High temperature $U(1)_A$ restoration in the chiral limit

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Fate of chiral symmetry in QCD above T_c

- Banks-Casher: $\langle\bar{\psi}\psi\rangle \propto \rho(0)$
  Chiral symmetry $\Longleftrightarrow$ Dirac spectrum near 0.

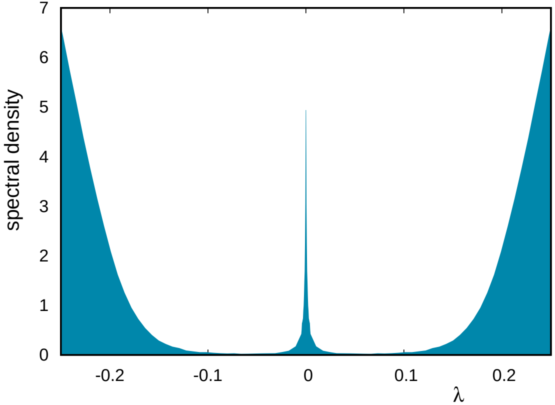
- Peak in the spectral density at 0 (quenched & dynamical).
  Edwards et al. (1999); Alexandru & Horváth (2019, 2021);
  Kaczmarek et al. (2023); Ding et al. (2021).

- Peak suppressed by light dynamical quarks.

- Contribution of the peak to χSB as $m \to 0$?
  (chiral condensate, $U(1)_A$ breaking)

The spectral density of the overlap at $T = 1.1\,T_c$

quenched, Wilson $\beta = 6.13$, $N_t = 8$, exact zero modes removed



Distribution of number of eigenvalues in peak consistent with mixing zero modes of free instanton gas.

Vig & TGK (2021).

Instanon based random matrix model (quenched)

- Model of Dirac operator in the subspace of zero modes
- Quenched – ideal instanton gas:
  - Choose n_i and n_a from independent Poisson distributions of mean $\chi V/2$.
  - Place (anti)instantons randomly in 3d box of size L^3 ($V = L^3/T$).
  - Construct $(n_\mathrm{i} + n_\mathrm{a}) \times (n_\mathrm{i} + n_\mathrm{a})$ random matrix:

$$\begin{array}{cc} n_\mathrm{i} & n_\mathrm{a} \end{array}$$
$$\left(\begin{array}{c|c} 0 & iW \\ \hline iW^\dagger & 0 \end{array} \right)$$

  - $w_{ij} = A \cdot \exp(-B \cdot r_{ij})$,

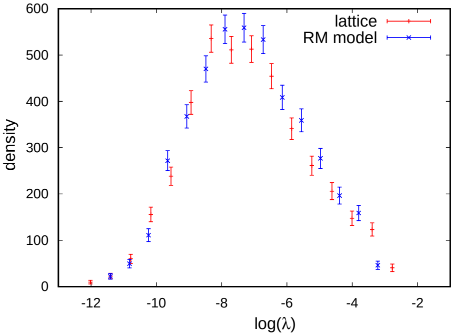
    r_{ij} is the distance of instanton i and antiinstanton j.

Fit parameters to quenched overlap spectrum

quenched, Wilson $T = 1.1\,T_c$, $N_t = 8$

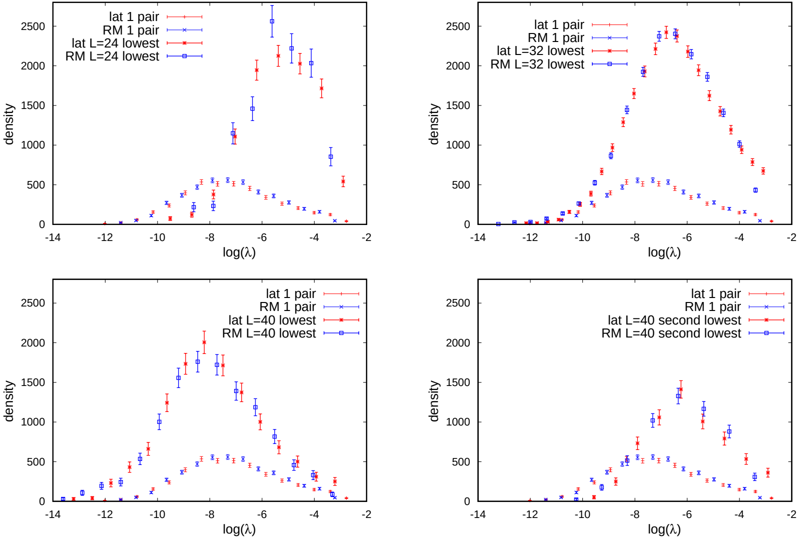
- Three parameters:
  - χ_0 – topological susceptibility – instanton density
  - A, B – parameters of the exponential mixing between zero modes
- Fit A, B to distribution of overlap spectrum

lowest eigenvalue on $32^3 \times 8$ configurations with only one IA pair



Perferct description of quenched overlap spectrum

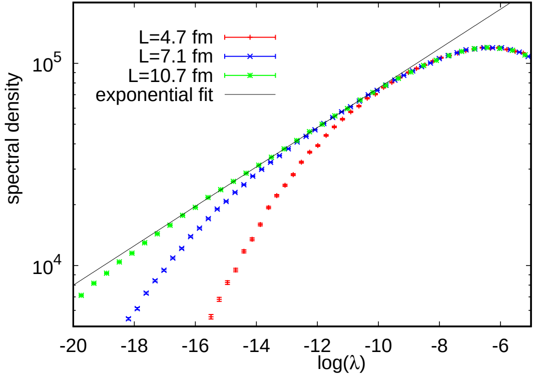
Distribution of lowest and 2nd lowest eigenvalues – different volumes



Quenched spectral density singular at the origin

(in the $V \to \infty$ limit)

RM model simulation, parameters from quenched $T = 1.1\,T_c$ overlap spectrum.

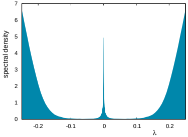


If $\rho(\lambda) \propto \lambda^{\alpha}$ and $y = \log(\lambda)$ then $\tilde{\rho}(y) \propto e^{(1+\alpha)y}$

$\alpha = -0.775(5)$

Include dynamical quarks

- On the lattice $\det(D+m)^{N_f}$ in Boltzmann weight
- $\det(D+m) = \prod_{\text{zmz}}(\lambda_i + m) \times \prod_{\text{bulk}}(\lambda_i + m)$
- Bulk weakly correlated with zero mode zone



- Approximate det with $\prod_{\text{zmz}}(\lambda_i + m)$
- Consistently included in RM model:

$$P(n_i, n_a) = \underbrace{e^{-\chi_0 V} \frac{1}{n_i!} \frac{1}{n_a!} \left(\frac{\chi_0 V}{2} \right)^{n_i + n_a}}_{\text{free instanton gas}} \cdot \det(D+m)^{N_f}$$

Suppression of the spectral peak

- If $|\lambda_i| \ll m \quad \Longrightarrow \quad \prod_i (\lambda_i + m) \approx m^{n_\mathrm{i}+n_\mathrm{a}}$

- $\left(\frac{\chi_0 V}{2}\right)^{n_\mathrm{i}+n_\mathrm{a}} \cdot \det(D+m)^{N_\mathrm{f}} \approx \left(\frac{m\chi_0 V}{2}\right)^{N_\mathrm{f}\cdot(n_\mathrm{i}+n_\mathrm{a})}$

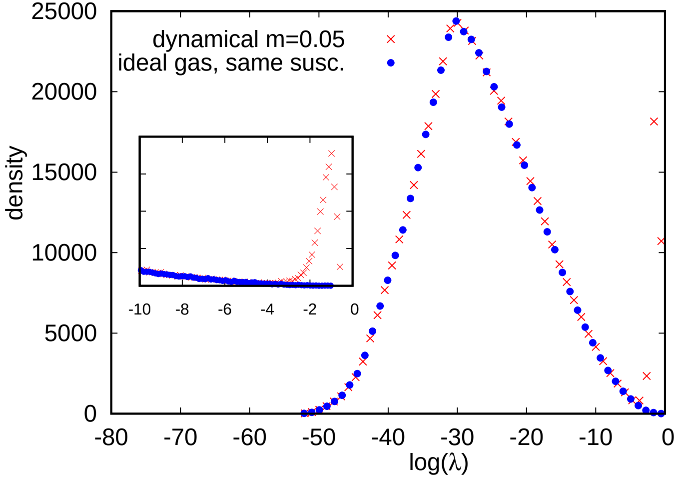
- Distribution of number of (anti)instantons still Poisson

- Free gas, but susceptibility suppressed as $\chi_0 \to m^{N_\mathrm{f}} \chi_0$

- Instanton gas more dilute $\Rightarrow$ $|\lambda_i|$ smaller

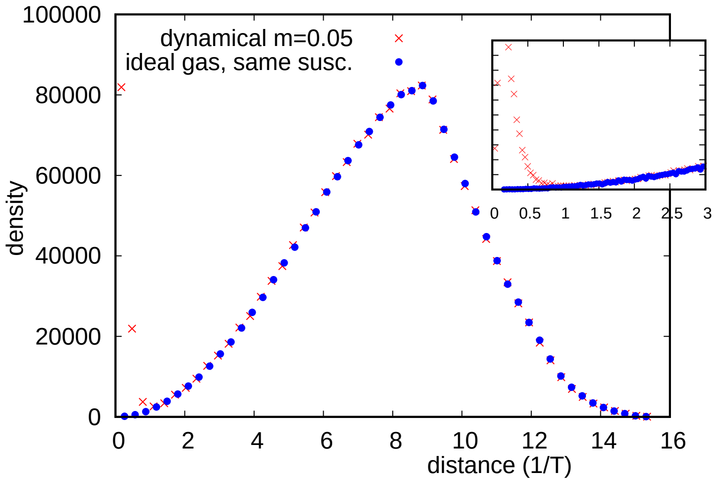
- Even in the chiral limit $|\lambda_i| \ll m \quad \Longrightarrow$ free instanton gas

Spectral density – full QCD vs. ideal instanton gas



Instanton-antiinstanton molecules

density of closest opposite charge pairs at given distance

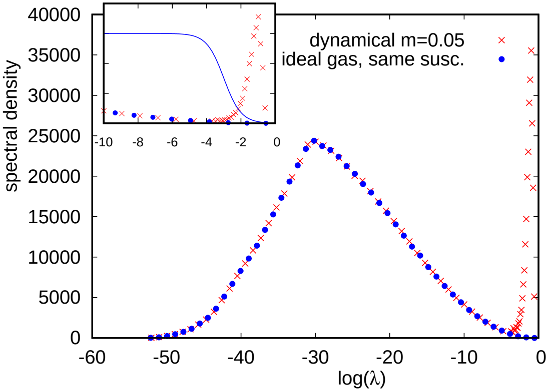


What about Banks-Casher?

Free instanton gas contribution dominates condensate

$$\langle \bar{\psi}\psi \rangle \approx \langle \sum_i \frac{m}{m^2+\lambda_i^2} \rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-}\\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_f}\chi_0 V} \cdot \frac{1}{m} = m^{N_f-1}\chi_0 V$$

$\lambda_i \ll m$



Fate of chiral symmetry as $m \to 0$

- Free IA gas eigenvalues $|\lambda_i| \ll m$ for any quark mass.

- $$\langle \bar{\psi}\psi \rangle \approx \langle \sum_i \frac{m}{m^2+\lambda_i^2} \rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-} \\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_f}\chi_0 V} \cdot \frac{1}{m} = m^{N_f-1}\chi_0 V$$

- $$\chi_\pi - \chi_\delta \approx \langle \sum_i \frac{m^2}{(m^2+\lambda_i^2)^2} \rangle \approx \underbrace{\begin{pmatrix}\text{avg. number of in-} \\ \text{stantons in free gas}\end{pmatrix}}_{m^{N_f}\chi_0 V} \cdot \frac{1}{m^2} = m^{N_f-2}\chi_0 V$$

- Contribution of IA molecules in $m \to 0$ limit: $|\lambda_i| \gg m$
  - For $N_f \leq 2$ numerically small ($\ll 1\%$ for realistic parameters).
  - Contribution to $\langle \bar{\psi}\psi \rangle \propto m$
  - Contribution to $\chi_\pi - \chi_\delta \propto m^2$

Conclusions

- Above T_c details of the quenched overlap spectrum described by simple free instanton random matrix model.
- Spectral density has singular peak at zero.
- With dynamical quarks:
  - Singular peak remains but gets suppressed.
  - Free instanton gas ($\rightarrow$ singular peak) + IA molecules.
  - For $N_f \leq 2$, as $m \rightarrow 0$ χSB dominated by singular spectral peak.
- Chiral limit with N_f degenerate light quarks:
  - $\langle \bar{\psi}\psi \rangle \approx m^{N_f - 1}$ agrees with small m expansion of the free energy Kanazawa and Yamamoto (2015)
  - $\chi_\pi - \chi_\delta \approx m^{N_f - 2}$