Quantum Computation of O(3) model using qumodes

[Collaborators: George Siopsis, Shane Thompson, and Felix Ringer]

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August 02, 2023

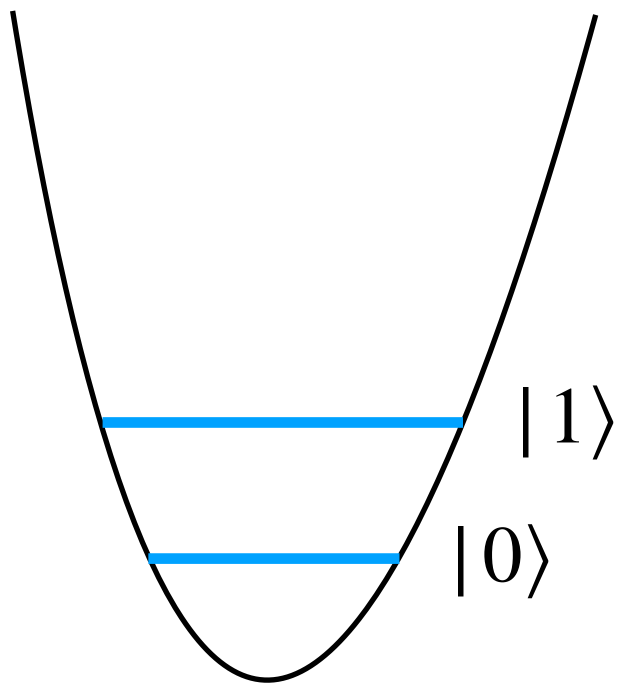
Quote

- The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction.

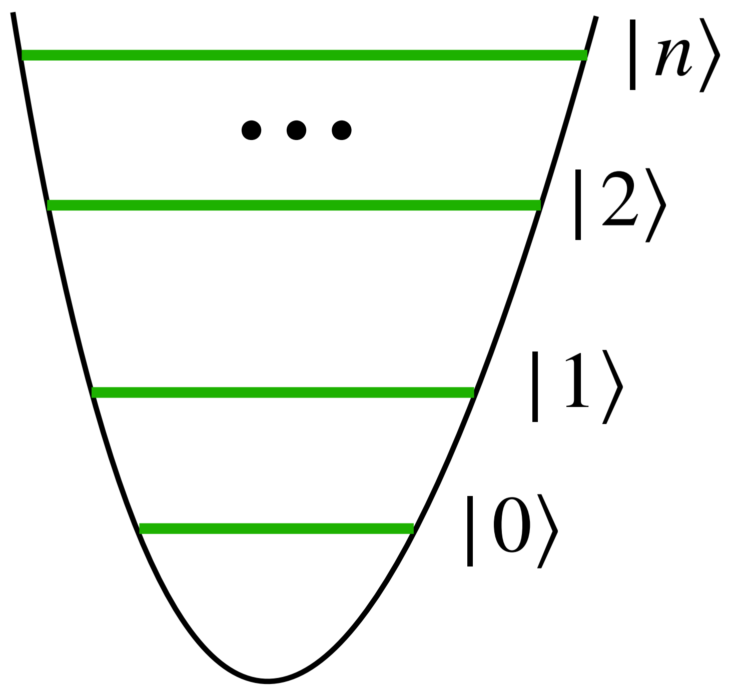
- Oscillators are also useful for quantum computation (subject of this talk!).

Overview

- Ways to universal quantum computation: qubits, qudits, and oscillators [also known as *qumodes*].
- Simple two-site example of Bose-Hubbard model and time evolution circuit.
- O(3) on a spatial lattice with two qumodes at each site
- Getting to scaling regime with photonic hardware in coming decade.

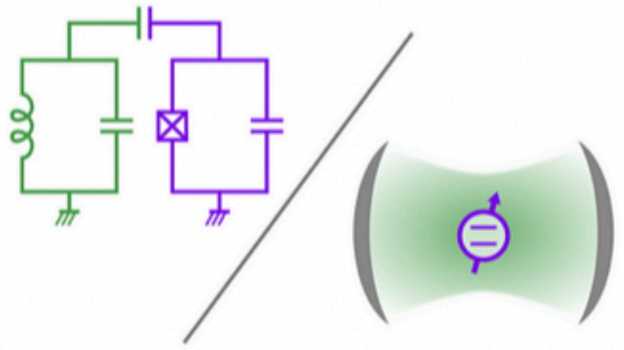
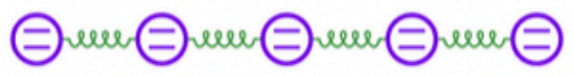
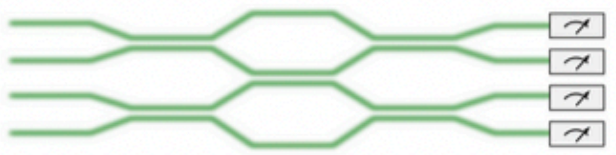


Qubit



Fock states

Qumodes

Circuit/Cavity QED 	Both circuit and cavity QED platforms involve an electromagnetic mode of a circuit/cavity interacting with a natural or "artificial" atom, such as a transmon or quantum dot. **Qumodes:** electromagnetic circuit/cavity modes **Qubits:** transmon, fluxonium, etc. (circuit QED) quantum dot, color center, cold atoms, etc. (cavity QED)
Trapped Ion 	Trapped ion architectures consist of qubits encoded in the electronic states of individual ions that couple through the collective motional modes of the ion chain/lattice. **Qumodes:** collective vibrational modes of the ions **Qubits:** (a pair of) electronic levels of an ion
Photonic 	Photonic quantum processors encode information using a network of guided optical modes, which can be leveraged either as a qumode or used to encode a qubit. Entangling gates are achieved through measurement. **Qumodes:** photonic modes **Qubits:** photon-encoded qubit (polarization, dual-rail, etc.)

CV vs. DV

	CV	Qubit
Basic element	Qumodes	Qubits
Relevant operators	Quadrature operators $\hat{x}, \hat{p}$ Mode operators $\hat{a}, \hat{a}^\dagger$	Pauli operators $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$
Common states	Coherent states $\|\alpha\rangle$ Squeezed states $\|z\rangle$ Number states $\|n\rangle$	Pauli eigenstates $\|0/1\rangle, \|\pm\rangle, \|\pm i\rangle$
Common gates	Rotation, Displacement, Squeezing, Beamsplitter, Cubic Phase	Phase Shift, Hadamard, CNOT, T Gate

Ref: Xanadu SF

arXiv: 1804.3159

State family	Displacement	Squeezing
Vacuum state $\|0\rangle$	$\alpha = 0$	$z = 0$
Coherent states $\|\alpha\rangle$	$\alpha \in \mathbb{C}$	$z = 0$
Squeezed states $\|z\rangle$	$\alpha = 0$	$z \in \mathbb{C}$
Displaced squeezed states $\|\alpha, z\rangle$	$\alpha \in \mathbb{C}$	$z \in \mathbb{C}$
$\hat{x}$ eigenstates $\|x\rangle$	$\alpha \in \mathbb{C}$, $x = 2\sqrt{\frac{\hbar}{2}}\mathrm{Re}(\alpha)$	$\phi = 0,\ r \to \infty$
$\hat{p}$ eigenstates $\|p\rangle$	$\alpha \in \mathbb{C}$, $p = 2\sqrt{\frac{\hbar}{2}}\mathrm{Im}(\alpha)$	$\phi = \pi,\ r \to \infty$

Gate	Unitary	Symbol
Displacement	$D_i(\alpha) = \exp(\alpha \hat{a}_i^\dagger - \alpha^* \hat{a}_i)$	D
Rotation	$R_i(\phi) = \exp(i\phi \hat{n}_i)$	R
Squeezing	$S_i(z) = \exp(\frac{1}{2}(z^* \hat{a}_i^2 - z \hat{a}_i^{\dagger 2}))$	S
Beamsplitter	$BS_{ij}(\theta, \phi) = \exp(\theta(e^{i\phi} \hat{a}_i \hat{a}_j^\dagger - e^{-i\phi} \hat{a}_i^\dagger \hat{a}_j))$	BS
Cubic phase	$V_i(\gamma) = \exp\left(i \frac{\gamma}{3\hbar} \hat{x}_i^3\right)$	V

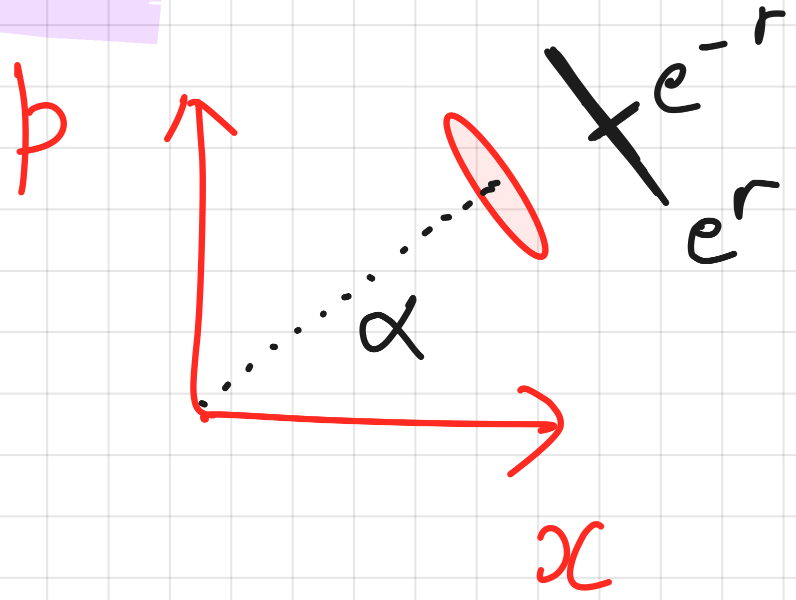
Recall: Coherent state is eigenstate of annihilation operator. This state has the dynamics most closely resembling the oscillatory behavior of a classical harmonic oscillator.

Basic CV state

Coherent States : States of light (photon) whose expectation value corresponds to classical EM wave.

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

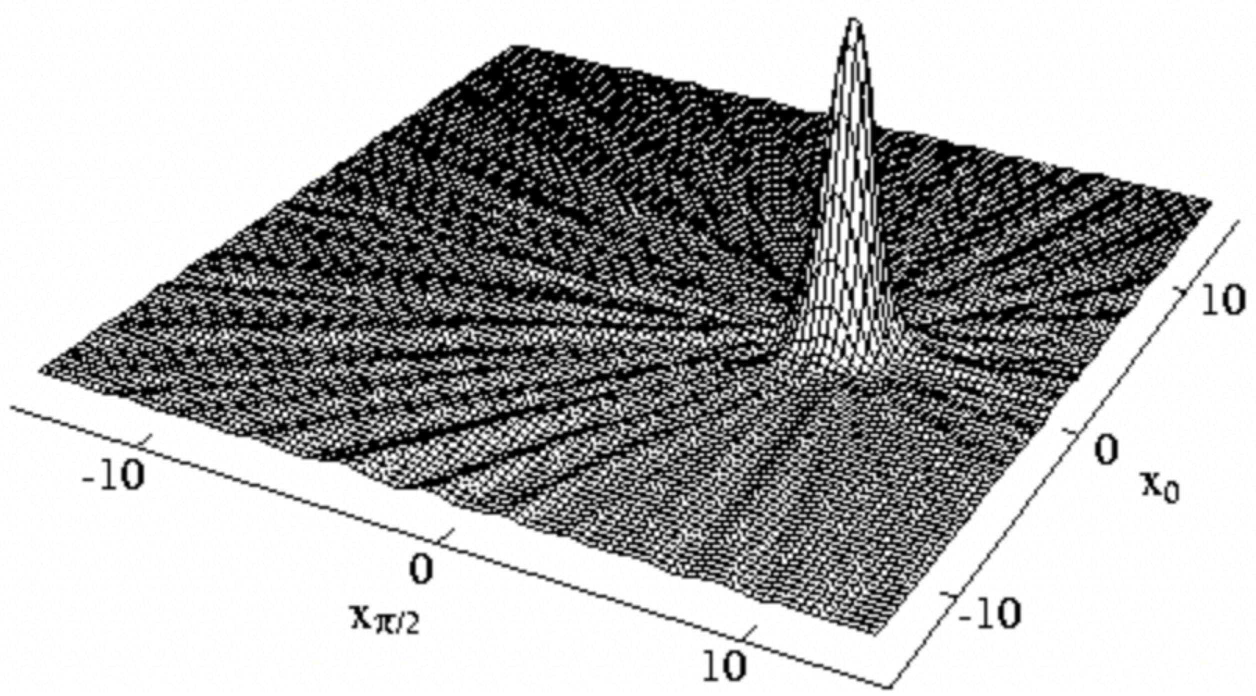
α is the displacement.



$$z = r e^{i\phi}$$

r → Sq. parameter

$|z\rangle \rightarrow$ squeezed state



Wigner distribution

Bose-Hubbard model

1801.06565

- For fermionic systems, like Ising model, the qubit approach is generally preferred but for models with bosonic degrees of freedom (where the local Hilbert space dimension is infinite), the more natural setting is one of oscillator (qumodes). Suppose, we consider the Bose-Hubbard model where the H is given by:

$$H = J\left[\sum_{\langle ij\rangle} a_i^\dagger a_j + H.c\right] + \frac{1}{2}U\sum_i \hat{n}_i(\hat{n}_i - 1)$$

where we have used create /annihilation operators and the number operators. The first term denotes the hopping of bosons between neighbouring sites and second term is the on-site potential term.

Two-site model

- We can write the time-evolution operator as:

$$e^{-iHt} = \left[BS\left(\theta,\phi\right)\left(K(r)R(-r)\otimes K(r)R(-r)\right)\right]^{N} + \mathcal{O}\left(t^2/N\right) \quad ;\theta = -Jt/N, \phi = \pi/2,\ r = -Ut/2N$$

where BS is the beam-splitter gate, K is the Kerr gate (non-Gaussian), and R is the rotation gate. These gates are qumodes equivalent of the qubit gates we saw before. For example, $K(\kappa) = \exp(i\kappa\hat{n}^2)$. Constructing these gates are major undertaking in quantum photonics labs where the photon is modelled as an oscillator.

$$H = \underbrace{J\left(\hat{a}_1^{\dagger}\hat{a}_2 + \hat{a}_2^{\dagger}\hat{a}_1\right)}_{\text{hopping}} + \frac{U}{2}\underbrace{\left(\hat{n}_1^2 - \hat{n}_1 + \hat{n}_2^2 - \hat{n}_2\right)}_{\text{on-site}}$$

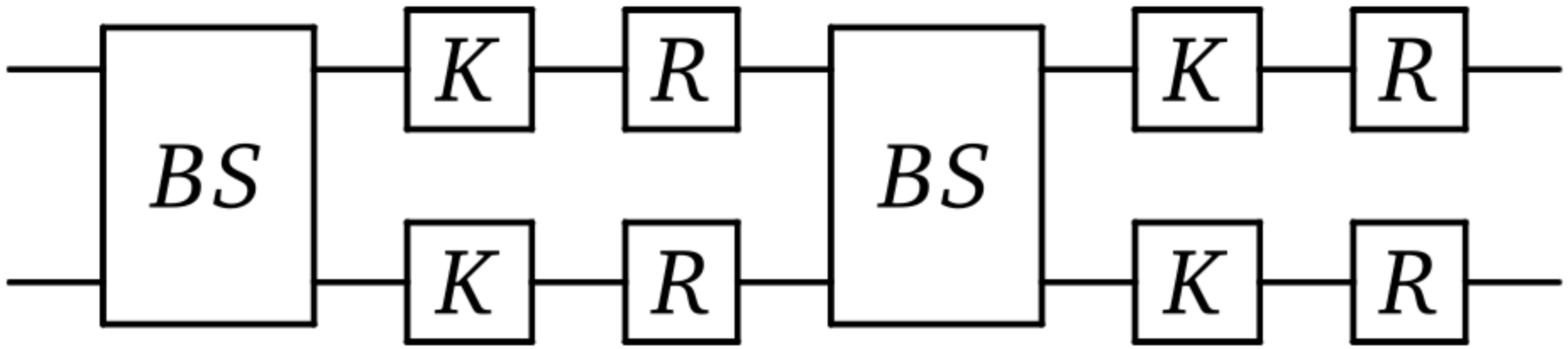
Use Lie-Product formula:

$$e^{A+B} = \mathop{Lt}_{N\to\infty}\left(e^{A/N}e^{B/N}\right)^N$$

We can write

$$e^{-iHt} = \left[\underbrace{e^{-\frac{iJt}{K}\left(a_1^{\dagger}a_2 + a_2^{\dagger}a_1\right)}}_{BS_{12}}\ \overbrace{e^{-\frac{iUt}{2K}\hat{n}_1^2}}^{K_1}\ \overbrace{e^{-(\,)}}^{K_2}\ \overbrace{e^{-(\,)}e^{-(\,)}}^{R_1, R_2} + O\left(t^2/K\right)\right]^K$$

Two-site model



O(3) model

- We would like to understand a simple model in 1-dimension (two Euclidean) possibly with properties similar to QCD such as - asymptotic freedom, mass gap, instantons etc. [Polyakov, '75].

- A good toy model is the O(3) model defined by the Hamiltonian [Hamer-Kogut-Susskind, '79]:

$$\hat{H} = \frac{1}{2\beta}\sum_{i} \mathsf{L}_i^2 - \beta \sum_{\langle i,j \rangle} \mathsf{n}_i \cdot \mathsf{n}_j,$$

where $\beta = 1/g^2$.

<u>O(3) model</u>

- This model was studied using qubit approach by Alexandru et. al [PRL 123, 090501, 2019] using ideas of fuzzy sphere (fuzzy qubitization) arguing that two qubits (i.e., $d = 4$) per site suffices. In general, the size of the Hamiltonian is $(l_{\text{max.}} + 1)^{2N} \times (l_{\text{max.}} + 1)^{2N}$, and they argued that keeping up to $l_{\text{max.}}$=1 works such that Hamiltonian is $4^N \times 4^N$.

- However, there are tensor network computations using MPS that have found that as we approach the continuum limit (i.e., $\beta \to \infty$), the higher representations are needed to get the correct Physics.

- As mentioned before, the continuous variable (qumodes) approach also is a pathway to universal quantum computation. It seems natural that this model should be studied in terms of bosonic oscillators. **<u>Bosons for Bosons.</u>**

- In the Bose-Hubbard example we considered, we had one set of creation and annihilation operator i.e., $a, a^\dagger$ at each lattice site.

- Consider that there are two sets of Bose operators at each site. Let us denote them by a, b respectively. In this case, we then have $\hat{n} = a^\dagger a + b^\dagger b$. It turns out that we can define operators at each site such as:

$$K_+ := a^\dagger b^\dagger, K_- := ba, K_3 := \frac{1}{2}(\hat{n} + \mathbb{I})$$

- These operators form representation theory of $\mathfrak{su}(1,1)$ algebra.

O(3) model

- We have to express the rotor Hamiltonian in terms of oscillators. This can be done using work due to [Schwinger 1952]. It turns out we need two modes per site (i.e., two oscillators). Note that for Bose-Hubbard, we needed just one per site!

- We make use of the relation:

$$|l, m\rangle = \frac{(a^\dagger)^{l+m}(b^\dagger)^{l-m}}{\sqrt{(l+m)!(l-m)!}}|0,0\rangle$$

- The kinetic term at each site becomes:

$$\mathsf{L}_i^2 = \frac{n_i}{2}\left(\frac{n_i}{2} + 1\right)$$

where $n_i = n_a + n_b$ with **a** and **b** being two oscillators at each site.

- The interaction term needs more work but it can also be written in terms of $a^\dagger, a, b^\dagger, b$. I will spare the details.

O(3) model

- The number operator at each site i.e., $n_i = n_a + n_b$ is related to the truncation over the angular momentum states in the rotor Hamiltonian as $n = 2l_{\text{max.}}$ at each site.

- The correct continuum limit is usually interpreted by computing the mass gap and observing its scaling with β. Tensor network computations using MPS methods [Bruckmann, 2018] have shown that $\beta \sim 1.3$ with $l_{\text{max.}} \sim 4{,}5$ can reproduce the continuum Physics reliably well.

- This is good because we do not need to consider very high photon number states and $|n = 2 \times 5 = 10\rangle$ should suffice.

- The current state of the art methods in photonics quantum experiments have create Fock states up to $|n = 15\rangle$ and is within limit of resources needed. However, the total number of modes (which depends on number of sites in O(3) model) would be a challenge. Implementation of time-evolution of this formulation is work in progress. State preparation in terms of techniques from quantum chemistry/nuclear physics.

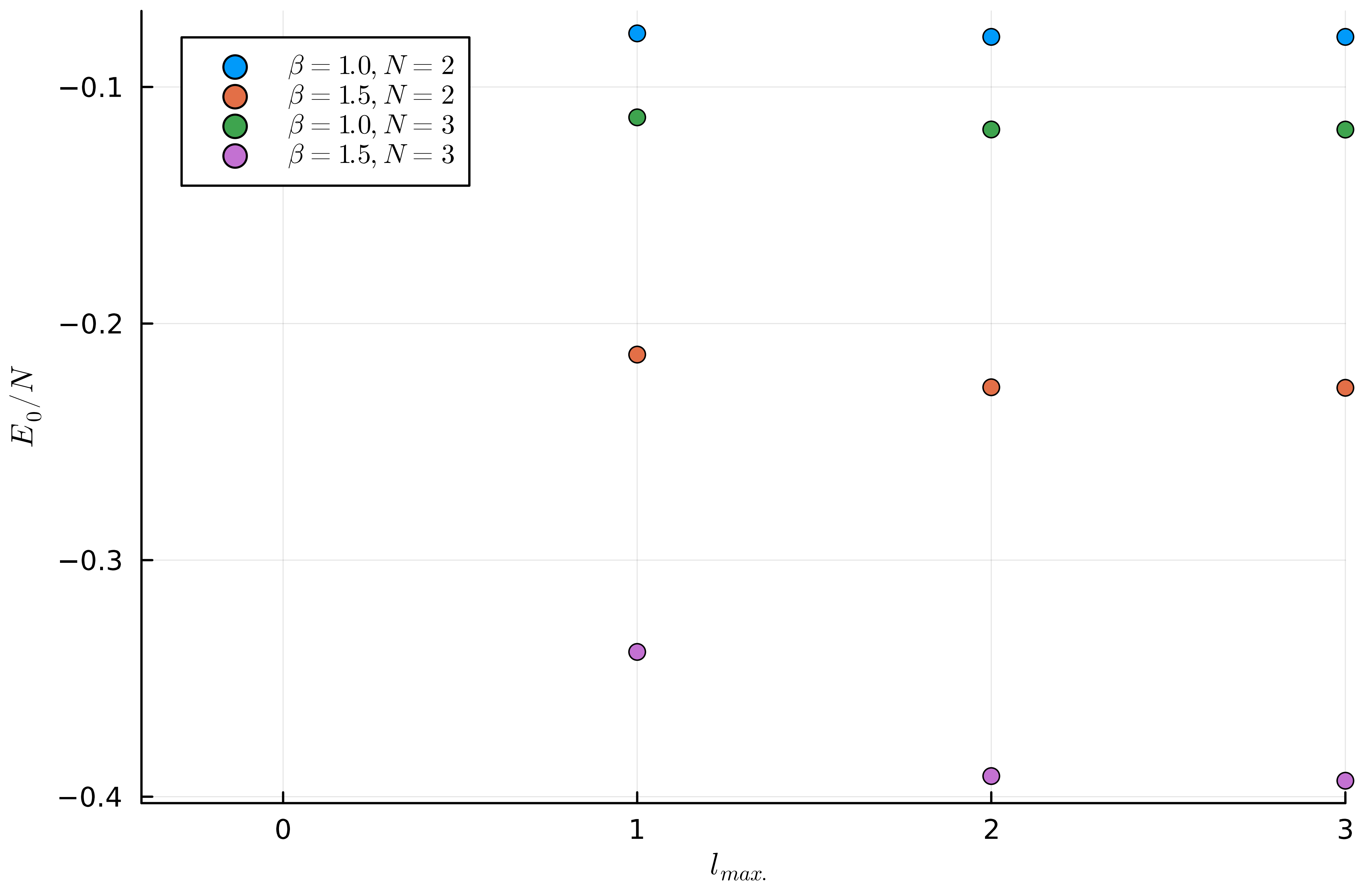
Summary

- We have formulated the O(3) model entirely in terms of oscillators and argued that it is suited for simulation using photonic hardware in coming years.
- We believe the continuum limit can be obtained by considering $\mathcal{O}(10)$ photons combined for both modes at each site which is within the reach of the state-of-the-art methods.
- We can use universal CV gate set to carry out the time-evolution
- CV approach to quantum computing is naturally suited for bosonic systems such as O(3) and scalar field theories. Addition of fermionic d.o.f is possible by considering hybrid methods.
- There is another way of implementing O(3) model for CV quantum computing (rather than the approach presented here). This should be on arXiv soon.

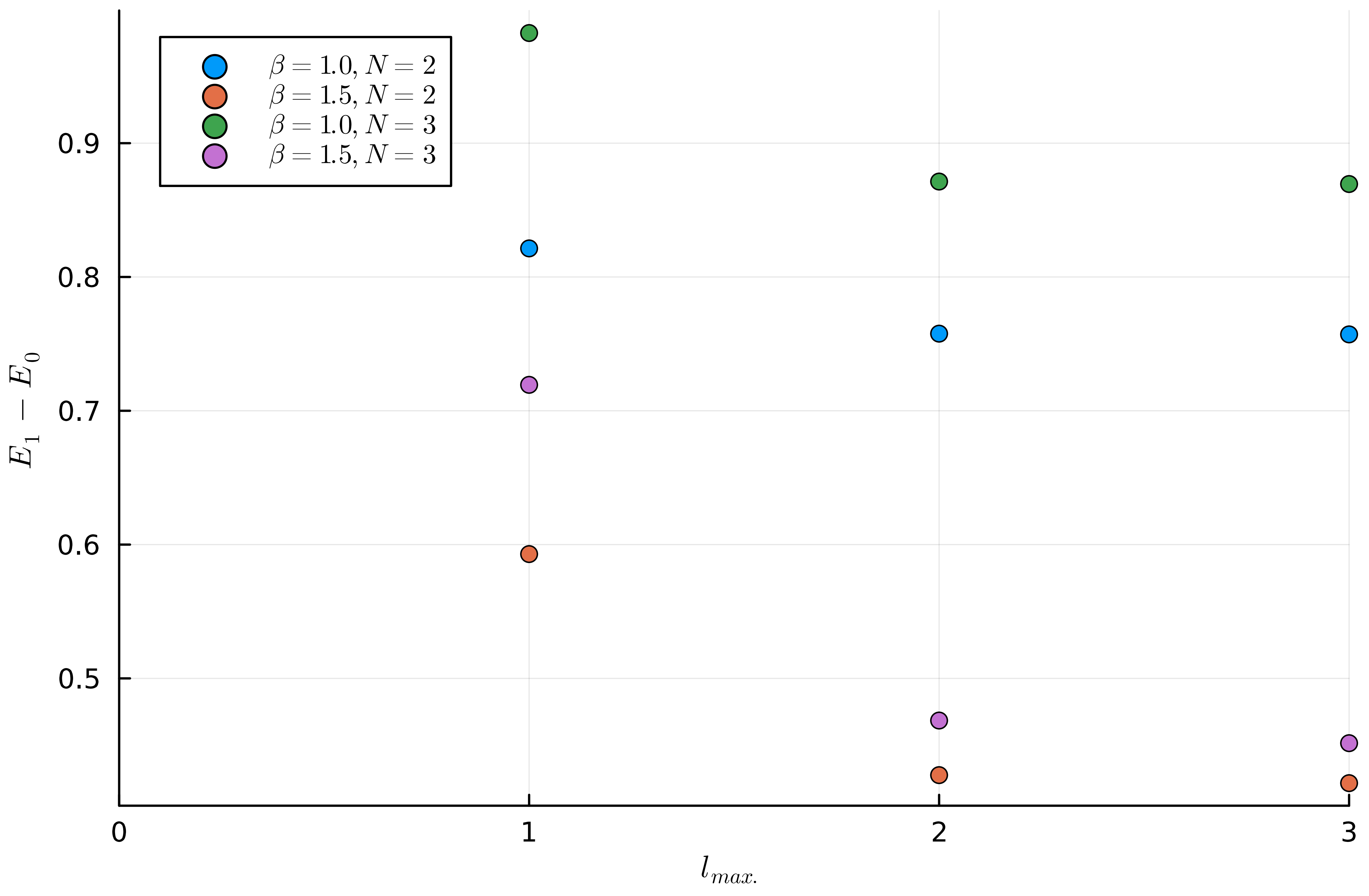
Thank you

Backup

O(3) model - ED



$\beta = 1.0, N = 2$
$\beta = 1.5, N = 2$
$\beta = 1.0, N = 3$
$\beta = 1.5, N = 3$
E_0/N
−0.1
−0.2
−0.3
−0.4
0
1
2
3
$l_{max.}$



$\beta = 1.0, N = 2$
$\beta = 1.5, N = 2$
$\beta = 1.0, N = 3$
$\beta = 1.5, N = 3$
$E_1 - E_0$
0.9
0.8
0.7
0.6
0.5
0
1
2
3
$l_{max.}$