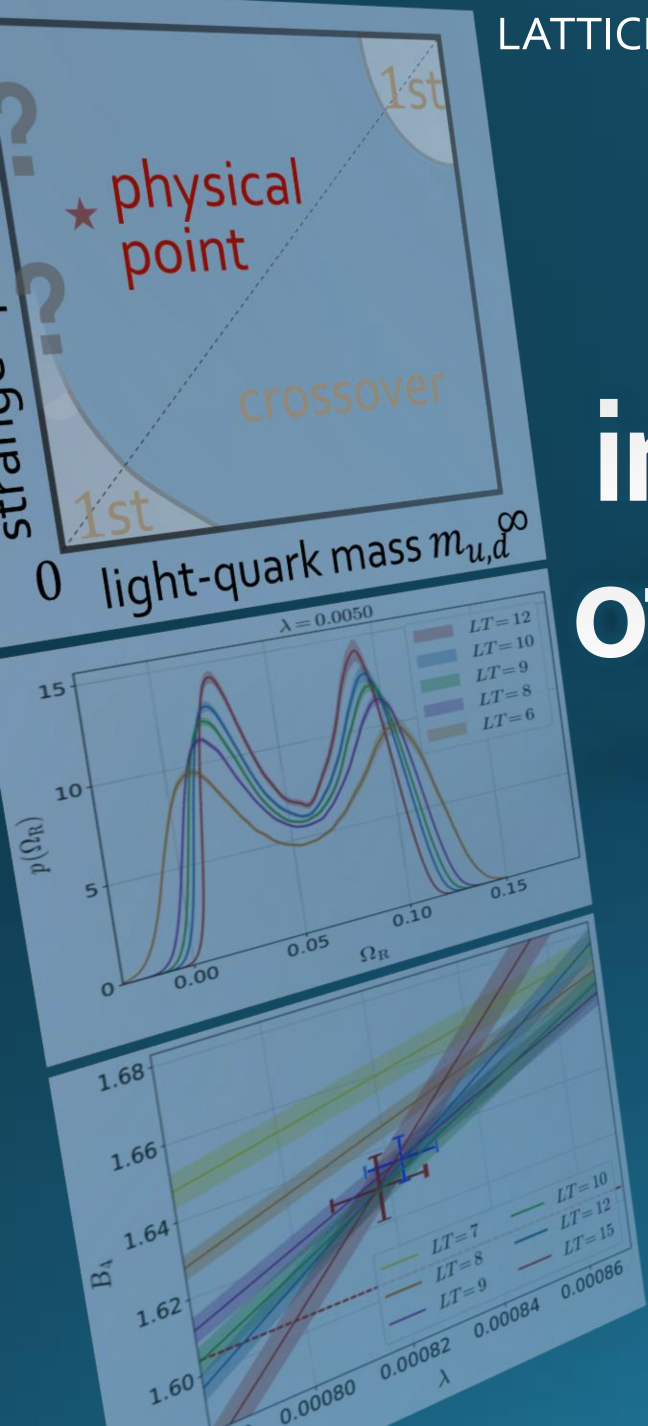


LATTICE 2023, FermiLab, Chicago, Aug. 1, 2023

Critical Point in heavy-quark region of QCD on fine lattices

Masakiyo Kitazawa

(YITP, Kyoto)

with R. Ashikawa, S. Ejiri, K. Kanaya

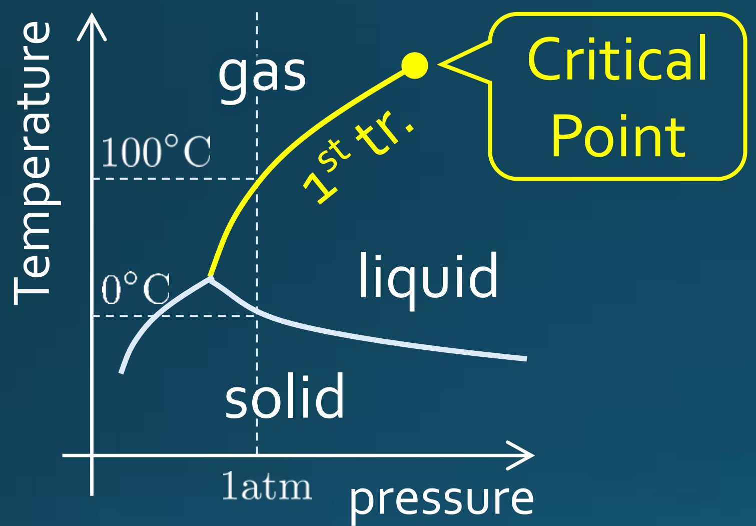
Ashikawa+, in preparation

Wakabayashi+, PTEP 2022, 033B05 (2022) [2112.06340]

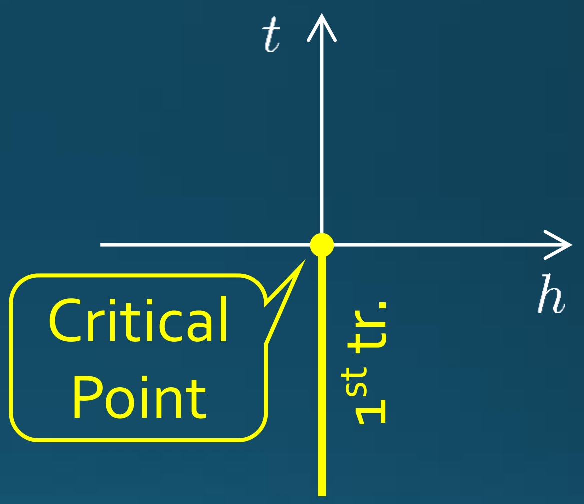
Kiyohara+, Phys. Rev. D104, 114509 (2021) [2108.00118]

Critical Points

- **Water**



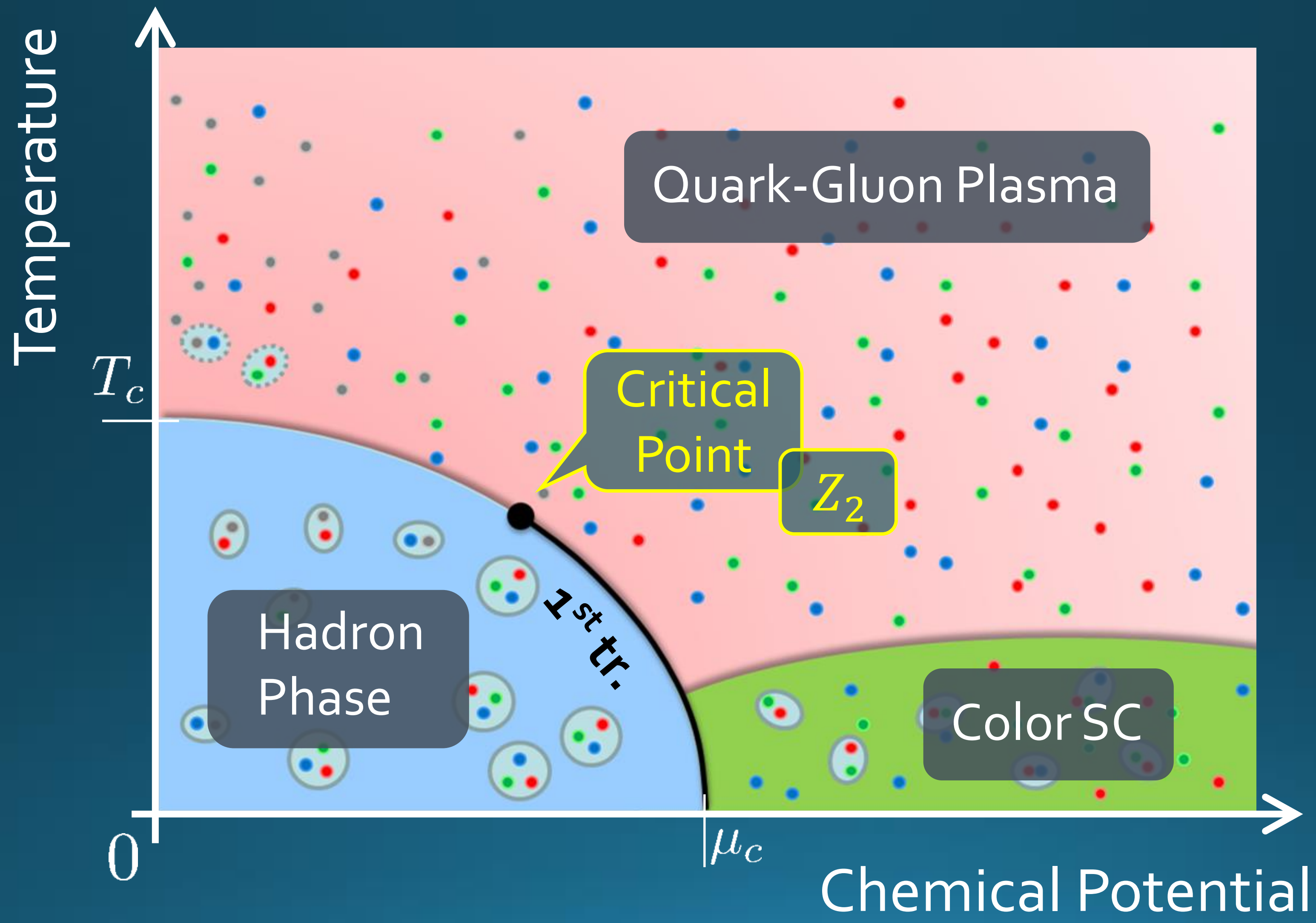
- **Ising Model**



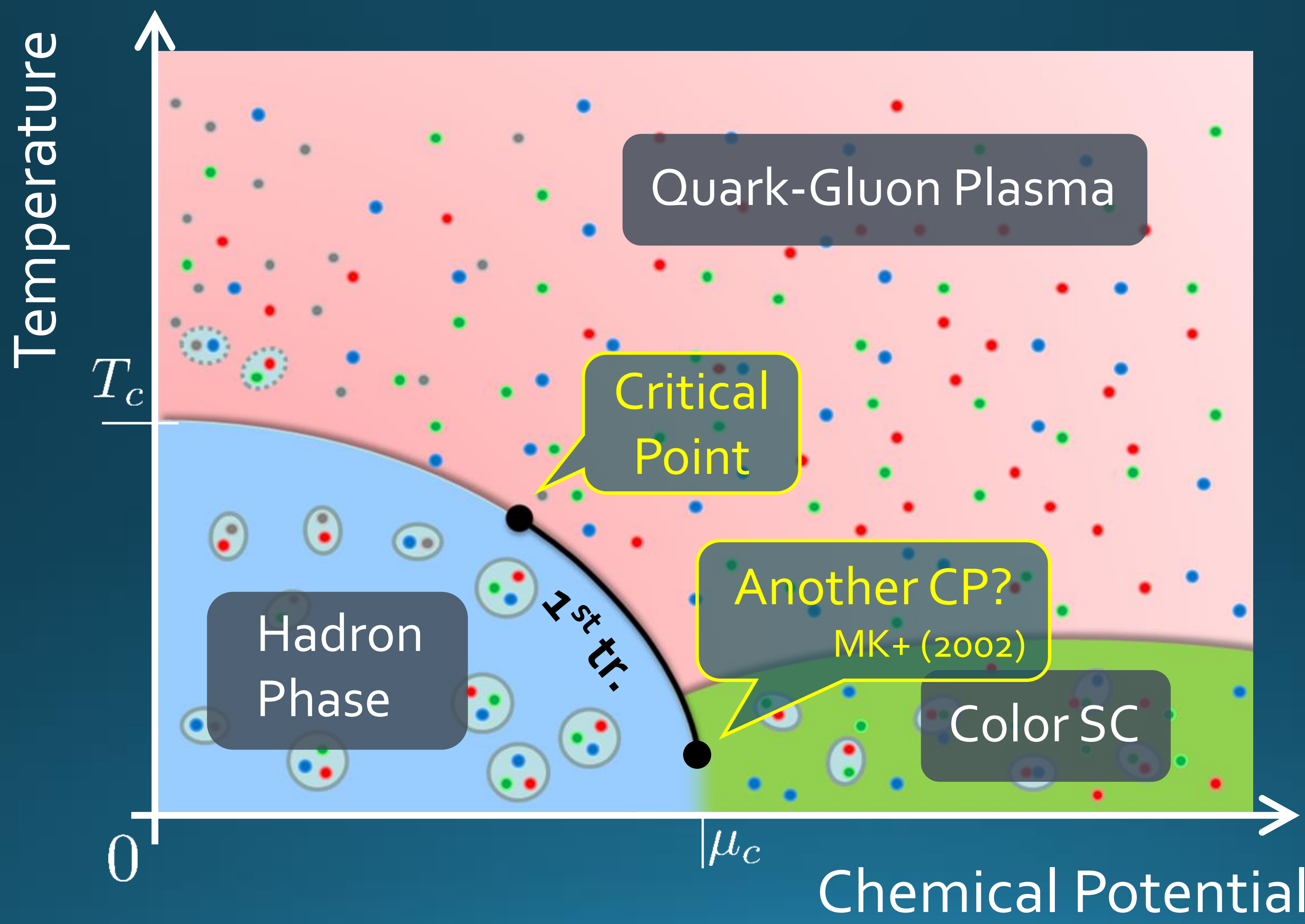
- CP: Second-order transition point.
- Singularities in thermodynamic quantities.
- These CPs belong to the same universality class (Z_2).

→ Common critical exponents. Ex. $C \sim (T - T_c)^{-\alpha}$

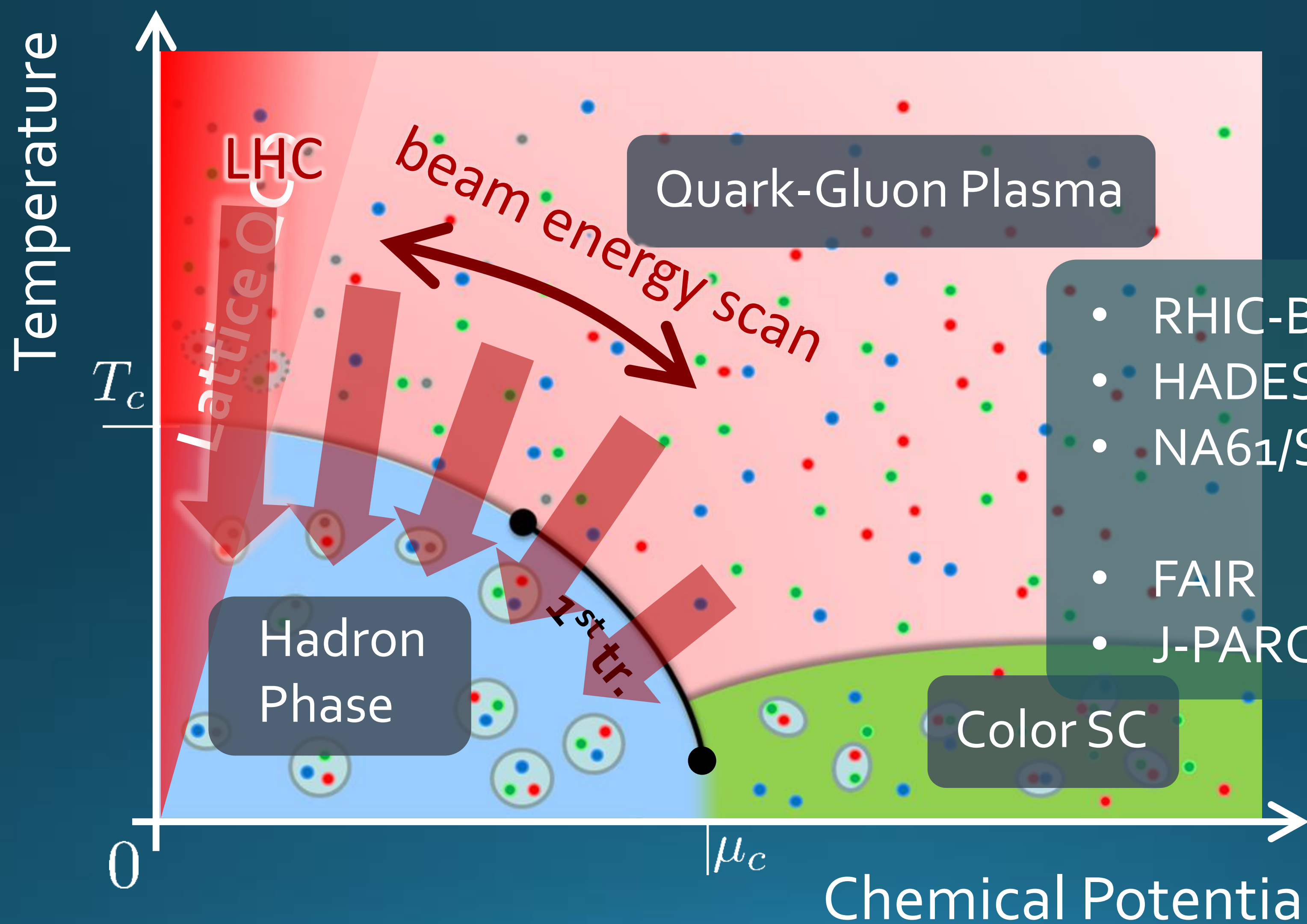
QCD Phase Diagram



QCD Phase Diagram



QCD Phase Diagram

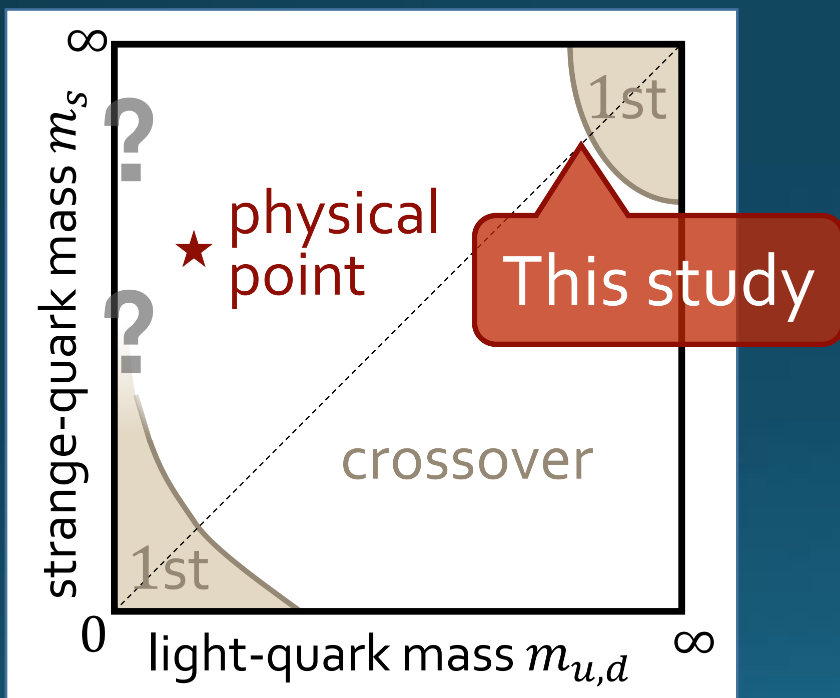


- RHIC-BES
- HADES
- NA61/SHINE
- FAIR
- J-PARC-HI

Varying Quark Masses @ $\mu_q = 0$

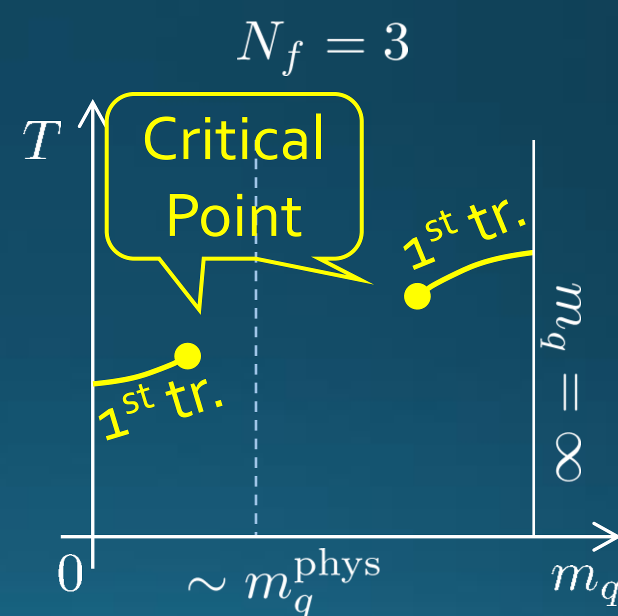
☐ Columbia plot

= order of phase tr. at $\mu_q = 0$



☐ Example

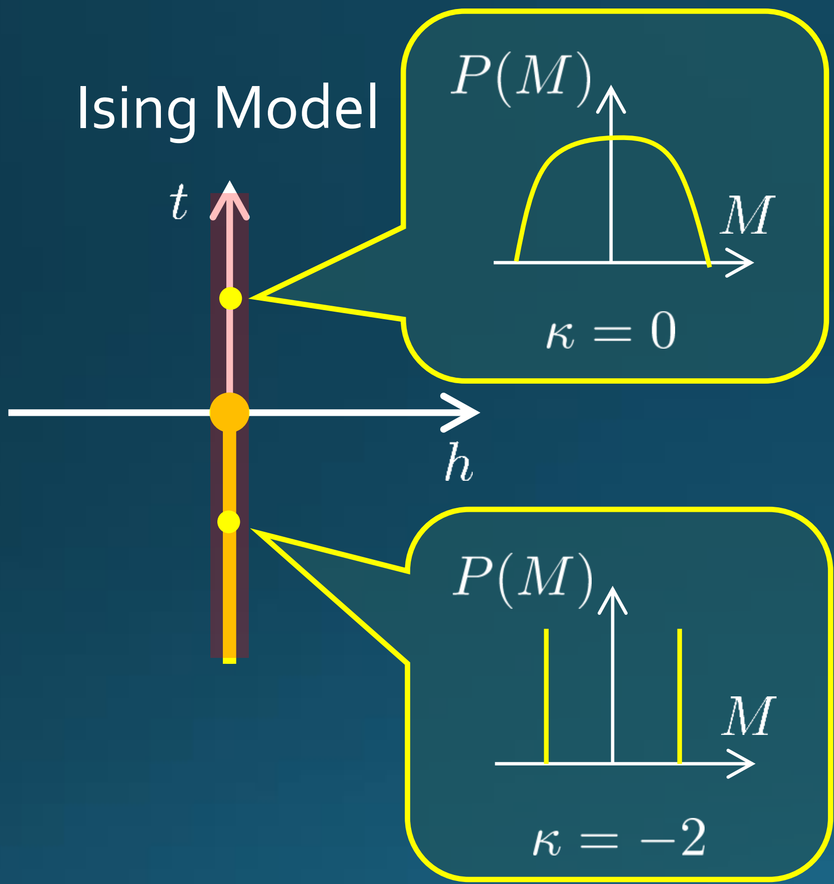
Phase diagram
in $T - m_q$ plane



Critical points on the Columbia plot → ☐ location? (existence?)
☐ universality class?

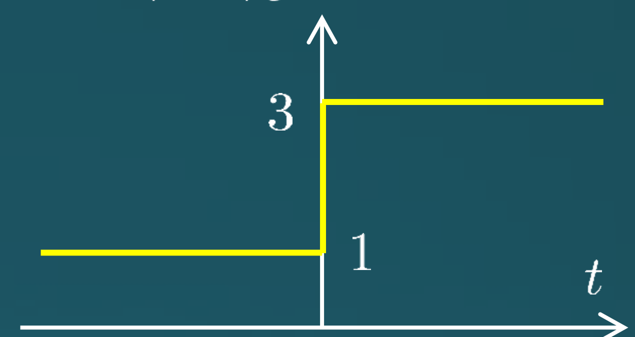
Binder Cumulant B_4

Ising Model



Binder Cumulant

$$B_4 = \frac{\langle M^4 \rangle_c}{\langle M^2 \rangle_c^2} + 3 = \kappa + 3$$



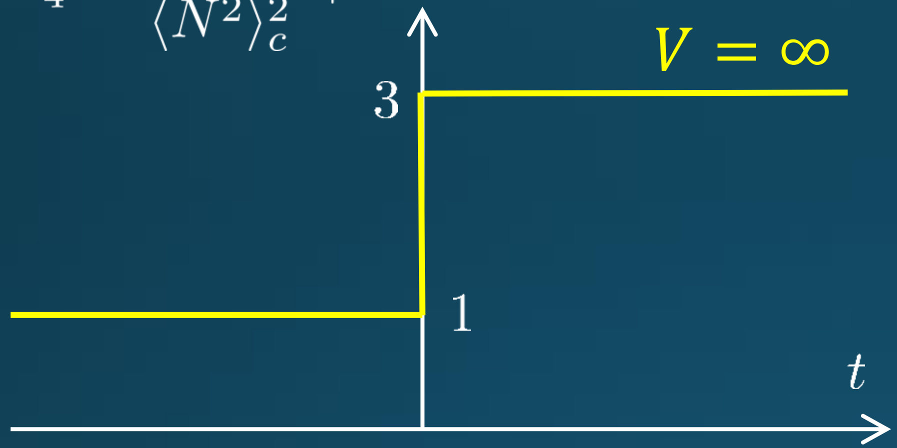
Kurtosis: $\kappa = \dfrac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2}$

- $\langle M^4 \rangle_{c,h=0}$ changes discontinuously at the CP.

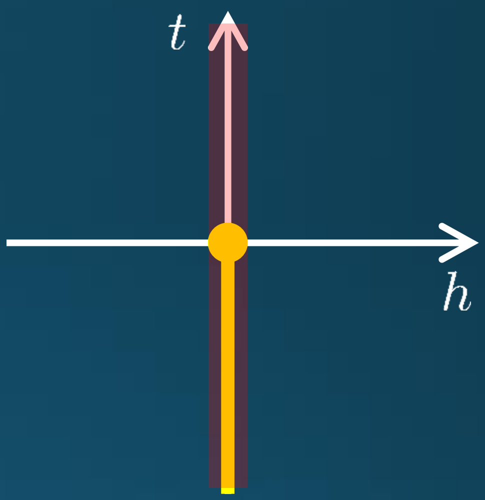
Finite-Volume Effects

Binder Cumulant

$$B_4 = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2} + 3$$



Ising Model

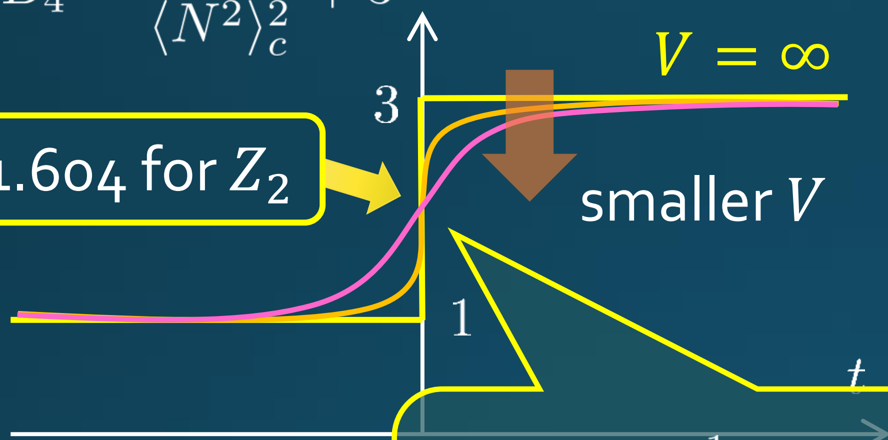


- Discontinuity of B_4 at the CP is smeared on finite V.
- B_4 obtained at various V have crossing at $t = 0$.

Finite-Volume Effects

Binder Cumulant

$$B_4 = \frac{\langle N^4 \rangle_c}{\langle N^2 \rangle_c^2} + 3$$

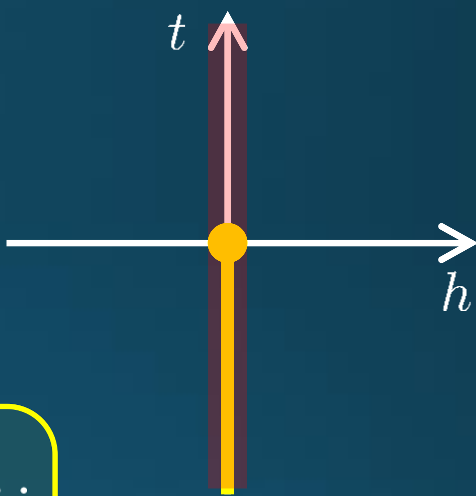


1.604 for Z_2

$$B_4(t, 0, L^{-1}) = b_4 + ctL^{y_t} + \cdots$$

Z(2) CP $\begin{cases} b_4 = 1.604 \\ \nu = 1/y_t = 0.630 \end{cases}$

Ising Model

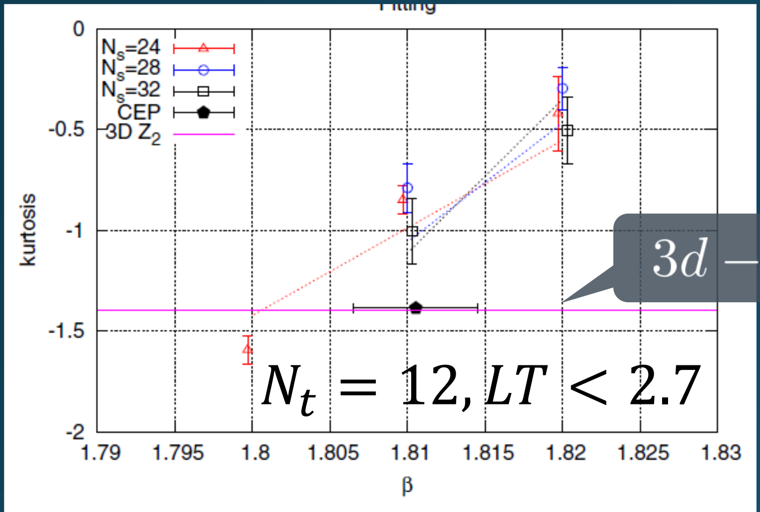


- ☐ Discontinuity of B_4 at the CP is smeared on finite V.
- ☐ B_4 obtained at various V have crossing at $t = 0$.

Lattice Studies of Binder-Cumulant

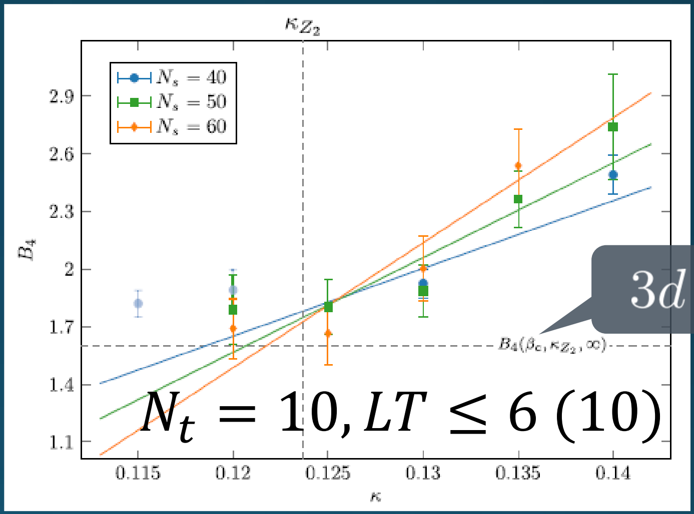
☐ Light-quark region

Kuramashi, Nakamura, Ohno, Takeda, '20



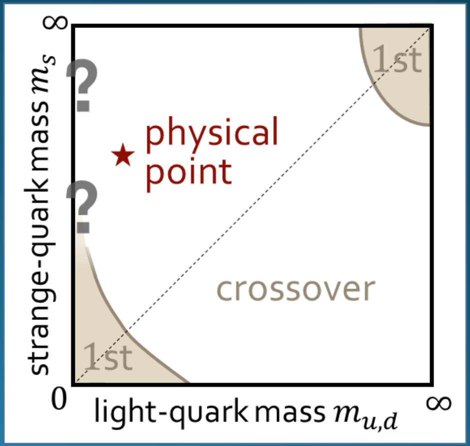
☐ Heavy-quark region

Cuteri , Philipsen , Schön, Sciarra, '21



☐ Statistically-significant deviation of the crossing point from the 3d-Ising value.

⇨ V may not be large enough?



Our Strategy

Kiyohara+, PRD104 (2021); Ashikawa+, in prep.

Numerical simulations on large spatial volume up to $LT = N_x/N_t = 15$

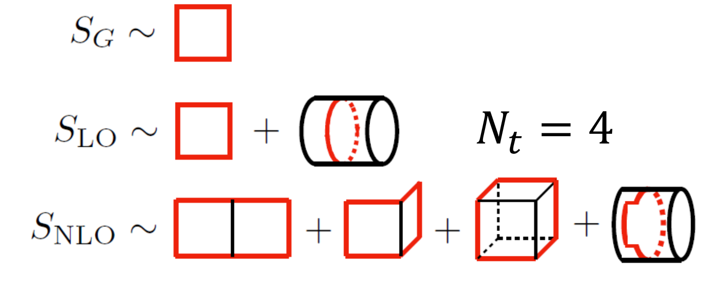
To realize it:

- ☐ CP in the heavy-quark region
- ☐ Simulations on coarse lattices ($N_t = 1/aT = 4 \to 6, 8$)
- ☐ Hopping parameter ($\kappa \sim 1/m_q$) expansion (HPE)

$$\kappa \sim \frac{1}{2m_q a}$$

$$\ln\det M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n]\kappa^n$$

$\mathrm{tr}[B_n]$ is given by the closed trajectories of length n.



Hopping Parameter Expansion

Wilson fermion

$$S_q = \sum_{x,y} \bar{\psi}_x M_{xy} \psi_y \qquad \kappa \sim \frac{1}{2m_q a}$$

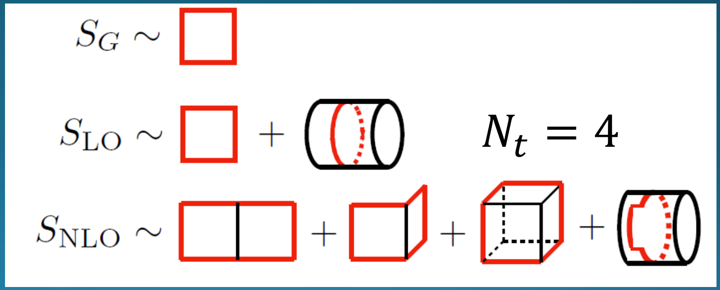
$$M_{xy} = \delta_{xy} - \kappa B_{xy} \qquad B_{xy} = \sum_{\mu=1}^{4} \left[(1-\gamma_\mu) U_{x,\mu} \delta_{y,x+\hat{\mu}} + (1+\gamma_\mu) U^\dagger_{y,\mu} \delta_{y,x-\hat{\mu}} \right]$$

nonzero only for neighboring (x, y)

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U_\mu \mathcal{O} e^{-S_g + \ln \det M(\kappa)}$$

$$\ln \det M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n] \kappa^n$$

$\mathrm{tr}[B_n]$ is given by the closed trajectories of length n.

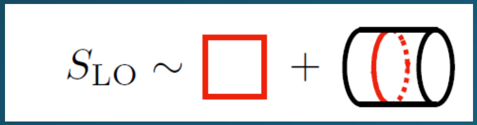


Numerical Setup for HPE

	LO	NLO	NNLO⋯
Wilson type	κ^4	κ^6	$\kappa^8 \dots$
Polyakov type	κ^{N_t}	κ^{N_t+2}	$\kappa^{N_t+4} \dots$

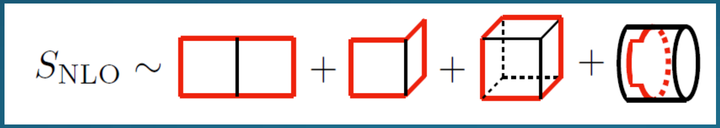
LO

Included in action of Monte-Carlo



NLO

Included by reweighting method



NNLO+

effective incorporation

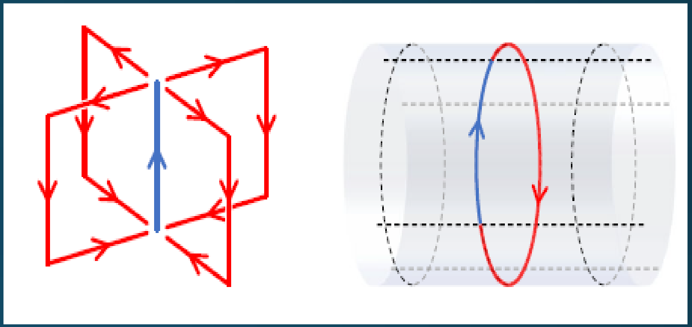
Wakabayashi+ ('22)

Kiyohara+ ('21)

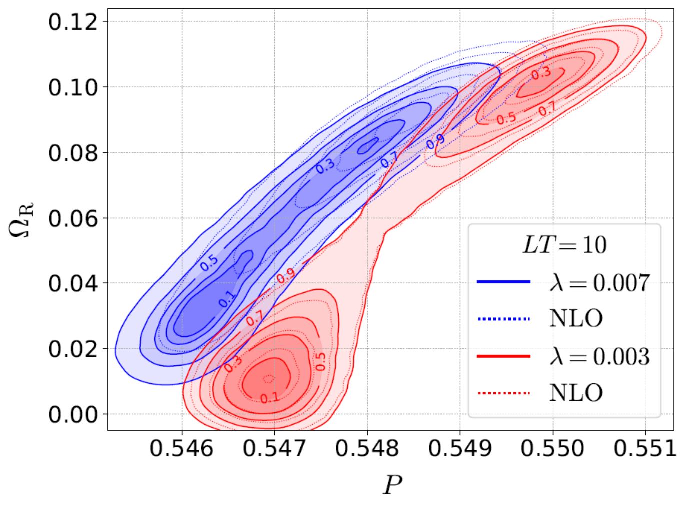
HPE: LO & NLO

Monte Carlo Simulation @ LO

- heat bath & over relaxation with modified staple
  - ➡ Numerical cost is almost the same as the pure YM!



$$\lambda = 64 N_c N_f \kappa^4$$



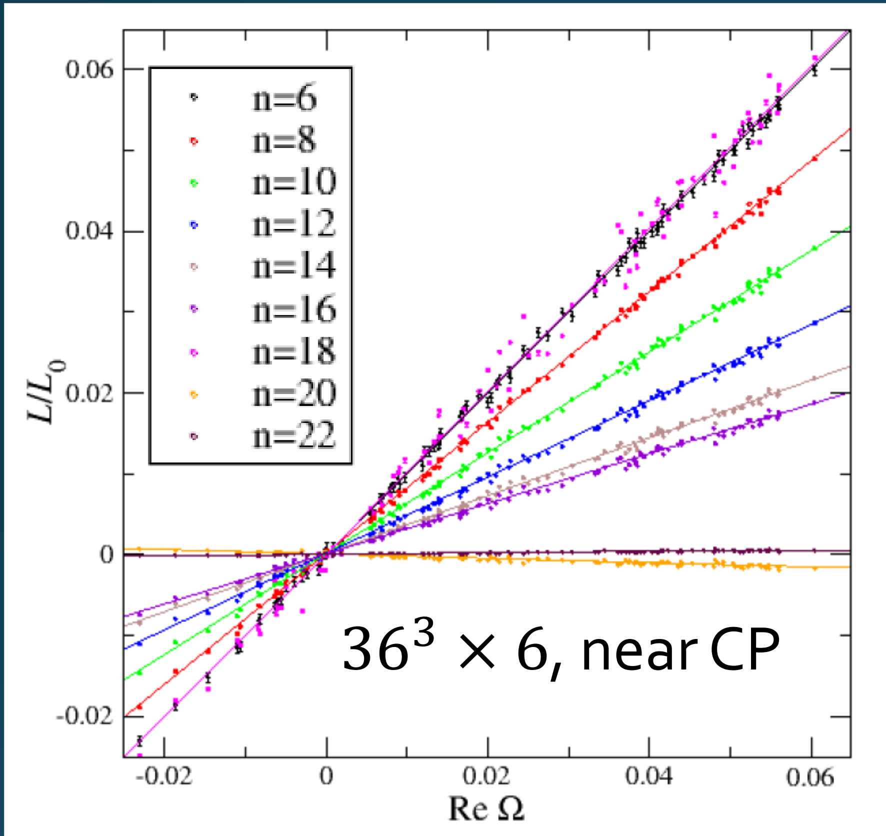
NLO by Reweighting

$$\langle \mathcal{O} \rangle_{\rm NLO} = \frac{\langle \hat{O} e^{-S_{\rm NLO}} \rangle_{\rm LO}}{\langle e^{-S_{\rm NLO}} \rangle_{\rm LO}}$$

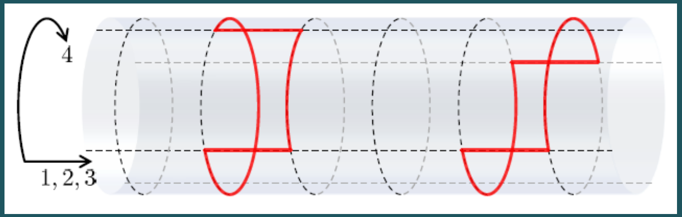
- Overlapping problem is well suppressed due to the LO confs.
  - ➡ Realize high statistical analysis

Polyakov-Types: Yet Higher Order

Wakabayashi+ ('22)



L_n: Winding loops of length n



Strong correlations of L_n between different n

$$L_n \simeq c_n \mathrm{Re}\Omega$$

Effects of winding terms: shift of HP: $\kappa \to \kappa_{\rm eff}$

$$\sum_n L_n \kappa_{\rm eff}^n = \mathrm{Re}\Omega \kappa_{\rm LO}^{N_t}$$

Numerical Simulations

$N_t = 4$
Kiyohara+, PRD104 ('21)
$N_t = 6, 8$
Ashikawa+, in prep.

- Coarse lattice: $N_t = 4, 6, 8$
- But **large spatial volume**:

  $LT = N_s\ /\ N_t \leq \mathbf{15}$
- High statistics (~$\mathbf{10^6}$ **measurements**)

- Hopping-param. (~$1/m_q$) expansion
- Monte-Calro with LO action
- 4~6 simulation points for reweighting
- Lattice size:

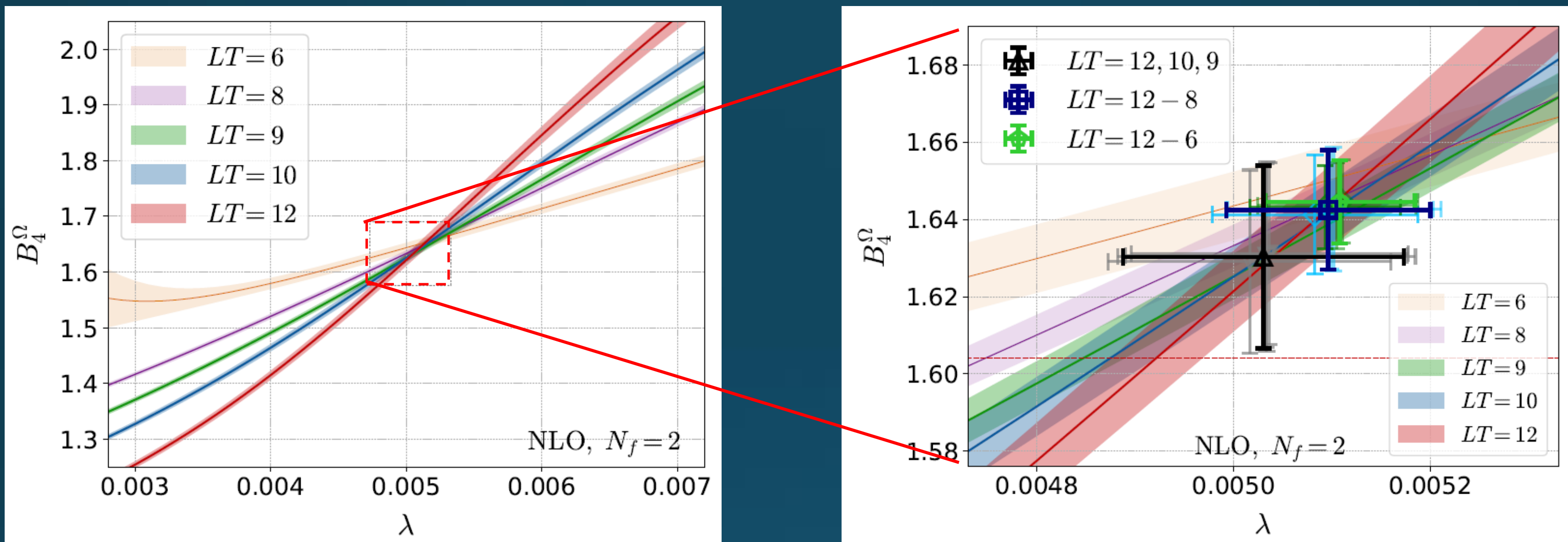
$N_t = 4 \quad LT = N_x/N_t = 6,8,9,10,12$

NEW

$N_t = 6 \quad LT = N_x/N_t = 6,7,8,9,10,12,15$

$N_t = 8 \quad LT = N_x/N_t = 6,8,10,12$ (in prog.)

Binder-Cumulant @ $N_t = 4$

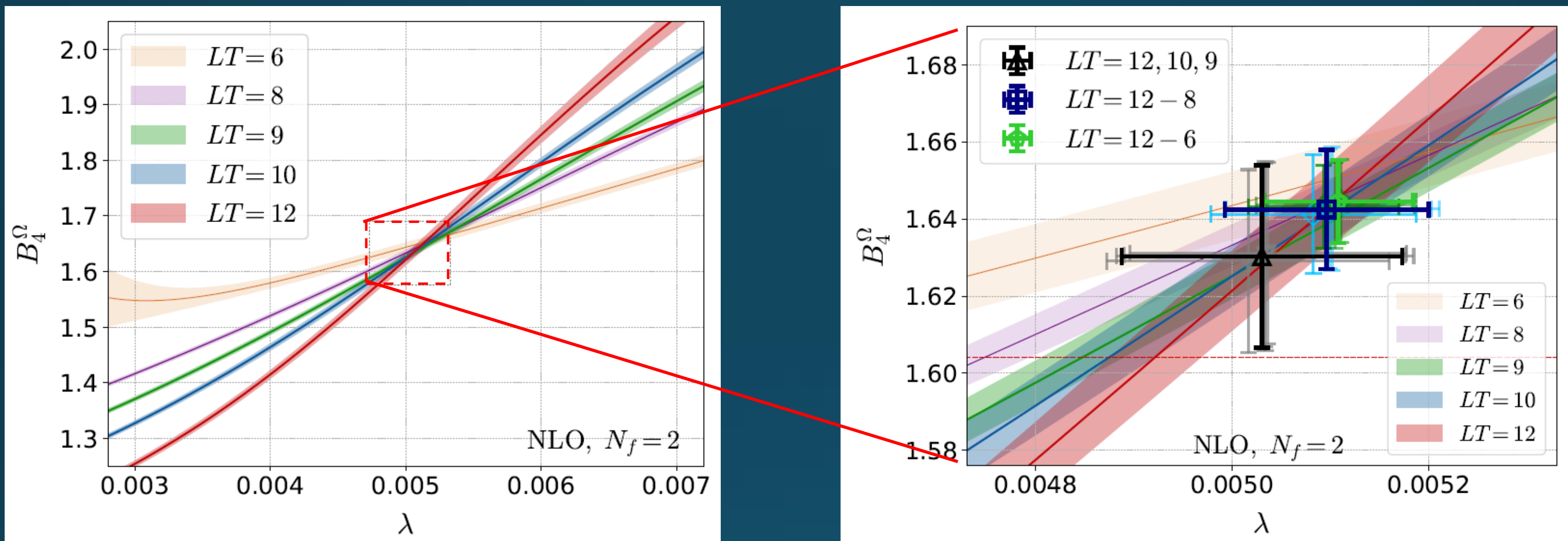


Fitting function

$$B_4(\lambda, LT) = b_4 + c(\lambda - \lambda_c)(LT)^{1/\nu}$$

params: b_4, c, λ_c, ν

Binder-Cumulant @ $N_t = 4$



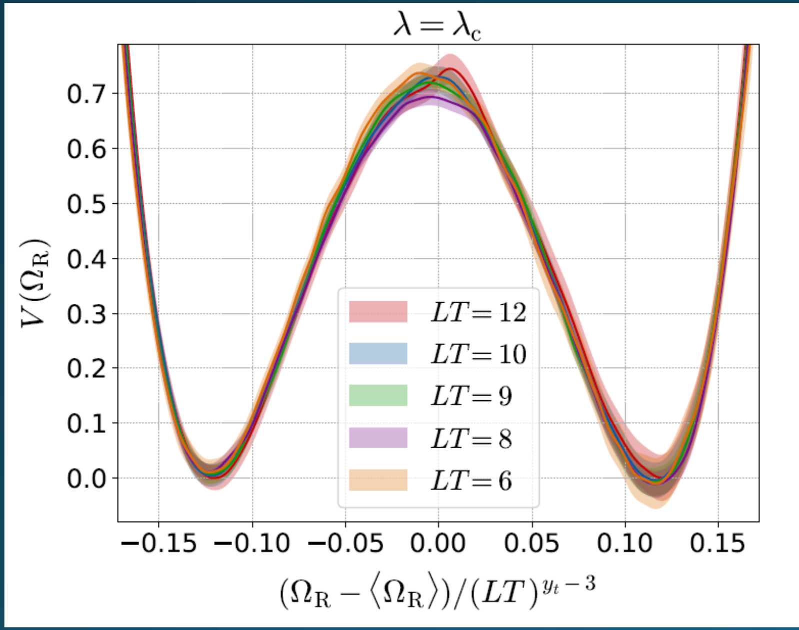
$LT \geq 9$	$B_4 = 1.630(24)(2),$	$\nu = 0.614(48)(3)$
$LT \geq 8$	$B_4 = 1.643(15)(2),$	$\nu = 0.614(29)(3)$
Z2	$B_4 = 1.604$	$\nu = 0.630$

- ☐ B_4 and ν are consistent with Z2 universality class only when $LT \geq 9$ data are used for the analysis.

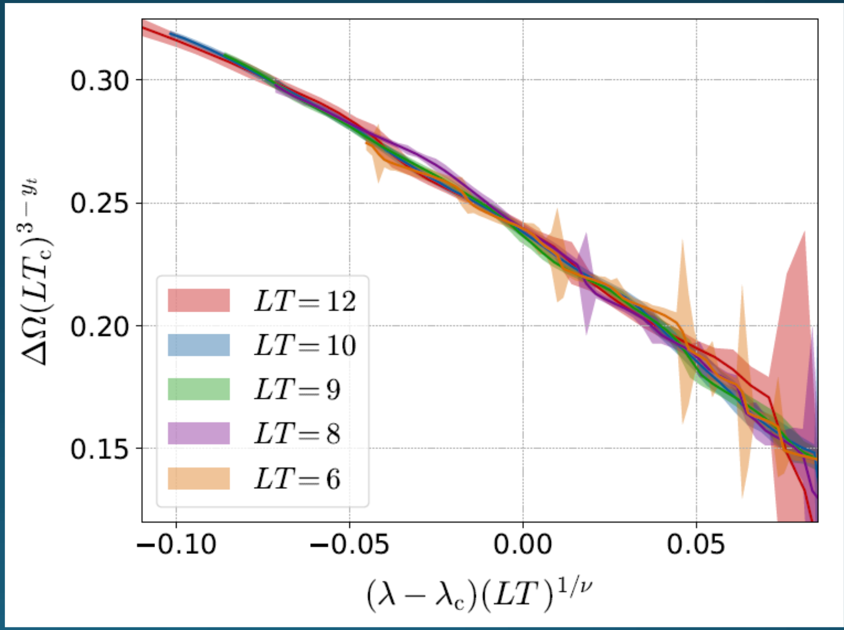
Other Scaling Analyses @ $N_t = 4$

Kiyohara+ ('21)

- **Effective Potential**



- **Gap of Peaks (~*M*)**

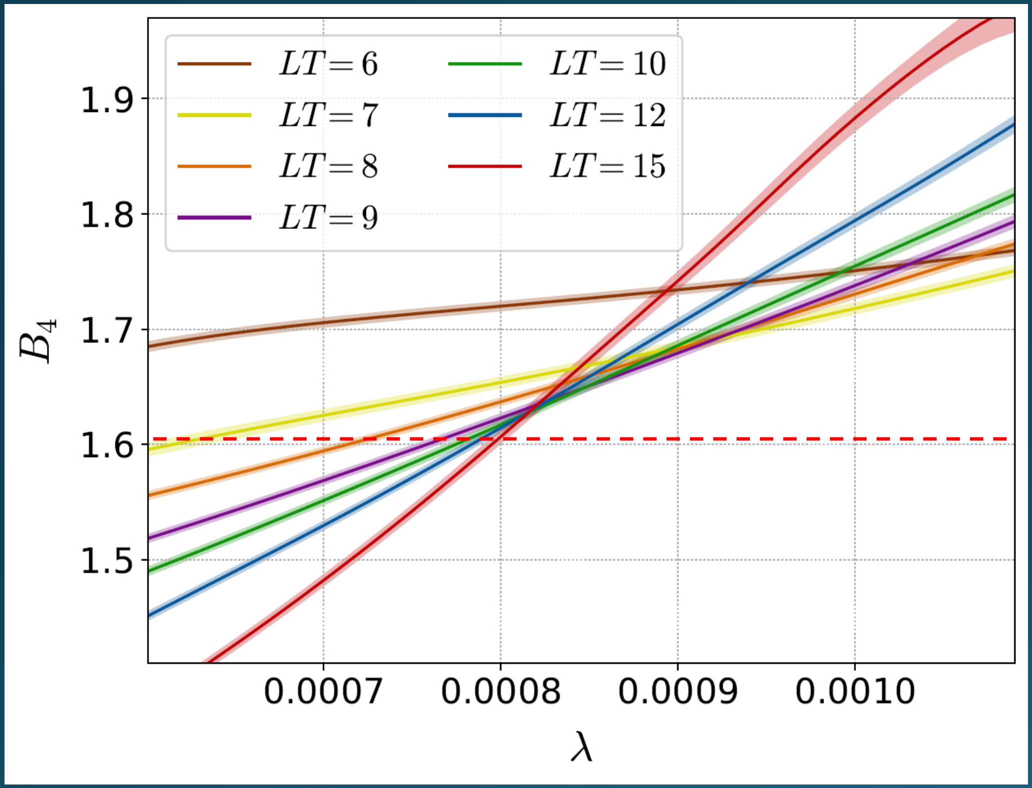


- Z2 scaling has been confirmed with high precision!
- Violation at small V comes from the tail of distribution.

NEW

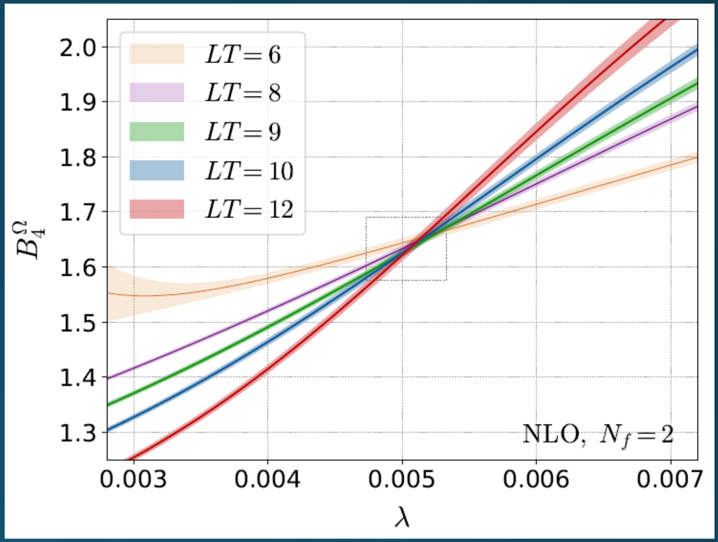
Binder Cumulant @ $N_t = 6$

$N_t = 6$



Ashikawa+, in prep.

$N_t = 4$

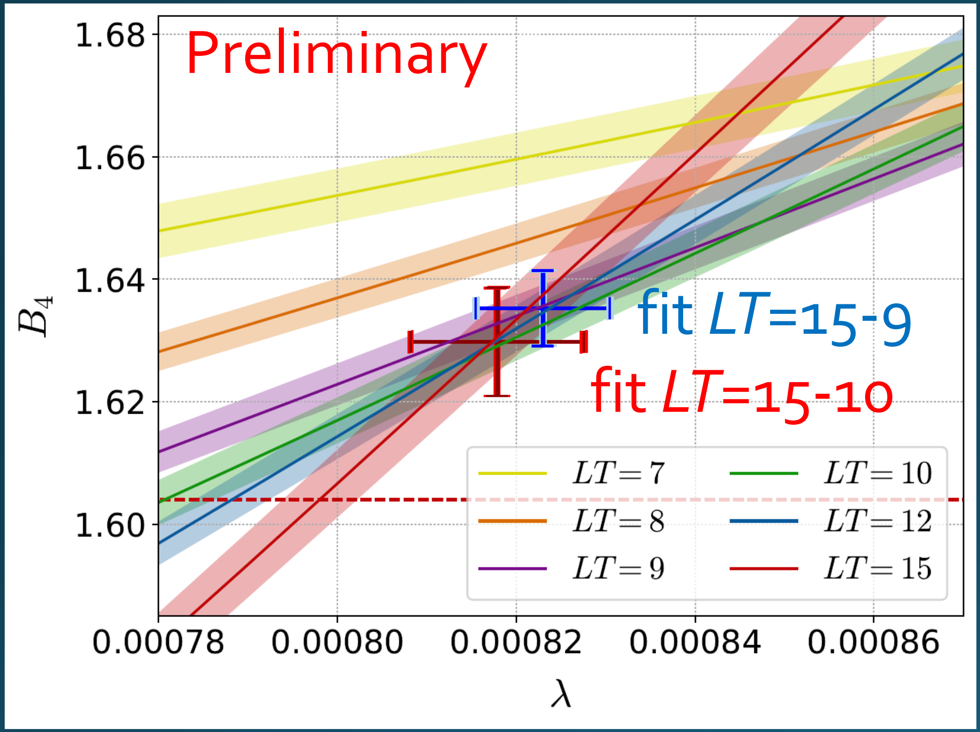


$$a = \frac{1}{N_t T}$$

- For $N_t = 6$, deviation from the finite-size scaling is more significant with the same V.

Binder Cumulant @ $N_t = 6$

$N_t = 6$



Fit result

$(N_t = 15,12,10)$	Z_2
$b_4 = 1.630(9)$	1.604
$\nu = 0.624(19)$	0.630

$\lambda_c^{\mathrm{NLO}} = 0.000818(10)$

Disagreement with $b_4^{Z_2}$?

☐ Critical HP

$\lambda_c^{\mathrm{NLO}} = 0.000818(10)$ ▶ $\lambda_c^{\mathrm{22th}} = 0.000704(8)$

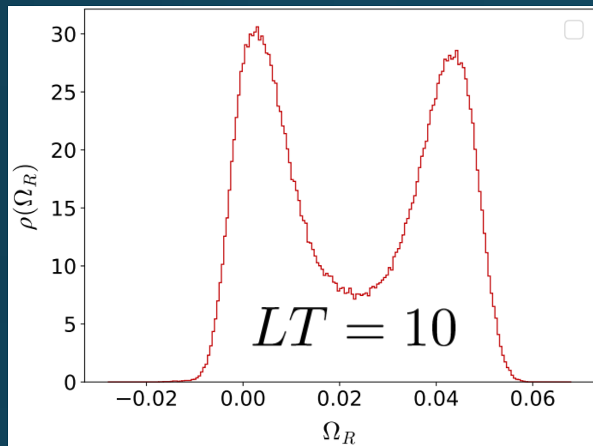
$\kappa_c^{\mathrm{NLO}} = 0.09003(19)$ ▶ $\kappa_c^{\mathrm{22th}} = 0.08781(17)$

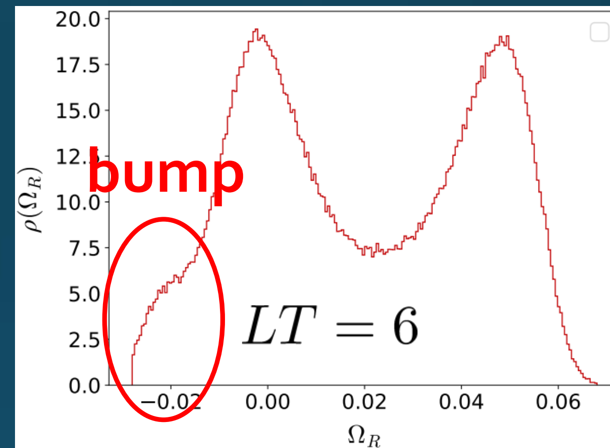
cf) Cuteri+ ('21) $\kappa_c = 0.0877(9)$

Violation of FSS & Remnant of Z(3)

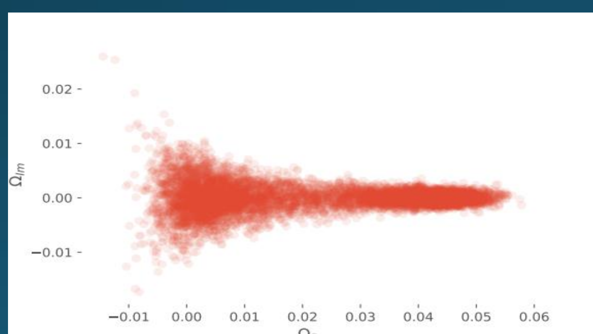
Probability Distribution of Polyakov loop

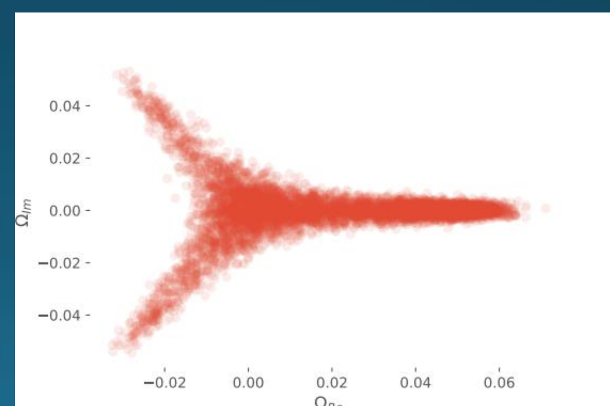
Real Part of Ω





Complex Plane





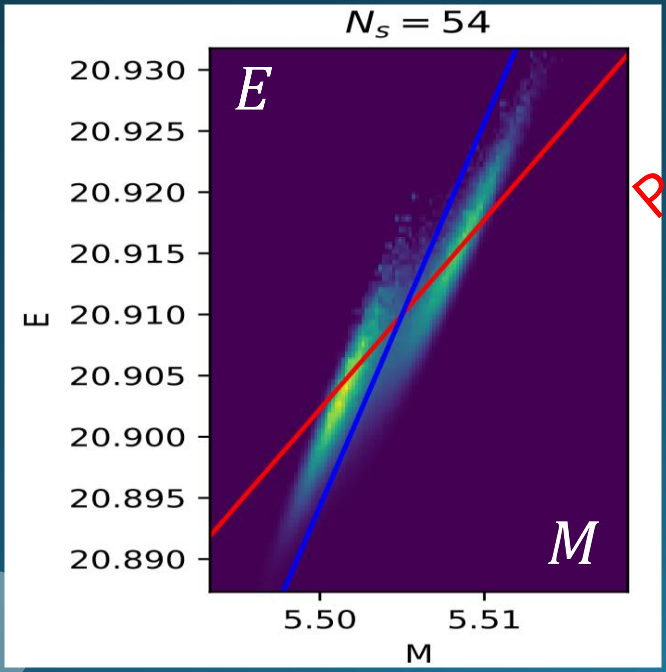
- ☐ Remnant of Z_3 is an origin of the violation of FSS

Extracting Magnetic-Like Obs.

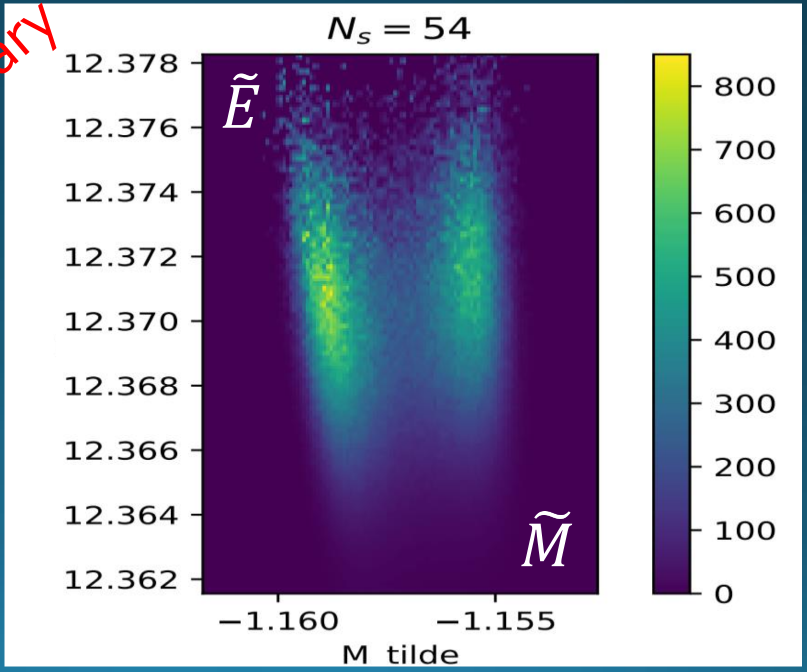
- $\Omega \neq$ magnetic observable in Ising model.
- Mixing of energy-like observable

$$\begin{pmatrix} \tilde{E} \\ \tilde{B} \end{pmatrix} = \begin{pmatrix} 1 & s \\ r & 1 \end{pmatrix} \begin{pmatrix} E \\ M \end{pmatrix}$$

- We construct new order paramters from
  - direction of 1st order transition
  - diagonality b/w *E* and *M* Karsch, Stickan ('00)

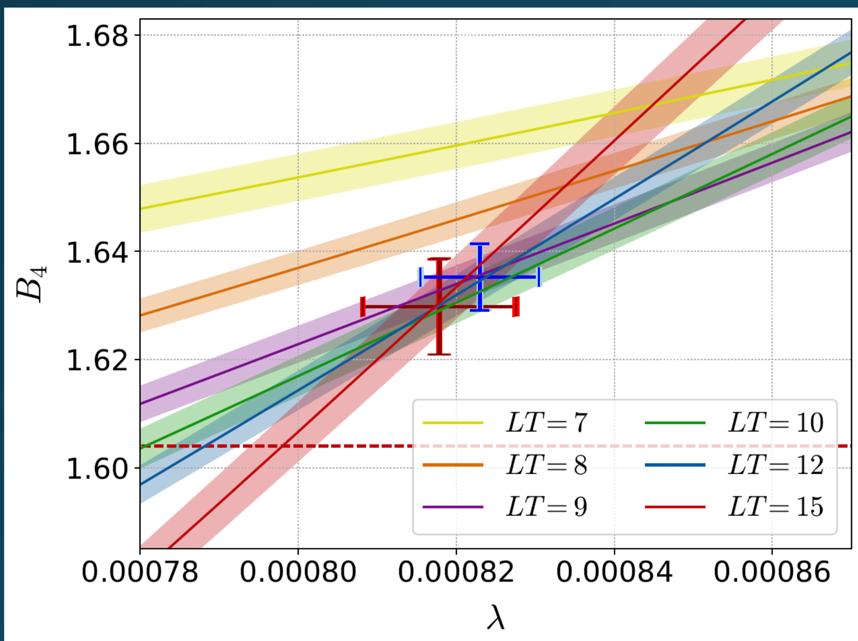


Preliminary

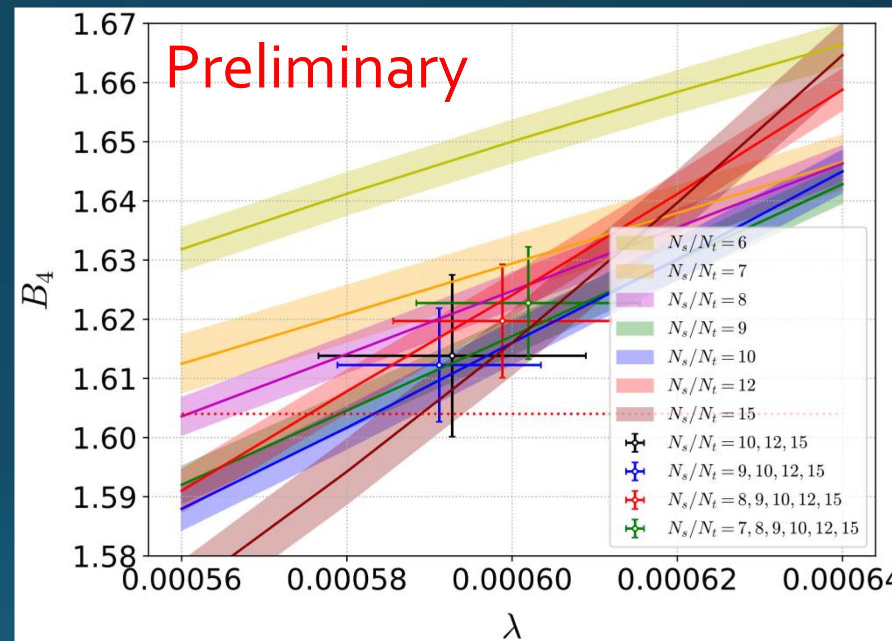


B4 of Magnetic-Like Obs.

B4: Polyakov loop



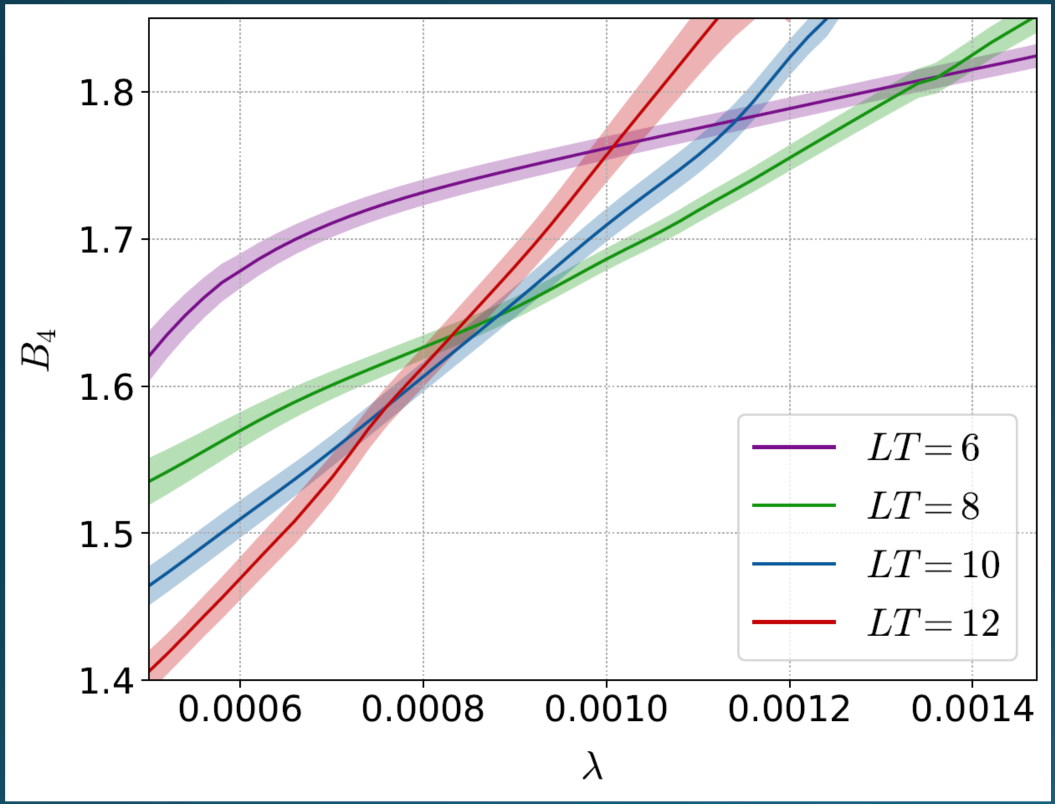
B4: magnetic $\widetilde{M}$



- ☐ Newly generated order parameter $\widetilde{M}$ gives the result consistent with the Z2 FSS.

Binder Cumulant @ $N_t = 8$

WHOT, in progress



$\lambda_c^{\mathrm{LO}} \simeq 0.0008$

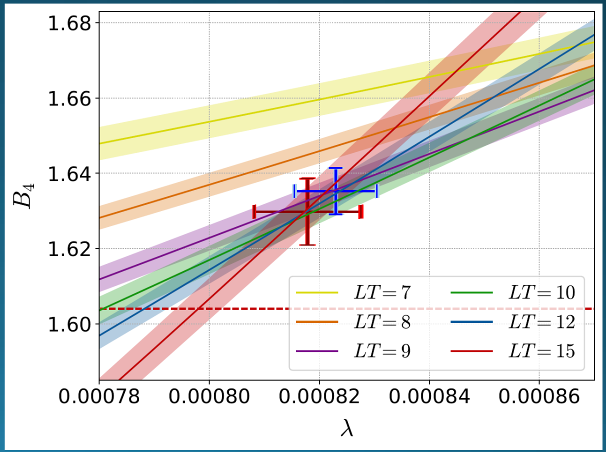
$\kappa_c^{\mathrm{LO}} \simeq 0.1378$

$\kappa_c^{\mathrm{22th}} \simeq 0.1087$

Similar result as $N_t = 6$.

Summary

- We investigated the critical point in heavy-quark QCD by the Binder cumulant analysis.
- Our Monte-Carlo analysis based on the HPE works quite effectively in the heavy-quark region.
- At $N_t = 6, 8$, the violation of FSS at finite V is more prominent than $N_t = 4$. In particular, $LT = N_x/N_t = 6$ would be too small to adopt the FSS analysis.
- Future:
  - yet larger N_t
  - mixing of energy-like observable,
  - finite density (Ejiri, Mon.)
  - etc.



Finite-Size Scaling

Infinite vol.: $F(t,h) = F(b^{y_t}t, b^{y_h}h)$

Finite vol.: $\tilde{F}(t,h,L^{-1}) = \tilde{F}(b^{y_t}t, b^{y_h}h, bL^{-1})$

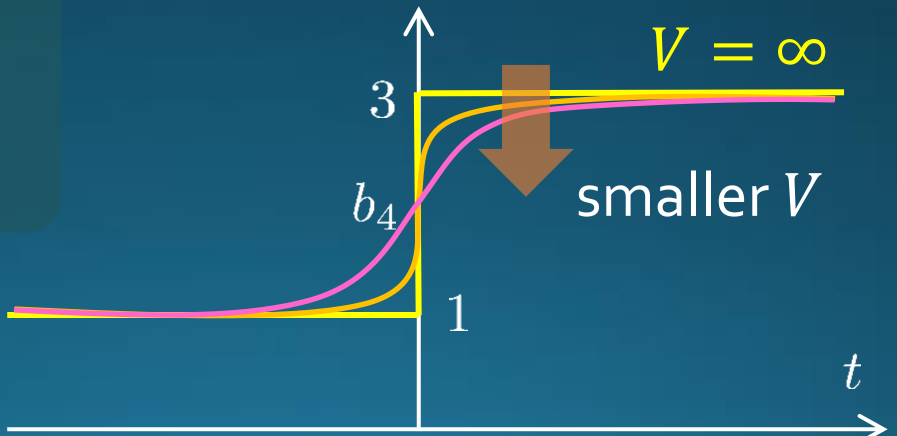
$= \tilde{F}(L^{y_t}t, L^{y_h}h, 1)$ $b = L$

$$\langle M^n \rangle_c = \frac{\partial^n F}{\partial h^n}$$

$B_4(t,0,L^{-1}) = b_4 + ctL^{y_t} + \cdots$

$Z(2)$ universality class:

$b_4 = 1.604, \quad \nu = 1/y_t = 0.630$

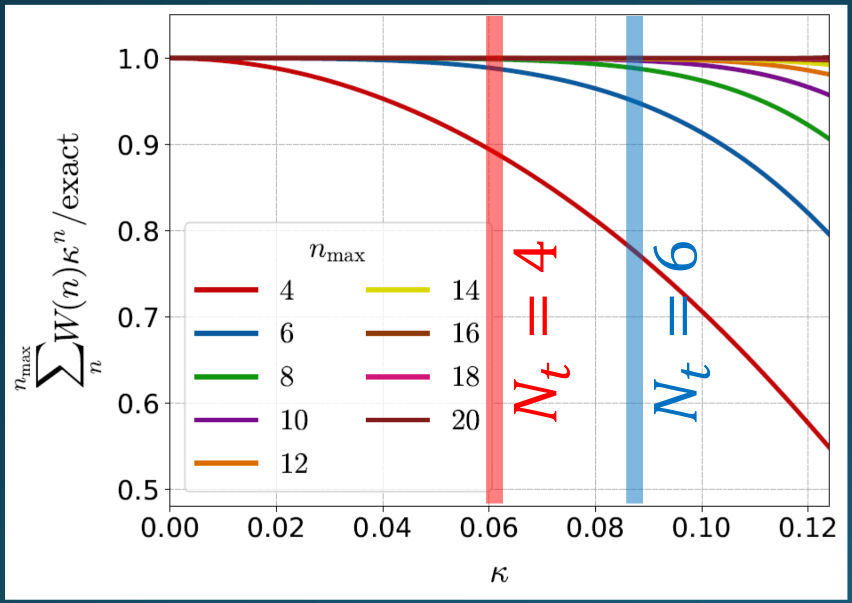


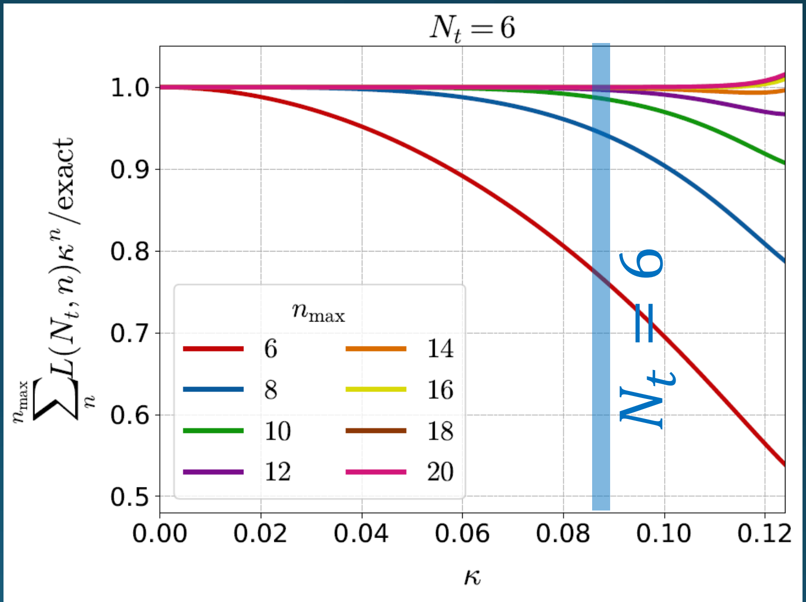
Convergence of HPE

Wakabayashi+ ('22)

- HPE of free lattice field (U=1)

Wilson-loop-type **Polyakov-loop-type**





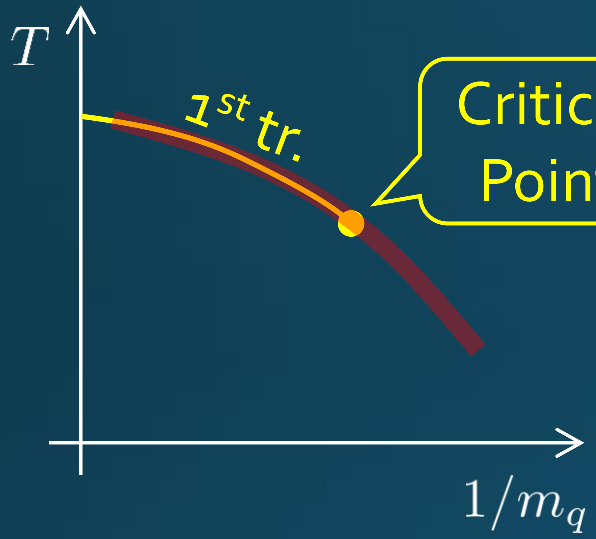
$N_t = 4\ \ \kappa_c = 0.0602(4)$ Kiyohara+,'21

$N_t = 6\ \ \kappa_c = 0.0877(9)$ Cuteri+, '21

NNLO and higher

Wakabayashi+ ('22)

Transition Line



Definitions of transition line

- ☐ Maximum of $\langle\Omega_R^2\rangle$
- ☐ Zero of $\langle\Omega_R^2\rangle$
- ☐ Minimum of B_4