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
- Branch-cut: Specific to non-local theory (No prediction unless we know $\mathcal{C}$ or get rid of it...)
- $\mathrm{QED}_{\mathrm{L}}^{\mathrm{IR}}$ Freedom to set $c_0^{\mathrm{IR}} = 0$, e.g. $\mathrm{QED}_{\mathrm{r}}$